ENERGY METHODS

A Tutorial Note
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VIRTUAL WORK

Consider a continuum subjected to surface tractions T_i and body forces X_i. Let the body be in static equilibrium and let the current stress field be $\sigma_{ij}(x, y, z)$ and the displacement field $u_i(x, y, z)$. Now assume that a virtual[1] displacement field, δu_i, is imposed on the system. Under this displacement field, conservation of mechanical energy requires that the work being performed by all external and internal forces equal zero.

$$\boxed{\delta W_{ext} + \delta W_{int} = 0}$$

The external forces of interest are the surface tractions and the body forces. Their virtual work is given by

$$\delta W_{ext} = \int_V X_i \delta u_i dV + \int_S T_i \delta u_i dS$$

Hence, the principle of virtual work states

$$\boxed{\delta W_{int} + \int_V X_i \delta u_i dV + \int_S T_i \delta u_i dS = 0}$$

VIRTUAL WORK OF INTERNAL FORCES, δW_{int}

In order to put the principle of virtual work to practical use it is necessary to calculate the virtual work of the internal forces as a function of the components of the stress tensor. We show how this can be done using the two most common methods.

First Method. This approach begins by isolating an infinitesimal element (dx,dy,dz) and assuming that the components of the stress are constant throughout its volume dV. If this infinitesimal element is subjected to a virtual displacement, δu_i, it is an easy task to calculate the virtual work performed by the external forces acting on its surface. We show this by first considering only the virtual displacement in the x-direction and the component of stress in that direction.

[1]Having the effect of the quantity but not in fact being the quantity

Let δu_i be the displacements at the center of the element. The virtual displacement of the negative x-face in the x-direction is then

$$\delta u_x(x - \frac{dx}{2}) = \delta u_x - \frac{\partial \delta u_x}{\partial x}\frac{dx}{2}$$

and the displacement of the positive x-face is

$$\delta u_x(x + \frac{dx}{2}) = \delta u_x + \frac{\partial \delta u_x}{\partial x}\frac{dx}{2}$$

On each of these two faces, the component of stress in the x-direction is σ_{xx} which creates a force equal to

$$dF_x = -\sigma_{xx} dy dz \qquad \text{on the negative x-face}$$

and

$$dF_x = +\sigma_{xx} dy dz \qquad \text{on the positive x-face}$$

The virtual work performed by these forces due to the component of the virtual displacement in the x-direction is

$$d(\delta W_{ext}) = -dF_x\left(\delta u_x - \frac{\partial \delta u_x}{\partial x}\frac{dx}{2}\right) + dF_x\left(\delta u_x + \frac{\partial \delta u_x}{\partial x}\frac{dx}{2}\right)$$

or, after substitution and simplification

$$d(\delta W_{ext}) = \sigma_{xx}\frac{\partial \delta u_x}{\partial x} dx dy dz$$

The gradient in the x-direction of the virtual displacement in the x-direction is the virtual strain in the x-direction, hence

$$d(\delta W_{ext}) = \sigma_{xx}\delta\epsilon_{xx} dx dy dz$$

In exactly the same manner we could determine the virtual work performed by all components of the stress tensor due to virtual displacements in all three coordinate directions. The result would show that

$$\begin{aligned} d(\delta W_{ext}) &= (\sigma_{xx}\delta\epsilon_{xx} + \sigma_{xy}\delta\epsilon_{xy} + \sigma_{xz}\delta\epsilon_{xz} \\ &+ \sigma_{yx}\delta\epsilon_{yx} + \sigma_{yy}\delta\epsilon_{yy} + \sigma_{yz}\delta\epsilon_{yz} \\ &+ \sigma_{zx}\delta\epsilon_{zx} + \sigma_{zy}\delta\epsilon_{zy} + \sigma_{zz}\delta\epsilon_{zz})\, dV \end{aligned}$$

or, because of symmetry of the stress tensor,

$$\begin{aligned} d(\delta W_{ext}) &+ (\sigma_{xx}\delta\epsilon_{xx} + \sigma_{yy}\delta\epsilon_{yy} + \sigma_{zz}\delta\epsilon_{zz} \\ &+ 2\sigma_{xy}\delta\epsilon_{xy} + 2\sigma_{yz}\delta\epsilon_{yz} + 2\sigma_{zx}\delta\epsilon_{zx})\, dV \end{aligned}$$

where

$$dV = dx dy dz$$

In index notation, this can be written as

$$d(\delta W_{ext}) = \sigma_{ij}\delta\epsilon_{ij} dV$$

Because of the conservation of mechanical energy this work must be the negative of the work performed by the internal forces. Hence

$$d(\delta W_{int}) = -\sigma_{ij}\delta\epsilon_{ij}dV$$

We now sum the internal work performed within all infinitesimal elements in our volume to obtain

$$\delta W_{int} = \int_V d(W_{int}) = -\int_V \sigma_{ij}\delta\epsilon_{ij}dV$$

This is our desired expression for the virtual work of all internal forces within the volume V.

