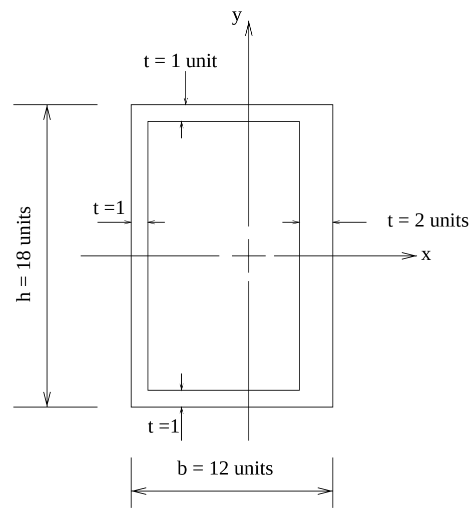
CE560 Advanced Mechanics of Solids

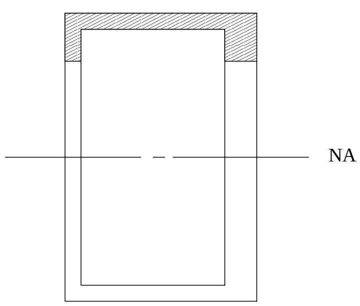
Shear-stress in Multiply-Connected, Thin-Wall Beams

Consider a beam with the cross-section shown. Assume it is subjected to a positive bending moment (compression in the upper half of the beam) and that this moment is increasing in magnitude out of the page. We know that the rate of change of the moment out of the page is the total shear force acting at the cross-section being considered, i.e.

$$\frac{dM}{dz} = V$$

Our problem is to determine the shear flow in the walls at this cross-section.

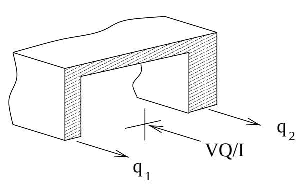




Elementary theory related to shear flow is still valid. Hence, for the cross-hatched section shown, the total unbalanced force per unit length of beam that is acting into the page due to increasing bending moment out of the page is

$$q = \frac{VQ}{I}$$

where Q is the moment of area of the cross-hatched area about the neutral axis (and centroidal axis) and I is the moment of inertia about the neutral axis. This unbalanced force must be balanced by shear forces acting at the cuts defining the cross-hatched area, hence

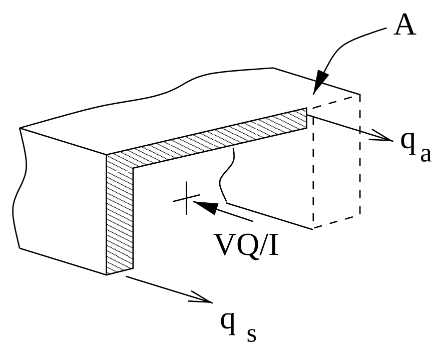


$$q = q_1 + q_2 = \frac{VQ}{I}$$

However, we do not know how the unbalanced force is distributed between the two walls. If the beam were symmetrical about the y-axis, then a reasonable assumption would be that

$$q_1 = q_2$$

Unfortunately, the beam shown in not symmetrical about the y-axis and such an assumption is not logical.



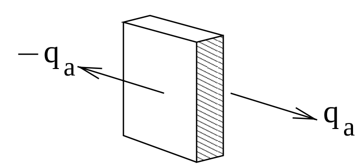
We conduct our analysis, therefore, by picking an arbitrary (but convenient) point in the cross-section and use the shear-flow at that point as a reference value for the shear-flow at any other point. Let the selected point be A and the the shear-flow at that point be designated as q_a. Let the shear-flow at any other point be designated as q_s. Then, at any point such as that shown we can write:

