

Lateral Stability of a Slender Cantilever Beam With End Load

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Consider the slender cantilever beam with an end load shown in Figure 1. The bending moment at any cross-section is in the x -direction. This is true regardless of how the beam deforms. If the beam deflects only in the y -direction (Fig 2a), then this bending moment will be about the major principal axis of the beam. In such a case, the deflections of the beam and energy of the system can be obtained easily from elementary theory. However, if the beam twists (Fig 2b) then the bending moment, which remains in the x -direction, will also have a component along the minor principal axis of a cross-section (Fig. 2c) and causes bending about this axis. The calculations for deflections and strain energy thus become more complex.

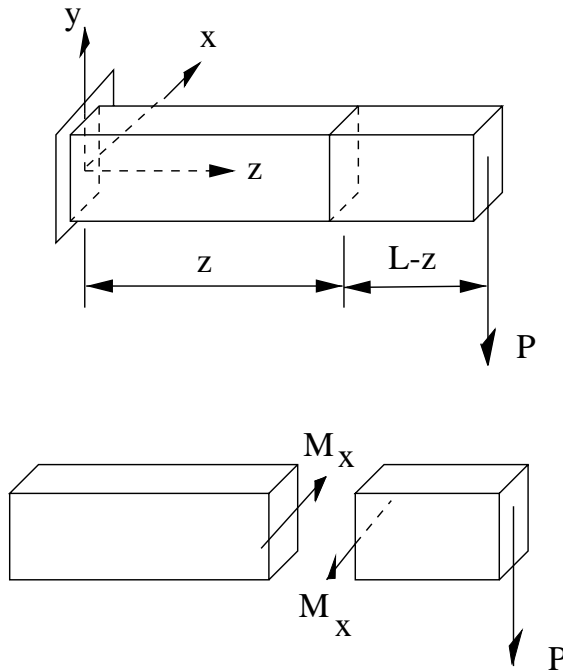


Figure 1: Slender Cantilever Beam

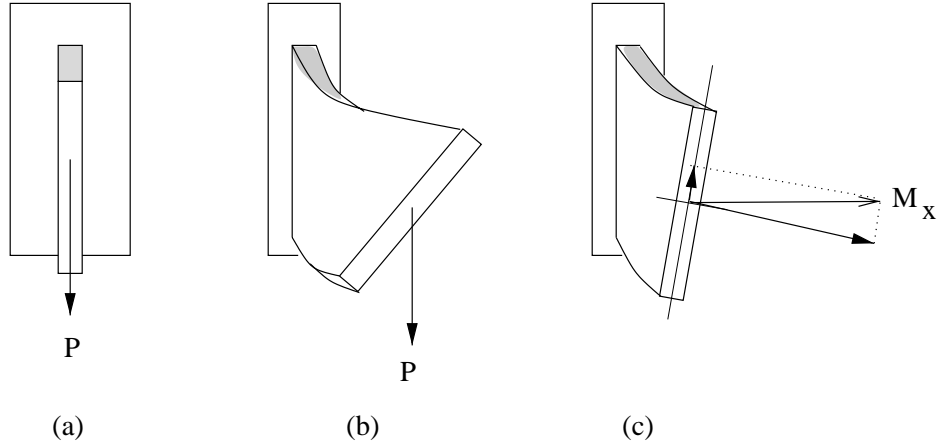


Figure 2: Lateral Buckling

The positions of the beam shown in Figures 2a and 2b should be considered as two possible equilibrium positions of the beam under the same load P . The second position is probably less desirable and its occurrence is referred to as lateral buckling. If the total mechanical energy associated with position b is less than that associated with position a , then the beam in position a is unstable and any small disturbance will cause it to move to position b . On the other hand, if the energy in position b is greater than that in position a , then the beam, after such a disturbance, will return to the lower energy level at a . Our problem is to determine what load P will cause the beam to become in neutral equilibrium in position a . We thus calculate the value of the load necessary to produce just enough positive work in moving from position a to position b to balance the increase in strain energy that takes place in moving from position a to b .