Second Method. This approach makes use of the divergence theorem to relate the surface tractions to the internal stress field.

Equilibrium at the surface of our continuum requires the following relationship to be met between the tractions and stresses.

$$T_i = \sigma_{ij}n_j$$

where n_j are the components of the outward normal vector to the surface. Hence

$$\delta W_{ext} = \int_V X_i\delta u_i dV + \int_S \sigma_{ij}\delta u_i n_j dS$$

We now use the divergence theorem to change the surface integral to a volume integral and obtain

$$\delta W_{ext} = \int_V X_i\delta u_i dV + \int_V \frac{\partial\left(\sigma_{ij}\delta u_i\right)}{\partial x_j}dV$$

which gives us

$$\delta W_{ext} = \int_V X_i\delta u_i dV + \int_V \frac{\partial\sigma_{ij}}{\partial x_j}\delta u_i dV + \int_V \sigma_{ij}\frac{\partial\delta u_i}{\partial x_j}dV$$

Because our body is in equilibrium at the instant being considered

$$\int_V X_i\delta u_i dV + \int_V \frac{\partial\sigma_{ij}}{\partial x_j}\delta u_i dV = \int_V \left(X_i + \frac{\partial\sigma_{ij}}{\partial x_j}\right)\delta u_i dV = 0$$

and

$$\delta W_{ext} = \int_V \sigma_{ij}\frac{\partial\delta u_i}{\partial x_j}dV$$

We next rewrite the above as

$$\delta W_{ext} = \frac{1}{2}\int_V \sigma_{ij}\frac{\partial\delta u_i}{\partial x_j}dV + \frac{1}{2}\int_V \sigma_{ij}\frac{\partial\delta u_i}{\partial x_j}dV$$

then interchange the dummy indices on the last term to obtain

$$\delta W_{ext} = \frac{1}{2}\int_V \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}dV + \frac{1}{2}\int_V \sigma_{ji}\frac{\partial \delta u_j}{\partial x_i}dV$$

and finally note that the symmetry of the stress tensor allows us to write the above as

$$\begin{aligned}
\delta W_{ext} &= \tfrac{1}{2}\int_V \sigma_{ij}\frac{\partial \delta u_i}{\partial x_j}dV + \tfrac{1}{2}\int_V \sigma_{ij}\frac{\partial \delta u_j}{\partial x_i}dV \\
&= \int_V \sigma_{ij}\left[\frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)\right]dV \\
&= \int_V \sigma_{ij}\delta\epsilon_{ij}dV
\end{aligned}$$

Again, the sum of the internal and external virtual work must equal zero, hence

$$\delta W_{int} = -\delta W_{ext} = -\int_V \sigma_{ij}\delta\epsilon_{ij}dV$$

which is the same expression as we obtained by our first method.
We are now in a position to write our final expression for virtual work. Our fundamental equation

$$\delta W_{ext} + \delta W_{int} = 0$$

can now be written as

$$\boxed{\int_V X_i\delta u_i dV + \int_S T_i \delta u_i dS - \int_V \sigma_{ij}\delta\epsilon_{ij}dV = 0}$$

There has yet been no reference to material properties; the principle of virtual work is therefore applicable to any material - linear, nonlinear, elastic, inelastic, etc. The expression simply states how surface tractions and body forces must be related to the stress field if the body is in equilibrium.

STRAIN ENERGY DENSITY IN ELASTIC SOLIDS

We now investigate the integrand in the last integral to determine if it represents a differential of a scalar function. That is, is there a scalar U_o which is a function of the strain such that

$$\delta U_o = \sigma_{ij}\delta\epsilon_{ij}$$

We determine this by noting that, if true, then

$$\delta U_o = \frac{\partial U_o}{\partial \epsilon_{ij}}\delta\epsilon_{ij}$$

and hence

$$\sigma_{ij} = \frac{\partial U_o}{\partial \epsilon_{ij}}$$

This requires that

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{mn}} = \frac{\partial \sigma_{mn}}{\partial \epsilon_{ij}}$$

The fact that this is true for elastic, isotropic materials is easily verified. In terms of Lame's constants, the consitutive equation for such a material is

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$

and we find, for instance, that

$$\frac{\partial \sigma_{xx}}{\partial \epsilon_{yz}} = 0 \qquad \frac{\partial \sigma_{yz}}{\partial \epsilon_{xx}} = 0$$

$$\frac{\partial \sigma_{xx}}{\partial \epsilon_{yy}} = \lambda \qquad \frac{\partial \sigma_{yy}}{\partial \epsilon_{xx}} = \lambda$$

etc. etc. etc.

