The rate-equilibrium method with hyperelastic based constitutive models

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ABSTRACT: The rate-equilibrium method is extended to incorporate hyperelastic based materials. For purely elastic responses, the method provides fast and accurate results. An elastic, viscoplastic constitutive equation with multiplicative decomposition of the deformation gradient is also illustrated. The rate of convergence is found to decrease rapidly as the viscoplastic response increases.

1 INTRODUCTION

Steady state analyses often provide the best approach for the study of forming processes. Because constitutive equations for most materials of interest are nonlinear, an iterative procedure must be used to obtain a solution. Usually, the iterations consist of solving the equilibrium equation for velocity (with stress assumed known), and then solving the constitutive equation for stress (with velocity assumed known). Under some circumstances the iterations converge to acceptable solutions; however, when the elastic response becomes significantly more pronounced than the non-elastic response, these algorithms tend to diverge. To overcome this difficulty, the rate-equilibrium equations have been used to obtain the velocity (Yu & Thompson 1989; Thompson & Yu 1990; Tsai & Thompson 1991; Tsai & Thompson 1992). This approach has been successful for constitutive models that incorporate a hypoelastic material. In the following, we report on results obtained from incorporating a hyperelastic material model with this approach.

2 PROBLEM STATEMENT

We seek an Eulerian, steady-state velocity field which satisfies specified boundary conditions and which, when substituted into an appropriate constitutive equation, generates a Cauchy stress field which is in equilibrium and agrees with specified surface tractions. The velocity field will be generated using the rate-equilibrium equation as described in the next section.

3 THE RATE EQUILIBRIUM EQUATION

The rate-equilibrium equation specifies that a material point which enters the control volume in equilibrium must stay in equilibrium while moving through the control volume. This is enforced by requiring that the material rate of change of the equilibrium vector is everywhere equal to zero, i.e.

$$D_t \left\{ \frac{\partial \sigma_{ij}}{\partial x_i} \right\} = -\frac{\partial u_k}{\partial x_k} \frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial}{\partial x_i} \left[u_k \frac{\partial \sigma_{ij}}{\partial x_k} - \frac{\partial u_i}{\partial x_k} \sigma_{kj} + \frac{\partial u_k}{\partial x_k} \sigma_{ij} \right] = 0 \tag{1}$$

where the term in brackets is the material derivative of the first Piola-Kirchhoff stress, $\dot{T}_{ij}$, evaluated in the reference configuration (Tsai & Thompson 1991).

The weak form of this equation, where the test functions are interpreted as variations in components of the velocity, is

$$\int_V \left[\frac{\partial \delta u_j}{\partial x_i} \dot{T}_{ij} + \delta u_j \frac{\partial u_m}{\partial x_m} \frac{\partial \sigma_{ij}}{\partial x_i} \right] dV = \int_S \delta u_j \dot{T}_{ij} \nu_i dS \tag{2}$$

with the natural boundary conditions being the components of the material rate of change of the surface tractions of the first Piola-Kirchoff stress.

4 CONSTITUTIVE EQUATIONS

Elastic Response: For pure elastic analyses, the material is assumed to be compressible neo-Hookean (Simo & Pister 1984; Bonet & Wood 1997). The Cauchy stress is given by

$$\sigma_{ij} = \frac{\mu}{J^e}\left(b^e_{ij} - \delta_{ij}\right) + \frac{\lambda}{J^e}\left(\ln J^e\right) \tag{3}$$

where μ and λ are material coefficients, and

$$b^e_{ij} = F^e_{ik} F^e_{jk} \tag{4}$$

$$J^e = \det(F^e_{ij}) \tag{5}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = F^e_{ik} F^p_{kj} \tag{6}$$

and x_i and X_i are the Eulerian (spatial) and Lagrangian (material) coordinates respectively.

Viscoplastic Response: For elastic-plastic behavior, a Bingham type material is assumed with multiplicative decomposition of the elastic and inelastic deformation gradients. The elastic response is again modeled with the neo-Hookean equation described above, while the inelastic response is given by,

$$D^p_{ij} = \text{sym}[L^p_{ij}] = \frac{1}{2\eta}\left\langle 1 - \frac{k}{\sigma_{eq}}\right\rangle \sigma'_{ij} \tag{7}$$

where

$$< x >= \begin{cases} 0, x \le 0 \\ x, x > 0 \end{cases} \tag{8}$$

$$\dot{F}^p_{ij} = L^p_{ik} F^p_{kj} \tag{9}$$

$$\sigma_{eq} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} \tag{10}$$

and σ'_{ij} is the deviatoric component of the Cauchy stress, k is the yield stress in simple tension, and η is the apparent viscosity taken as a constant.

5 STRESS CALCULATIONS

After a velocity field has been created using the rate-equilibrium equations, it is necessary to calculate a corresponding stress field. For a pure elastic material, this is accomplished in the following manner:

1. For current velocity field, solve

   $$\dot{F}_{ij} = L_{ik} F_{kj}$$

   to obtain nodal values of F_{ij}. This is a global solution using the method of weighted residuals over the entire domain.

2. With F_{ij} known at each node, calculate the Cauchy stress using Eq. 3. This is a local solution, performed node by node.

For elastic-viscoplastic materials the stress is found by solving Eqs. 7 and 3 iteratively. The steps are:

1. With the current state of stress, solve (Eq. 7) locally to obtain L^p_{ij} at each node. Assume:

   $$L^p_{ij} = \text{sym}(L^p_{ij}) = D^p_{ij}$$

2. With L^p_{ij} now specified at each node, solve globally

   $$\dot{F}^p_{ij} = L^p_{ik} F^p_{kj}$$

3. At each node, perform a local multiplicative decomposition of the deformation gradient:

   $$F_{ij} = F^e_{ik} F^p_{kj}$$

   to obtain nodal values of F^e_{ij}.