The strain energy in position b is comprised of three components:

1. energy due to bending about the ξ axes,
2. energy due to bending about the η axes, and
3. energy due to twisting about the z -axis.

We will make the assumption that position b is very close to position a ; hence, the strain energy due to bending about the ξ -axis is the same in both positions. Therefore, we need only calculate the strain-energy in position b due to bending

about the η -axis and the strain-energy due to twisting about the z-axis to determine the change in strain-energy from a to b . Once found, we will calculate the magnitude of the load P necessary to produce enough work from a to b to account for this increase in strain energy. This then will be the critical load that will allow the beam to move from one position to the other with no increase in total potential energy of load and beam. Any load greater than this will cause a decrease in total potential energy for positions away from a while any load less than this will cause an increase in total potential energy while moving away from a .

Before proceeding, you should understand two points. The first is that the rotation and the side-sway are related - you can't have one without the other. Any side sway will cause the applied load P to move away from the z-axis thus causing a twisting moment about the axis and rotation. Any rotation of the cross-section will produce a component of moment about the minor principal axis thus side-sway.

The second point you should understand is that the beam in the position shown in Figure 2a is in equilibrium; thus, by virtual work, any infinitesimal motion from this position will not change the total energy of the system. Although we will be considering that our change to position b is small, it will be finite, not infinitesimal. For example, we will assume $\sin(\beta) \approx \beta$ but not zero.

We now turn our attention to the actual calculations needed to solve our problem. Figure 3a shows an arbitrary cross-section of the beam looking toward the built-in end. The cross-section itself remains rectangular and makes an angle with the y-axis equal to β . For each cross-section, two directions in space will be considered that are equal to the directions of the major and minor axes of the cross-section. These are shown as ξ and η . Figure 4 shows the components of the bending moment in these two directions. Because we assume β small, we can write

$$M_\eta = M_x \sin \beta \approx M_x \beta$$

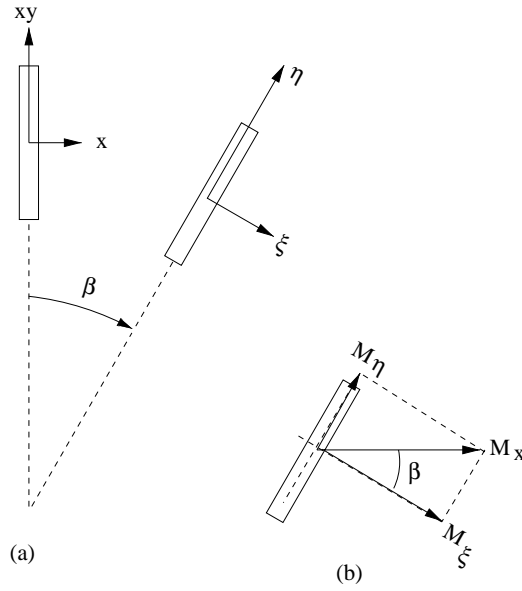


Figure 4 Components of applied moments

The equations between curvature and moment about principal axes are still valid at every cross-section, thus

$$EI_{\eta} \frac{d^2 u_{\xi}}{dz^2} = M_x \beta$$

Note that this gives us the relationship between the angle of twist and the side-sway.