$$q_s + q_a = \frac{VQ}{I}$$

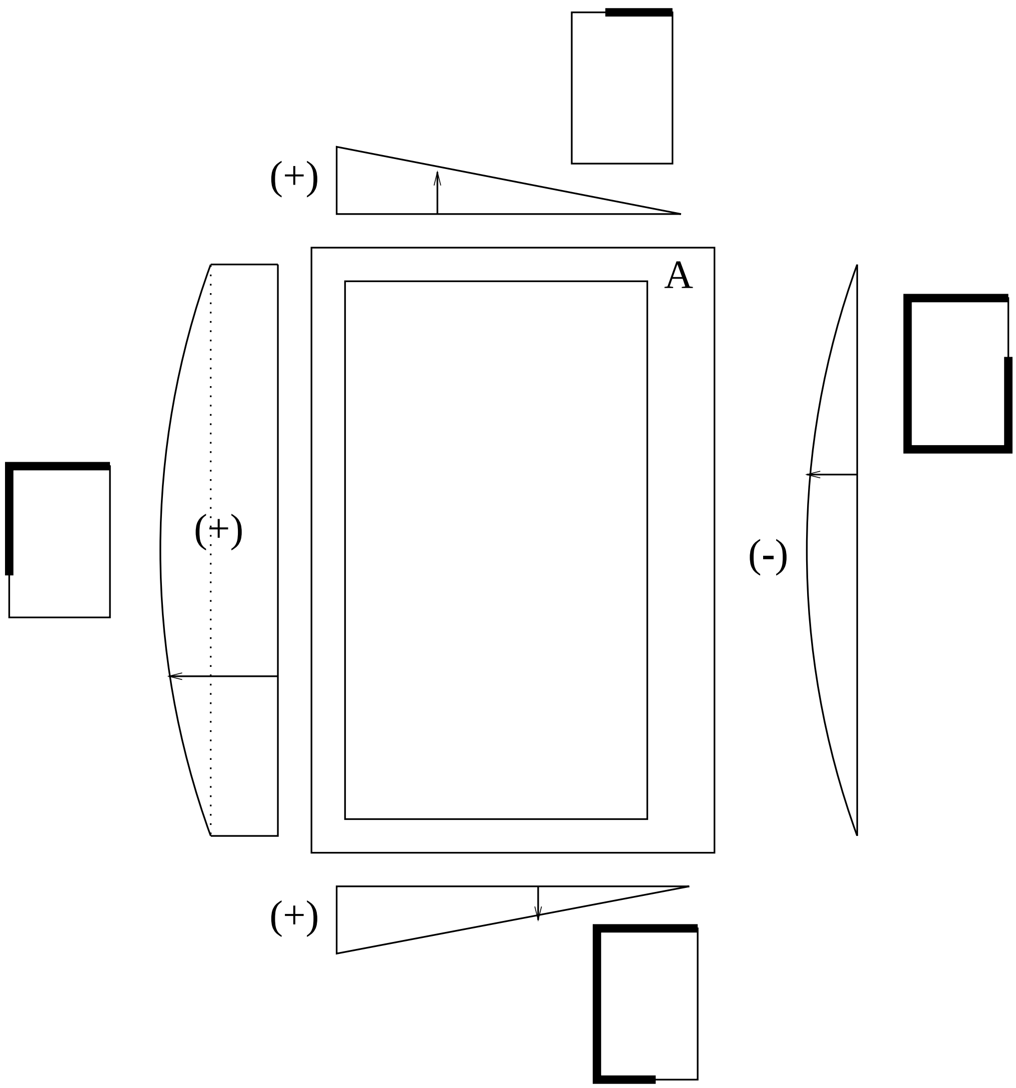
which gives us

$$q_s = \frac{VQ}{I} - q_a$$

Note that for a section very close to point A, $Q \approx 0$ and then $q_s \rightarrow -q_a$. This situation is shown on the right.



The following figure shows a plot of Q as defined above for all points along the walls of the cross-section. When multiplied by V/I, it is equal to the total unbalance force, q, on these sections. If q_a is then subtracted from this total q, the remainder would equal the shear-flow at the corresponding point, i.e. q_s.



Plot of Q around wall centerline starting at point A.

The figure on the previous page would be sufficient to determine the shear flow at any point if we knew the value of q_a. Our next task, therefore, is to determine this value. We do this by noting that the shear flow will cause deformations of the walls in the z-direction. If the wall was continuous before the loads were applied, then this deformation must be singled valued. That is, if we begin at any point along the wall and move around the wall, adding (integrating) the increments of displacements in the z-direction, we must end with a net change of zero if we come back to the point where we started. Hence,

$$\oint_C dw = \oint_C \frac{dw}{ds} ds = 0$$

where C is a complete path that comes back to its starting point, w is the displacement in the z-direction, and ds is an increment of length along the path C. To perform this integration we will assume:

Deformations in the plane of the cross-section due to the shear V produce only rigid-body translation of the cross-section. (No rotation and no distortion in the plane of the cross-section). Hence, the displacements in the x-direction, u, and the y-direction, v, are functions only of z.