4. With F^e_{ij} now specified at each node, calculate the Cauchy stress at each node using Eq. 3.

5. Compare new stress with that used in Step 1 above; if sufficiently close, return to rate-equilibrium equation for calculation of new velocity field; if not, return to Step 1 above.

6 EQUILIBRIUM CORRECTION

Although the rate-equilibrium equation should assure equilibrium throughout the control volume, the finite element approximations for velocity and stress can prevent this from happening to the degree of accuracy desired. Therefore, an equilibrium check is included as part of the algorithm. The most obvious correction to be made when equilibrium is not satisfied to the degree of accuracy desired is to refine the mesh. However, to avoid the large computational times associated with this approach, several other alternatives were investigated, none of which proved totally satisfactory.

The method currently used is to check equilibrium at the centroid of each element after each calculation of the stress field as described in the previous section. The material derivative of the resulting equilibrium residual is then calculated and its negative is used as the right hand side of Eq. 1. Although this procedure does lead to equilibrium being satisfied at the centroid of each element, it can lead to spatial oscillations in the stress field. This entire area of research continues to be investigated.

7 ILLUSTRATIONS

For illustration, the rolling of a slab is considered. A roll radius equal to 2.75 the thickness of the slab. The elastic constants in Eq. 3 were selected so that $\lambda/\mu = 1.5$. For both examples shown below, the free surface was adjusted to correspond to a streamline.

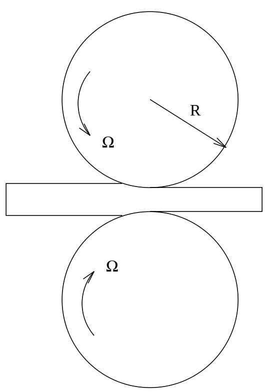

Figure 1: Slab rolling

Figure 2 shows contours of the equivalent stress, σ_{eq}, for a hyperelastic material undergoing a 25% reduction. The contour interval equals 0.1μ, with the largest value being 0.803μ. As one would expect for a purely elastic material, the contours are symmetric and the results were found to be independent of the roll speed.

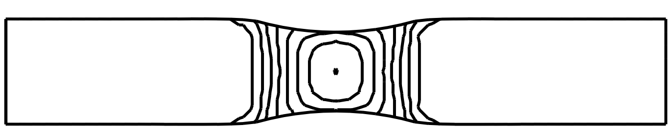

Figure 2: Rolling of a hyperelastic material

The second example is the rolling of the Bingham type material previously discussed with a 10% reduction. The parameters used for the results shown in Fig. 3 were

$$\frac{k}{\mu} = 0.00325 \qquad \frac{\eta\Omega}{\mu} = 0.0159$$

where Ω is the angular velocity of the roll. The contours shown are for the horizontal component of the Cauchy stress (σ_{xx}). The contour interval equals 0.0154μ. The maximum compressive component equals 0.067μ and the maximum tensile component equals 0.056μ. The contour extending into the billet is zero, thus showing the separation of the residual compressive stress in the inner core of the slab and the residual tensile stress in the outer regions. The magnitude of the maximum residual stress is less than 3% of the magnitude of the stress appearing under the roll.

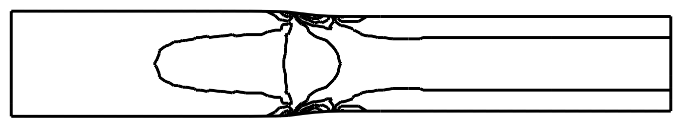

Figure 3: Rolling of a Bingham type material

8 CONCLUDING REMARKS

The incorporation of a hyperelastic constitutive equation into the rate equilibrium method for determining steady state flows of material through a control volume has been found to be a stable and relatively fast method of analysis. When coupled with a viscoplastic component, using a multiplicative decomposition of the deformation gradient, the algorithm was again found to be stable, but relatively slow. The rate of convergence becomes unacceptably slow as the the non-elastic components approach those to be found in most metal forming processes. Work continues to be conducted on ways to increase the speed of convergence. In addition, it has been found that equilibrium may not be satisfied to a degree of accuracy desired unless very fine meshes are used with their increased computational cost. In an effort to circumvent this, research continues to be conducted for ways to add correction factors to the rate equilibrium equation which will drive the solution toward stress fields which more accurately satisfy equilibrium.

REFERENCES

Bonet, J. & R. D. Wood (1997). *Nonlinear Continuum Mechanics for Finite Element Analysis.* Cambridge University Press.

Simo, J. & K. Pister (1984). Remarks on rate constitutive equations for finite deformation problems: computational implications. *Computational Methods in Applied Mechanics and Engineering 46*, 201–215.

Thompson, E. & S.-W. Yu (1990). A flow formulation for rate equilibrium equations. *Int. J. for Num. Meth. in Eng. 30*, 1619–1632.

Tsai, L. & E. G. Thompson (1991). Flow formualtion of the rate equilibrium equations applied to elastic, viscoplastic materials. In R. Batra & B. Armaly (Eds.), *Developments in Mechanics, Volume 16*, pp. 243–244. University of Missouri - Rolla.

Tsai, L. & E. G. Thompson (1992). Elastic, viscous, steady-state flows by the rate equilibrium method. In R. W. J.-L. Chenot & O. Zienkiewicz (Eds.), *NUMIFORM92*, pp. 109–115. A.A.Balkema.

Yu, S.-W. & E. Thompson (1989). A direct Eulerian finite element method for steady state elastic plastic flow. In E. Thompson, R. Wood, O. Zienkiewicz, & A. Samuelsson (Eds.), *NUMIFORM89*, pp. 95–103. A.A.Balkema.