We are now in a position to write our expression for strain energy due to bending about the minor principal axis and torsion about the z-axis.

$$U = \frac{1}{2} \int_0^{\ell} \frac{M_{\eta}^2}{EI_{\eta}} dz + \frac{1}{2} \int_0^{\ell} \frac{T_z^2}{JG} dz$$

where JG is the torsional rigidity of the cross-section. From Figures 1 and 3, we have

$$M_{\eta} = M_x \beta = P(\ell - z)\beta$$

and if we write the strain energy for torsion as

$$\frac{1}{2} \int_0^\ell \frac{T_z^2}{JG} dz = \frac{1}{2} \int_0^\ell JG \left(\frac{d\beta}{dz} \right)^2 dz$$

our expression becomes

$$U = \frac{1}{2} \int_0^\ell \frac{(P(\ell - z)\beta)^2}{EI_\eta} dz + \frac{1}{2} \int_0^\ell JG \left(\frac{d\beta}{dz} \right)^2 dz$$

We now have an expression for the increase in strain energy in terms of one variable, $\beta(z)$. Our next task is to determine the work done by P in terms of this same variable.

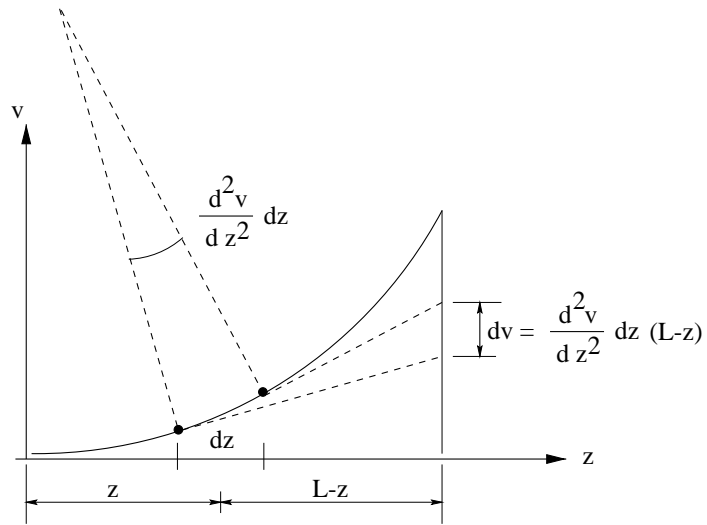
In order to determine the work done by P we first determine its displacement in the y -direction. Recall from elementary mechanics of solids the moment-area method given for determining the deflection of beams. This method is represented in Figure 5a where the deflection in the y -direction, v , is shown as a function of z .

The change in the slope of this curve in the distance dz produces a “stroke” on a line passing through $z = L$ in the y -direction equal to

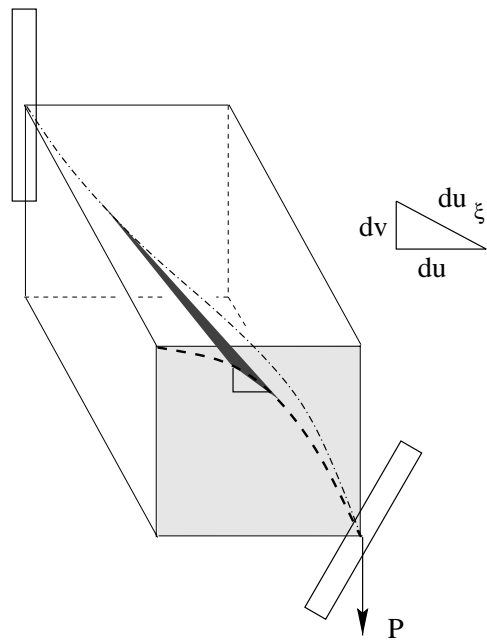
$$d(\Delta y) = \frac{d^2 v}{dz^2} dz (L - z)$$

The sum of these “strokes” will be the total deflection, Δy in the y -direction at the end of the beam. In our case, we do not have a convenient expression for the second derivative of v . However, we do have one for u_ξ , namely

$$\frac{d^2 u_\xi}{dz^2} = \frac{M_\eta}{EI_\eta} = \frac{M_x \beta}{EI_\eta} = \frac{P(\ell - z)\beta}{EI_\eta}$$



(a)



(b)

