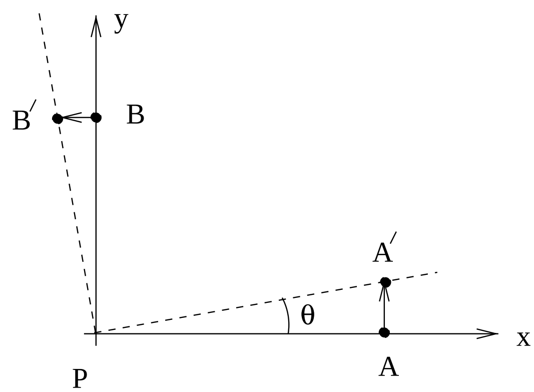
The Rotation Tensor

Two Dimensional Case

Pure, two dimensional rotation can be described by the single angle, θ, as shown.



Clearly, for pure rotation, all lines eminating from a generic point P remain the same length and simply rotate about P. Therefore,

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0.0 \\ \frac{\partial v}{\partial y} &= 0.0 \end{aligned}$$

where $u =$ the displacement in the positive $x-$direction and $v =$ the displacement in the positive $y-$direction. This gives the displacement gradient matrix the form:

$$\left[\begin{array}{c|c} 0.0 & \frac{\partial u}{\partial y} \\ \hline \frac{\partial v}{\partial x} & 0.0 \end{array} \right]$$

If the rotation is small, then the relative displacement of any point with respect to PA is appoximately perpendicular to the line joining the point to P. That is

$$\begin{aligned} \widehat{AA'} &= \frac{\partial v}{\partial x} dx \approx \theta dx \\ \widehat{BB'} &= \frac{\partial u}{\partial y} dy \approx -\theta dy \end{aligned}$$

Hence:

$$\begin{aligned}\frac{\partial v}{\partial x} &\approx \theta \\ \frac{\partial u}{\partial y} &\approx -\theta\end{aligned}$$

With this approximation, the displacement gradient matrix becomes:

$$\left[\begin{array}{c|c} 0.0 & -\theta \\ \hline +\theta & 0.0 \end{array}\right]$$

From this we see that any small rigid body rotation gives rise to a skew symmetric displacement gradient in the vicinity of a point. Likewise, any skew symmetric displacement gradient is a result of a pure rotation. Finally, any skew symmetric component of a displacement gradient can be thought of as the average rotation about the point P. Hence, if the displacement gradient is factored into a symmetric component and a skew symmetric component as follows:

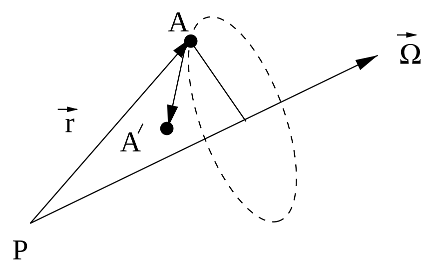
$$\left[\begin{array}{c|c} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \hline \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array}\right] = \left[\begin{array}{c|c} \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \hline \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) \end{array}\right]$$

$$+ \left[\begin{array}{c|c} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ \hline \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 \end{array}\right]$$

the symmetric component is the strain tensor, and the skew symmetric component is the rotation matrix.

Three Dimensional Case

The two dimensional case is easily extended to three dimensions by considering the following figure:



where Ω is the rotation vector and $\vec{r}$ is the relative position vector of point A with respect to an generic point P. The displacement of A to A' due to the rotation Ω is:

$$\widehat{AA'} = \vec{u} = \vec{\Omega} X \vec{r}$$

which, when expanded, gives the following matrix relationship:

$$\left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\} = \left[\begin{array}{ccc} 0 & -\Omega_z & +\Omega_y \\ +\Omega_z & 0 & -\Omega_x \\ -\Omega_y & +\Omega_x & 0 \end{array} \right] \left\{ \begin{array}{c} r_x \\ r_y \\ r_z \end{array} \right\}$$

Again, we see that pure rotation results in a skew symmetric matrix, and that any skew symmetric matrix represents a pure rotation matrix. Note, of course, that the rotations are again assumed small. The displacement matrix in three dimension can, therefore, be factored into a symmetric component representing strain and a skew symmetric component representing rotations. In indicial notation, we have:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ij} + \omega_{ij}$$