Torsion of Non-Circular, Simply-connected Cross-sections

Actions and/or Assumptions	Consequences and Comments
Assume projections of cross-sections on the x-y plane rotate as rigid bodies. Assume displacement in the axial direction is the same for all cross-sections. This is due to Saint Venant, 1853.	$$u = -\alpha z y$$ $$v = +\alpha z x$$ $$w = +\alpha \Psi(x, y)$$ Where α = angle of twist per unit length of rod. $$(\sigma) = \begin{pmatrix} 0 & 0 & \sigma_{xz} \\ 0 & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{pmatrix}$$ $\Sigma F_y = 0$ and $\Sigma F_x = 0$ are identically satisfied.
Define the following stress function: $$\sigma_{zx} = +\frac{\partial \Phi}{\partial y}$$ $$\sigma_{zy} = -\frac{\partial \Phi}{\partial x}$$ This is due to Prandtl, 1903.	$\Sigma F_z = 0$ is identically satisfied. All equilibrium equations are now satisfied.
Assume linear elastic material and relate Prandtl's stress function and Saint Venant's assumed displacements.	$$\sigma_{zx} = G\left(-\alpha y + \frac{\partial w}{\partial x}\right) = +\frac{\partial \Phi}{\partial y}$$ $$\sigma_{zy} = G\left(+\alpha x + \frac{\partial w}{\partial y}\right) = -\frac{\partial \Phi}{\partial x}$$
Eliminate w from above equation.	One equation and one unknown: $$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2G\alpha$$ This equation guarantees that Prandtl's stress function is compatible with Saint Venant's assumed displacements, hence strains are compatible.
Enforce that shear stress is tangent to surface.	Φ is constant along boundary of cross-section
$\Sigma M_z = 0$	$$T = 2\int_A \Phi dA$$

Torsion of Non-Circular, Multiply-connected Cross-sections

Equations	Description
$\sigma_{zx} = \frac{\partial \Phi}{\partial y}$ $\sigma_{zy} = -\frac{\partial \Phi}{\partial x}$	Prandtl's stress function. Insures equilibrium equations are satisfied
$\Phi =$ Constant on boundary	Insures stresses are tangent to boundaries thus all boundary condtions are satisfied.
$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\mathrm{G}\alpha$	Insures compatibility of strains derived from Prandtl's stress function.
$T = 2\int_A \Phi dA - 2\Phi_0 A_0 + \sum_{i=1}^{n} 2\Phi_i A_i$	From summation of moments about the z-axis. A = Solid area of cross-section. n = number of opennings in cross-section Φ_i = value of stress function on boundary of i^{th} hole. A_i= Area of i^{th} opennings. $i = 0$= Outside boundary.
$\int_{C_i} \tau dS = 2\mathrm{G}\Theta A_i$	From enforcement of condition that the displacement in the z-direction is single-valued. τ = shear stress tangent to boundary of opennings.