Figure 6 Deflection curves on the plane $Z = \ell$

Now at any point z along the axis of the beam, an increment of dz will paint a “stroke” on a plane perpendicular to the z -axis and passing through $z = L$. Because the direction of these strokes will change for each cross-section between $z = 0$ to $z = L$, these strokes could “paint” a rather fanciful curve on the plane as shown in Figure 5b. However, if for each incremental “stroke”, we take its component in the y -direction and add each of these components, we will end up with the total displacement of the end of the beam in the y -direction. The component in the y -direction is

$$d(\delta y) = -\frac{P(\ell - z)\beta}{EI_\eta} dz(\ell - z)\sin(\beta) \approx -\frac{P(\ell - z)\beta}{EI_\eta} dz(\ell - z)\beta$$

and therefore

$$-\delta_y = \int_0^\ell dy = \int_0^\ell \frac{P(\ell - z)\beta}{EI_\eta} \beta(\ell - z) dz$$

where we have emphasized that the displacement in the negative y -direction is the displacement in the direction of the load P . The work done by this load is therefore

$$W = P \int_0^\ell \frac{P(\ell - z)\beta}{EI_\eta} \beta(\ell - z) dz$$

or

$$W = \int_0^\ell \frac{P^2(\ell - z)^2\beta^2}{EI_\eta} dz$$

We now set the work done by P equal to the increase in strain energy

$$W = U$$

to obtain

$$\int_0^\ell \frac{P^2(\ell - z)^2\beta^2}{EI_\eta} dz = \frac{1}{2} \int_0^\ell \frac{(P(\ell - z)\beta)^2}{EI_\eta} dz + \frac{1}{2} \int_0^\ell JG \left(\frac{d\beta}{dz} \right)^2 dz$$

which simplifies to

$$P^2 = (EI_\eta)(JG) \frac{\int_0^\ell (\beta')^2 dz}{\int_0^\ell (\ell - z)^2 \beta^2 dz}$$

The critical P is the smallest value that can be obtained from the above equation when considering all possible functions for β that satisfy the boundary conditions. If not all possible functions are considered, then the smallest value of P found from

the functions considered can be considered the best approximation obtainable from these functions. If only one function is considered, the value obtained for the critical P is good only if the function used is a good representation of the true β .

EXAMPLE

We now select a function which we think might be a good approximation for β . We know that it must be zero at the wall ($z = 0$) and that at $x = \ell$ its gradient in the z -direction must be zero since the $T_z = 0$ at this point. Thus we start with the latter condition and assume

$$\frac{d\beta}{dz} = \alpha(\ell - z)$$

where α is an unknown constant. Integrating this expression we obtain

$$\beta = -\frac{1}{2}\alpha(\ell - z)^2 + C_1$$

If this is to be zero at $z = 0$, then

$$C_1 = \frac{\ell}{2}\alpha$$

and we have

$$\beta = \alpha[z(2\ell - z)]$$

where we have incorporated a one-half into the unknown α . Substitution of this expression and its derivative with respect to z into our expression for P^2 , gives us

$$P_{cr} = \left(\sqrt{\frac{105}{6}}\right) \left(\frac{\sqrt{(EI_\eta)(JG)}}{\ell^2}\right) = 4.183 \frac{\sqrt{(EI_\eta)(JG)}}{\ell^2}$$

A more accurate analysis shows that the correct coefficient should be 4.01, thus our approximate answer is in error by only 4%.

QUESTIONS

1. If we were to choose a more complex approximation for β , e.g.

$$\beta = \alpha[z(2\ell - z)] (\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3)$$

what would change about the above procedure? Outline what you would do in this case.

2. We have taken a simple cantilever beam subjected to an end load and assumed two different equilibrium positions. Are not solutions to mechanics of material problems unique? How can there be two possible equilibrium positions? Why haven't we worried about this when working other problems?

3. Give an argument based on physical grounds why we should expect approximate solutions to the critical load by the above method to be larger than the actual value.