In general,

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{mn}} = 2G\delta_{im}\delta_{jn} + \lambda\delta_{ij}\delta_{mn}$$

and

$$\frac{\partial \sigma_{mn}}{\partial \epsilon_{ij}} = 2G\delta_{mi}\delta_{nj} + \lambda\delta_{mn}\delta_{ij}$$

Clearly, the last terms on the right hand sides of these equations are equal. Because the Kronecker delta is symmetric, the first terms on the right hand side are also equal. Therefore, a strain-energy density function does exist for linear, isotropic elastic materials such that:

$$\sigma_{ij} = \frac{\partial U_o}{\partial \epsilon_{ij}}$$

Because this relationship is true for all states of strain, the value of U_o is independent of the strain path taken to arrive at any given state. We can therefore calculate its value by assuming a path where all components of strain increase proportionately. Each component of stress during loading is then linearly proportional to its corresponding component of displacement. Thus

$$U_o = \int dU_o = \int \sigma_{ij} d\epsilon_{ij} = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$$

The above expression can be verified as follows:

$$\begin{aligned}
\frac{\partial U_o}{\partial \epsilon_{mn}} &= \frac{1}{2}\frac{\partial \sigma_{ij}}{\partial \epsilon_{mn}}\epsilon_{ij} + \frac{1}{2}\sigma_{ij}\frac{\partial \epsilon_{mn}}{\partial \epsilon_{ij}} \\
&= \frac{1}{2}\left(2G\frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}} + \lambda\delta_{ij}\frac{\partial \epsilon_{kk}}{\partial \epsilon_{mn}}\right)\epsilon_{ij} + \frac{1}{2}\sigma_{ij}\frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}} \\
&= \frac{1}{2}\left(2G\delta_{im}\delta_{jn} + \lambda\delta_{ij}\delta_{mn}\right)\epsilon_{ij} + \frac{1}{2}\sigma_{ij}\delta_{im}\delta_{jn} \\
&= \frac{1}{2}\left(2G\epsilon_{mn} + \lambda\delta_{mn}\epsilon_{ii}\right) + \frac{1}{2}\sigma_{mn} \\
&= \sigma_{mn}
\end{aligned}$$

We now have the following ways of expressing the virtual work performed by internal forces:

$$\boxed{\begin{aligned} \delta W_{int} &= -\delta \int_V \left(\frac{1}{2}\sigma_{ij}\epsilon_{ij}\right) dV \\ &= -\int_V \sigma_{ij}\delta\epsilon_{ij} dV \\ &= -\int_V \frac{\partial U_o}{\partial \epsilon_{ij}}\delta\epsilon_{ij} dV \end{aligned}}$$

THE PRINCIPLE OF STATIONARY POTENTIAL ENERGY

The strain-energy density function is the negative of the work done (per unit volume) by the internal forces in an elastic body through the displacements u_i. Thus, because the system of internal forces is conservative, it is equal to the work these forces would perform in returning to the unstrained position. We can therefore consider

$$U = \int_V U_o dV$$

as the total work that would be done by all internal forces if our body were returned to its unstrained position. Hence, it can be interpreted as the potential energy of the internal forces with respect to the unstrained position.

If now the external forces are likewise conservative, then we can let the negative of the work they perform during the deformation of our solid be the potential energy of the external forces with respect to the unstrained position. The total potential energy of our elastic solid in its deformed configuration can then be written as

$$PE = (U - W_{ext})$$

If the external forces are constant, then

$$W_{ext} = \int_V X_i u_i dV + \int_S T_i u_i dS$$

We have, therefore,

$$\begin{aligned} \delta(PE) &= \delta U - \delta W_{ext} \\ &= \int_V \frac{\partial U_o}{\partial \epsilon_{ij}}\delta\epsilon_{ij} dV - \int_V X_i \delta u_i dV - \int_S T_i \delta u_i dS \\ &= \int_V \sigma_{ij}\delta\epsilon_{ij} dV - \int_V X_i \delta u_i dV - \int_S T_i \delta u_i dS \end{aligned}$$

which must be zero according to our principle of virtual work. Hence, we have for elastic materials the principle of stationary potential energy

$$\boxed{\delta(PE) = \delta(U - W_{ext}) = 0}$$

If our elastic body is in a stable configuration, then the potential energy will be a minimum; if the configuration is unstable the potential energy will be a maximum.