With this assumption we can write

$$\begin{aligned} \epsilon_{zx} &= \frac{1}{2}\left(\frac{\partial u(z)}{\partial z} + \frac{\partial w(x,y,z)}{\partial x}\right) &= \frac{1}{2G}\sigma_{zx} \\ \epsilon_{zy} &= \frac{1}{2}\left(\frac{\partial v(z)}{\partial z} + \frac{\partial w(x,y,z)}{\partial y}\right) &= \frac{1}{2G}\sigma_{zy} \end{aligned}$$

We now note that

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

and that we can write

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\sigma_{zx} - \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial y} &= \frac{1}{G}\sigma_{zy} - \frac{\partial v}{\partial z} \end{aligned}$$

Because the displacements u and v are functions only of z, the last terms in these expressions are constant at any given cross-section. With this in mind, we can write

$$dw = \frac{1}{G}\left(\sigma_{zx}dx + \sigma_{zy}dy\right) - C_1 dx - C_2 dy$$

Integration of $\int dx$ and $\int dy$ along any closed path is zero. Hence:

$$\oint C_2 dx = C_2 \oint dx = 0$$

We now have

$$\oint dw = \frac{1}{G} \oint (\sigma_{zx} dx + \sigma_{zy} dy) = \frac{1}{G} \oint \tau ds$$

where τ is the resultant shear stress in the direction of integration. Hence, because this integration must be zero, we conclude

$$\oint \tau ds = 0$$

For thin-wall sections, we can assume that the shear stress is uniformly distributed across the thickness of the wall and equal to the shear flow divided by the wall thickness. We therefore have:

$$\oint \left(\frac{q_s}{t}\right) ds = 0$$

Finally, this gives us

$$\oint \left(\frac{(\frac{VQ}{I} - q_a)}{t}\right) ds = 0$$

$$\oint \left(\frac{VQ}{It}\right) ds - \oint \left(\frac{q_a}{t}\right) ds = 0$$

or, because V, Q and q_a are constant

$$\frac{V}{I} \oint \left(\frac{Q}{t}\right) ds - q_a \oint \left(\frac{1}{t}\right) ds = 0$$

This is one equation for the one unknown, q_a. Hence its solution provides us with the information necessary to determine the shear flow at each point in the walls of our beam.

Numerical Example

We now calculate q_a using the nominal dimensions (the dimensions to the centerlines of the walls) of the cross-section shown on the first page. The integration of

$$\oint \left(\frac{Q}{t}\right) ds$$

is simply the sum of the areas under each of the curves shown on page 3 divided by the their corresponding wall thickness. Thus

$$\begin{aligned}
\oint \left(\tfrac{Q}{t}\right) ds \;=\; & +\tfrac{1}{2}(10.5)\left[8.5(10.5)(1)\right]/1 && \text{top} \\
& +\tfrac{2}{3}(17)\left[4.25(8.5)(1)\right]/1 + (17)\left[8.5(10.5)(1)\right]/1 && \text{left side} \\
& +\tfrac{1}{2}(10.5)\left[8.5(10.5)(1)\right]/1 && \text{bottom} \\
& -\tfrac{2}{3}(17)\left[4.25(8.5)(2)\right]/2 && \text{right side} \\
\;=\; & 2454
\end{aligned}$$

The next integral is simply the length of each side divided by its thickness. Hence

$$\begin{aligned}
\oint \left(\tfrac{1}{t}\right) ds \;=\; & +10.5/1 && \text{top} \\
& +17/1 && \text{left side} \\
& +10.5/1 && \text{bottom} \\
& +17/2 && \text{right side} \\
\;=\; & 46.5
\end{aligned}$$