STRAIN ENERGY & WORK OF EXTERNAL FORCES

It is important that one realize that the total potential energy of a conservative force has no particular meaning. It is the change in the potential energy as the force moves from one point to another that is significant. The potential energy of the external forces as stated above is based on the assumption that these forces are constant and that their reference position is the unstrained (unloaded) position of the structure. Other reference positions would be equally valid, e.g. the deformed position of the structure. Therefore, it may not alway be appropriate to equate external and internal potential energies. What is important, however, is that their sum has a stationary value if all forces, internal and external, are in equilibrium.

To clarify this, consider the elementary example of a spring-mass system. If the mass of weight w and spring are held in the zero deflection position of the spring and suddenly released, the work done by the internal forces in the spring as it arrives at the position associated with static equilibrium, u, would be

$$W_{int} = -\frac{1}{2}ku^2 = -\frac{1}{2}wu$$

The work done by the external force of gravity, w, would be

$$W_{ext} = wu$$

The difference in these two quantities would equal the kinetic energy of the system as it reached the static equilibrium position of the spring-mass system.

Therefore, a system analyzed for static equilibrium is a system, if loaded as described, that has come to rest after a period of time due to internal and external friction. It could also be a system where the external loads are so slowly applied that the kinetic energy is zero at each instant during the loading. In the latter case, the total work done by the internal forces during loading would be the negative of the work done by the external forces.

This will prove a convenient way to calculate the potential energy of the internal forces (strain energy) in terms of the external loads. For example, if a structure is loaded only by point forces, the strain energy at the equilibrium position is simply

$$U = \frac{1}{2}\sum_{1}^{n} P_{(i)}u_{(i)}$$

where $P_{(i)}$ is the magnitude of the i^{th} point force, $u_{(i)}$ is the total displacement of the force in the direction of loading, and n is the total number of forces.

Three important notes. First, if the external forces on a structure were actually proportional to the deformation of the structure at the point of application, then all deformed positions of the structure would be in equilibrium. As the structure moved from one position to another, the loads would change so as to make it always in equilibrium. This, of course, is seldom the case.

The second point is: For linear materials, the above expression does not necessarily represent the work done by the actual forces acting on the structure. Rather, the expression represents the work that would be done by a set of forces applied slowly to structure that now have the magnitude of the actual forces on the structures. Hence, the expression represents the negative of the work done by the internal forces, or the total strain energy in the structure. If the loads were all constant, such as those produce by gravity, then their work would actually be twice this value.

The third point is: for nonlinear materials, the above expression represent neither the strain energy of the system nor the work done by external forces that have been slowly applied.

MAXWELL'S LAW OF RECIPROCITY

If C_{ab} represents the deflection at point a in a given direction (say e_a) due to a unit load at a point b in a given direction (say e_b), and if C_{ba} is the deflection at point b in the direction e_b due to a unit load at point a in the direction e_a, then

$$\boxed{C_{ab} = C_{ba}}$$

To prove this, let our elastic structure (prevented from rigid body motion by suitable constraints) have the unit load at a placed on it slowly. The work done by that load will be

$$U_a = \frac{1}{2}(1.0)C_{aa}$$

Note, C_{aa} agrees with our definition stated above. We next place the unit load at point b on our structure. During this application, the unit load at point a, which remains constant in magnitude, will move and thus do work as well as the unit load at point b. The final energy will be

$$U_{ab} = \frac{1}{2}(1.0)C_{aa} + 1.0C_{ab} + \frac{1}{2}(1.0)C_{bb}$$

If, on the other hand, the loads were placed on our structure in the reverse order, the final energy would be

$$U_{ba} = \frac{1}{2}(1.0)C_{bb} + (1.0)C_{ba} + \frac{1}{2}(1.0)C_{aa}$$

Because the strain energy is independent of the strain path (order of loading), these two quantities must be equal, thus

$$\boxed{C_{ab} = C_{ba}}$$

INFLUENCE COEFFICIENTS

The coefficients, C_{aa}, C_{ab}, C_{ba} and C_{bb} defined in the last discussion are referred to as influence coefficients. For any structure, a set of influence coefficients can be defined. Hence, let C_{ij} be the deformation of our solid at point j in the direction e_j due to a unit load at point i in the direction e_i. Maxwell's law of reciprocity states that this set of influence coefficients is symmetric, i.e.

$$C_{ij} = C_{ji}$$

In terms of influence coefficients, the deflection at a point i can be written as

$$u_i = \sum_{1}^{n} C_{ij} P_j$$

where the subscripts do not designate vector components but only the labels attached to the points.

CASTIGLIANO'S FIRST THEOREM

The relationship between the internal virtual work and the external virtual work allows us to consider variations in strain energy in terms of the magnitude of the external loads and variations in the displacements at their points of application. Thus, if the external loads on our elastic body are point forces, $P_1, P_2, P_3 \cdots P_n$, then

$$\delta U = P_1 \delta u_1 + P_2 \delta u_2 + P_3 \delta u_3 \cdots + P_n \delta u_n$$

where the subscripts represent the force and displacement designations and not vector components. It is understood that the displacements referred to are in the directions of the forces at each of the points.

Because the principle of virtual work must apply for any arbitrary virtual displacement, it applies for a virtual displacement field that is zero at all points of loading but one. Let the point where it is not zero be designated as "a", then

$$\delta U = P_a \delta u_a$$

Thus,

$$\boxed{\frac{\partial U}{\partial u_a} = P_a}$$

That is, the derivative of the internal potential energy (strain energy) with respect to a displacement is equal to the component of the load acting at that point in the direction of the displacement. This is Castigliano's first theorem.

CASTIGLIANO'S SECOND THEOREM

For elastic structures under the influence of a discrete number of point loads, the influence coefficients can be used to write the total strain energy as a function of either the loads or the deformations. When considered a function of the loads, Castigliano's second theorem states

$$\boxed{\frac{\partial U}{\partial P_a} = u_a}$$

where u_a is the deformation at point a in the same direction as P_a.

To prove this theorem, consider the structure with all the loads acting. Let the strain energy of the structure in this state be U. Now let one of the loads, say P_a, be increased by the amount dP_a. The final strain energy will then be

$$U + dU = U + \frac{\partial U}{\partial P_a} dP_a$$

Now consider that the increment dP_a was placed on the structure first, before the other loads were applied. Then during the addition of the loads $P_1, P_2, \cdots P_n$, the incremental load dP_a remains constant and will perform an amount of work

$$dU = (dP_a)u_a$$

where u_a represents, as before, the deformation at point a due to all the loads (including P_a). The total energy in this last case is

$$U + dU = \frac{1}{2}(dP_a)(du_a) + U + (dP_a)(u_a)$$

where du_a is the displacement at a due to the incremental load. Because strain energy is independent of the order of loading, we have

$$U + dU = U + \frac{\partial U}{\partial P_a} dP_a = \frac{1}{2}(dP_a)(du_a) + U + (dP_a)(u_a)$$

hence

$$\frac{\partial U}{\partial P_a} dP_a = \frac{1}{2}(dP_a)(du_a) + (dP_a)(u_a)$$

If we divide through by dP_a we obtain

$$\frac{\partial U}{\partial P_a} = \frac{1}{2}(du_a) + u_a$$

which, because we can make du_a as small as we wish, gives us

$$\frac{\partial U}{\partial P_a} = u_a$$

and we have proved our theorem.

THE RAYLEIGH-RITZ METHOD

Strictly speaking, this is not an energy method but rather a method for approximating solutions based on an energy method. The potential energy of an elastic structure must have a stationary value with respect to displacements at the point of equilibrium. The Rayleigh-Ritz method use of this fact to obtain approximations to the displacement of the structure. Assume it is known that the series

$$u_i = \sum_{1}^{n} A_{ij} \Phi_j(x, y, z)$$

could give an accurate approximation of the displacement field, u_i if the parameters A_{ij} were properly chosen. It is assumed that the functions $\Phi_i(x, y, z)$ are known, e.g. trigonometric functions. The problem is, therefore, one of finding a proper set of values for the parameters A_{ij}. The Rayleigh-Ritz method states that a proper set would be the set that gives the potential energy of the system a stationary value. This is usually a minimum value, hence the method is interpreted as out of all possible displacement configurations that are possible to obtain by the above approximation, the best one is the one that gives the lowest value to the potential energy.

The above equation represents three approximation for displacements - one for each direction in space. There are, therefore, $3n$ coefficients to be solved for by minimizing the potential energy with respect to each. That is,

$$\frac{\partial (PE)}{\partial (A_{ij})} = 0$$

Hence there are always enough equations to solve for the unknown parameters. Once these parameters are determined the resulting equations, u_i, are those which give the lowest possible value for the potential energy possible with the assumed approximation.

THE FOREST

Energy methods create a virtual forest of formulas and concepts. In the previous sections we walked through that forest examining each tree in detail. However, as the adage says, sometimes its hard to see the forest because of the trees. Here is a summary that should help you see the beautiful forest.