With these values our equation is simply

$$\frac{V}{I}(2454) - q_a(46.5) = 0$$

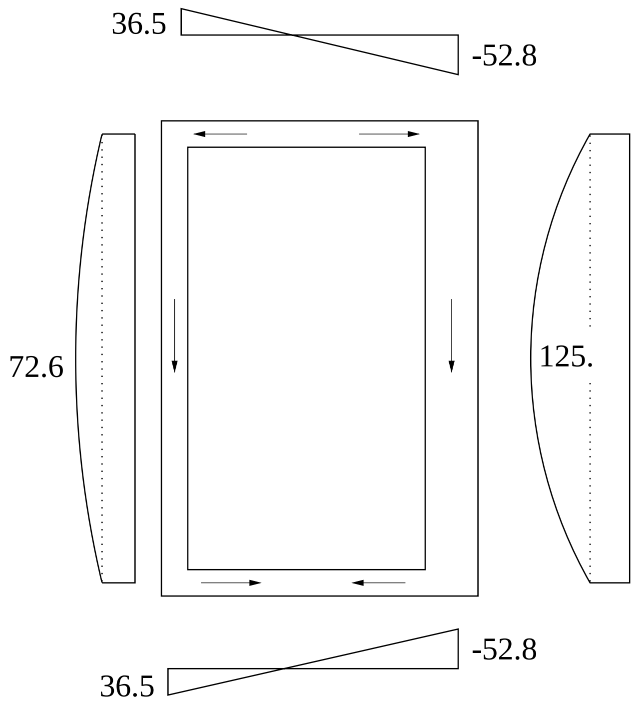
which gives

$$q_a = 52.8\frac{V}{I}$$

With this value, the shear flow at any point in the cross section is given as

$$q_s = \frac{V}{I}\left(Q - 52.8\right)$$

where Q is the the moment of area with respect to the neutral axis of the area between point A and whatever point is being considered. When q_s is positive it has the direction shown in the figures on page 2. The shear flow thus calculated is



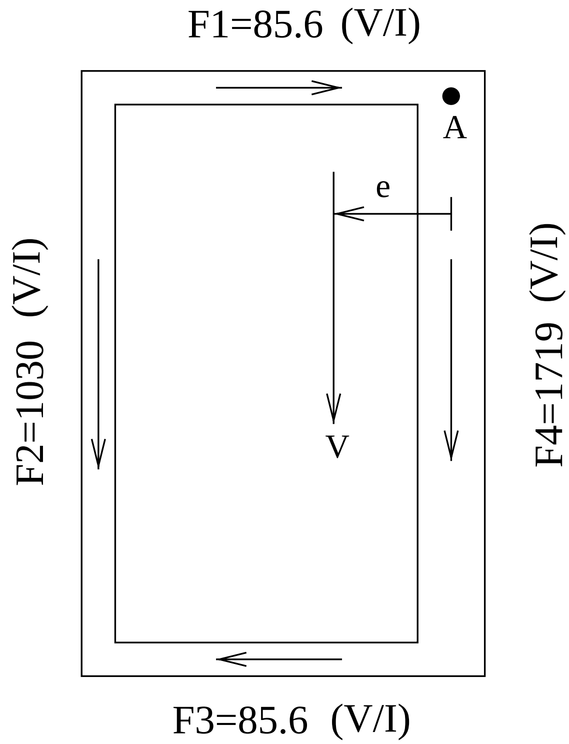
Note: shear flow equals numbers shown times (V/I)

Location of Resultant (The Center of Shear)

We now locate the line of action of the resultant of the shear flows shown. We select an arbitrary but convenient point to take moments - in our case, point A. The total shear force in each each segment of boundary is simply the area of the corresponding curve shown in the figure. Hence, we have:

$$\begin{aligned} F1 &= \frac{V}{I}\left\{+\frac{1}{2}(10.5)(36.5-52.8)\right\} \\ &= -85.6 \qquad \text{top} \\ F2 &= \frac{V}{I}\left\{\frac{2}{3}(17)(36.1)+(17)(36.5)\right\} \\ &= 1030\frac{V}{I} \qquad \text{left side} \\ F3 &= \frac{V}{I}\left\{\frac{1}{2}(10.5)(36.5-52.8)\right\} \\ &= -85.6\frac{V}{I} \qquad \text{bottom} \\ F4 &= \frac{V}{I}\left\{-\frac{2}{3}(17)(72.5)-17(52.8)\right\} \\ &= -1719\frac{V}{I} \qquad \text{right side} \end{aligned}$$

Because we have taken shear flow counter-clockwise as positive, the above values have the directions shown below



Moments about point A gives us:

$$\mathrm{M}_A = \frac{V}{I}\left\{1030(10.5) - 85.6(17)\right\} = 9360\frac{V}{I}$$

The total downward shear is

$$V = \frac{V}{I}(1030 + 1719) = \frac{V}{I}(2749)$$

Hence, the line of action of the resultant must be a distance e away from point A given by

$$e = \frac{\mathrm{M}_a}{V} = \frac{9360}{2749} = 3.40$$

NOTE: According to the above equation for total shear, V, what must be the value for the moment of inertia? Check your answer.