VIRTUAL WORK

The basic principle behind all of our equations is virtual work. It is a direct consequence of Newton's second law of motion.

$$\boxed{\delta W_{ext} + \delta W_{int} = 0}$$

VIRTUAL WORK IN CONTINUUM MECHANICS

The principal of virtual work as used in continuum mechanics relates the internal stress field to surface tractions and body forces. The following equation is valid for all continua regardless of the material.

$$\boxed{\int_V X_i \delta u_i dV + \int_S T_i \delta u_i dS - \int_V \sigma_{ij} \delta \epsilon_{ij} dV = 0}$$

PRINCIPLE OF STATIONARY POTENTIAL ENERGY

For structures made of linear elastic materials and loaded by conservative forces, the principle of stationary potential energy is equivalent the principle of virtual work. It is given by

$$\boxed{\delta(PE) = \delta(U - W_{ext}) = 0}$$

$$\boxed{U = \int_V \frac{1}{2} \sigma_{ij} \epsilon_{ij}}$$

$$\boxed{W_{ext} = \int_V X_i u_i dV + \int_S T_i u_i dS}$$

$$\boxed{\delta(PE) = \int_V \sigma_{ij} \delta \epsilon_{ij} dV - \int_V X_i \delta u_i dV - \int_S T_i \delta u_i dS}$$

MAXWELL'S LAW OF RECIPROCITY

In a linear, elastic structure the total energy is independent of the order of loading. A consequence of this is that the displacement at a point B due to a unit load at a point A is the same as the displacement at point A due to the unit load at point B. The displacement and the force at each point is in the same direction; however, the direction chosen for point A does not have to be the same as that at point B.

CASTIGLIANO'S FIRST THEOREM

A direct consequence of the principle of minimum potential energy is that the derivative of the strain energy with respect to the displacement at a point of application of a load is equal to the load. The direction of the load and component of displacement used must be the same.

$$\boxed{\frac{\partial U}{\partial u_a} = P_a}$$

CASTIGLIANO'S SECOND THEOREM

For linear, elastic structures under the influence of a discrete number of point loads, the derivative of the strain energy with respect to a load is equal to the displacement in the direction of loading at the point of loading.

$$\boxed{\frac{\partial U}{\partial P_a} = u_a}$$

With the introduction of the concept of complementary energy this theorem can be generalized to include nonlinear materials. However, this was a tree we did not have time to examine. Perhaps on another day we can venture back into the forest and look at it.

SOME ADDITIONAL PROOFS

Positive-definiteness of the Strain Energy Density

Positive definiteness means that the strain energy density is equal to or greater than zero for all states of strain and is zero only if all components of strain are zero. This is seen as follows:

$$\begin{aligned} V \quad = \tfrac{1}{2}\{ \quad & \sigma_{xx}\epsilon_{xx} + \sigma_{xy}\epsilon_{xy} + \sigma_{xz}\epsilon_{xz} \\ + \quad & \sigma_{yx}\epsilon_{yx} + \sigma_{yy}\epsilon_{yy} + \sigma_{yz}\epsilon_{yz} \\ + \quad & \sigma_{zx}\epsilon_{zx} + \sigma_{zy}\epsilon_{zy} + \sigma_{zz}\epsilon_{zz}\} \end{aligned}$$

We write our constitutive equations in terms of Lame's constants:

$$\begin{aligned} \sigma_{xx} &= 2\mathrm{G}\epsilon_{xx} + \lambda e \\ \sigma_{yy} &= 2\mathrm{G}\epsilon_{yy} + \lambda e \\ \sigma_{zz} &= 2\mathrm{G}\epsilon_{zz} + \lambda e \\ \sigma_{xy} &= 2\mathrm{G}\epsilon_{xy} \\ \sigma_{xz} &= 2\mathrm{G}\epsilon_{xz} \\ \sigma_{yx} &= 2\mathrm{G}\epsilon_{yx} \\ \sigma_{yz} &= 2\mathrm{G}\epsilon_{yz} \\ \sigma_{zx} &= 2\mathrm{G}\epsilon_{zx} \\ \sigma_{zy} &= 2\mathrm{G}\epsilon_{zy} \end{aligned}$$

where

$$e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$$

Substitution of the constitutive equations into the expression for V gives us:

$$\begin{aligned} V \quad = \tfrac{1}{2}\{2\mathrm{G}(\quad & \epsilon_{xx}^2 + \epsilon_{xy}^2 + \epsilon_{xz}^2 \\ + \quad & \epsilon_{yx}^2 + \epsilon_{yy}^2 + \epsilon_{yz}^2 \\ + \quad & \epsilon_{zx}^2 + \epsilon_{zy}^2 + \epsilon_{zz}^2) \\ + \quad & \lambda e^2\} \end{aligned}$$

or, because of the symmetry of the strain tensor,

$$V = \mathrm{G}(\epsilon_{xx}^2 + \epsilon_{xy}^2 + \epsilon_{xz}^2 + 2\epsilon_{xy}^2 + 2\epsilon_{yz}^2 + 2\epsilon_{zx}^2) + \frac{1}{2}\lambda e^2$$

Clearly, because all terms that appear in this expression are squared, we have proved the positive-definiteness of the strain energy density function.

Clapeyron's Theorem

This theorem states that the strain energy of a linear elastic body (structure), in static equilibrium, under the action of constant surface tractions and body forces, is equal to half the work these tractions and body forces would perform in moving through their respective displacements. That is:

$$2U = \int_V (B_x u + B_y v + B_z w)\, dV + \int_S (T_x u + T_y v + T_z w)\, dS$$

We begin the proof of this theorem by writing the equations of equilibrium, satisfied at every point in the body

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0$$

We now multiply the above three equations by the displacements u, v, and w, respectively and add them to obtain:

$$u\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + B_x\right) +$$

$$v\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + B_y\right) +$$

$$w\left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z\right) = 0$$

We next rearrange the above (note, transpose the above format) to obtain:

$$u\frac{\partial \sigma_{xx}}{\partial x} + v\frac{\partial \sigma_{xy}}{\partial x} + w\frac{\partial \sigma_{xz}}{\partial x} +$$

$$u\frac{\partial \sigma_{yx}}{\partial y} + v\frac{\partial \sigma_{yy}}{\partial y} + w\frac{\partial \sigma_{yz}}{\partial y} +$$

$$u\frac{\partial \sigma_{zx}}{\partial z} + v\frac{\partial \sigma_{zy}}{\partial z} + w\frac{\partial \sigma_{zz}}{\partial z} +$$

$$uB_x + vB_y + wB_z = 0$$

Next, make use of

$$d(AB) = dA(B) + A(dB)$$

to obtain:

$$+\frac{\partial}{\partial x}(u\sigma_{xx}) + \frac{\partial}{\partial x}(v\sigma_{xy}) + \frac{\partial}{\partial x}(w\sigma_{xz})$$

$$+\frac{\partial}{\partial y}(u\sigma_{yx}) + \frac{\partial}{\partial y}(v\sigma_{yy}) + \frac{\partial}{\partial y}(w\sigma_{yz})$$

$$+\frac{\partial}{\partial z}(u\sigma_{zx}) + \frac{\partial}{\partial z}(v\sigma_{zy}) + \frac{\partial}{\partial z}(w\sigma_{zz})$$

$$-\frac{\partial u}{\partial x}\sigma_{xx} - \frac{\partial v}{\partial x}\sigma_{xy} - \frac{\partial w}{\partial x}\sigma_{xz}$$

$$-\frac{\partial u}{\partial x}\sigma_{yx} - \frac{\partial v}{\partial x}\sigma_{yy} - \frac{\partial w}{\partial x}\sigma_{yz}$$

$$-\frac{\partial u}{\partial x}\sigma_{zx} - \frac{\partial v}{\partial x}\sigma_{zy} - \frac{\partial w}{\partial x}\sigma_{zz}$$

$$+uB_x + vB_y + wB_z = 0$$

If we now use the definitions of strain:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yx} = \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) \qquad \epsilon_{zx} = \frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$$

$$\epsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zy} = \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})$$

$$\epsilon_{xz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \qquad \epsilon_{yz} = \frac{1}{2}(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}) \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

and note its symmetry as well as the symmetry of the stress tensor and group terms to obtain:

$$\begin{aligned}
&+\frac{\partial}{\partial x}(u\sigma_{xx} + v\sigma_{xy} + w\sigma_{xz}) + \frac{\partial}{\partial y}(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz}) \\
&+\frac{\partial}{\partial z}(u\sigma_{zx} + v\sigma_{zy} + w\sigma_{zz}) \\
&-\epsilon_{xx}\sigma_{xx} - \epsilon_{xy}\sigma_{xy} - \epsilon_{xz}\sigma_{xz} - \epsilon_{yx}\sigma_{yx} - \epsilon_{yy}\sigma_{yy} - \epsilon_{yz}\sigma_{yz} \\
&-\epsilon_{zx}\sigma_{zx} - \epsilon_{zy}\sigma_{zy} - \epsilon_{zz}\sigma_{zz} \\
&+uB_x + vB_y + wB_z = 0
\end{aligned}$$

Next, we integrate the above expression over the domain V to obtain:

$$\begin{aligned}
&\int_V (\frac{\partial}{\partial x}(u\sigma_{xx} + v\sigma_{xy} + w\sigma_{xz}) + \frac{\partial}{\partial y}(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz}) \\
&\qquad + \frac{\partial}{\partial z}(u\sigma_{zx} + v\sigma_{zy} + w\sigma_{zz}))dV \\
&- \int_V (\epsilon_{xx}\sigma_{xx} + \epsilon_{xy}\sigma_{xy} + \epsilon_{xz}\sigma_{xz} + \epsilon_{yx}\sigma_{yx} + \epsilon_{yy}\sigma_{yy} \\
&\qquad + \epsilon_{yz}\sigma_{yz} + \epsilon_{zx}\sigma_{zx} + \epsilon_{zy}\sigma_{zy} + \epsilon_{zz}\sigma_{zz})dV \\
&+ \int_V (uB_x + vB_y + wB_z)dV \qquad = 0
\end{aligned}$$

We can use Green's theorem to change the first integral to a surface integral, hence:

$$\begin{aligned}
&\int_V (\frac{\partial}{\partial x}(u\sigma_{xx} + v\sigma_{xy} + w\sigma_{xz}) + \frac{\partial}{\partial y}(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz}) \\
&\qquad + \frac{\partial}{\partial z}(u\sigma_{zx} + v\sigma_{zy} + w\sigma_{zz}))dV \\
&= \int_S (\nu_x(u\sigma_{xx} + v\sigma_{xy} + w\sigma_{xz}) + \nu_y(u\sigma_{yx} + v\sigma_{yy} + w\sigma_{yz}) + \nu_z(u\sigma_{zx} + v\sigma_{zy} + w\sigma_{zz}))dS \\
&= \int_S (u(\nu_x\sigma_{xx} + \nu_y\sigma_{yx} + \nu_z\sigma_{zx}) + v(\nu_x\sigma_{xy} + \nu_y\sigma_{yy} + \nu_z\sigma_{zy}) + w(\nu_x\sigma_{xz} + \nu_y\sigma_{yz} + \nu_z\sigma_{zz})dS
\end{aligned}$$

or, simply,

$$\int_S (uT_x + vT_y + wT_z)dS$$

Finally, we note that the second integral is simply twice the strain energy. Hence we have:

$$\int_S (uT_x + vT_y + wT_w)dS - 2U + \int_V (uB_x + vB_y + wB_z)dV = 0$$

Thus we have what we set out to prove, namely

$$2U = \int_S (uT_x + vT_y + wT_w)dS + \int_V (uB_x + vB_y + wB_z)dV$$

Uniqueness of Solution

With the establishment of the positive definiteness of the strain energy density function (and hence the strain energy) and Clapeyron's theorem, uniqueness of solution to problems in linear elasticity can be established. Consider an elastic solid with prescribed body forces, surface tractions and surface displacements. Now consider the possibility of two solutions for the stress field and the displacement field, say $(\sigma^{(\mathbf{1})}, \mathbf{u}^{(\mathbf{1})})$ and $(\sigma^{(\mathbf{2})}, \mathbf{u}^{(\mathbf{2})})$. Because our governing equations and boundary conditions are linear, we can subtract the second solution from the first and obtain another solution corresponding to the new boundary conditions obtained after the subtraction. If we now apply Clapeyron's theorem to our new solution we have

$$2U = \int_S (uT_x + vT_y + wT_w)dS + \int_V (uB_x + vB_y + wB_z)dV$$

where

$$uT_x = (u^{(1)} - u^{(2)})(T_x^{(1)} - T_x^{(2)}) \qquad \text{etc.}$$

and U is a function of the strain components $(\epsilon^{(\mathbf{1})} - \epsilon^{(\mathbf{2})})$. Because the subtraction removes all body forces, and on any surface, either the difference in traction or the difference in displacement is zero, the two integrals in the above equation are identically zero. Hence, we have

$$2U = 0$$

However, U is a positive definite function of strain and can be zero only when the strain is zero everywhere within the body. We conclude, therefore, that

$$\epsilon = \epsilon^{(1)} - \epsilon^{(2)} = 0$$

If rigid body motion is prevented by suitable constraints, then the displacement field

$$\mathbf{u} = \mathbf{u}^{(\mathbf{1})} - \mathbf{u}^{(\mathbf{2})} = \mathbf{0}$$

everywhere. Therefore, the two possible solutions proposed for our problem must be identical proving uniqueness of solution.

Clapeyron's Theorem - A second proof

Because the governing equations (equilibrium, stress/strain, strain/displacement) are all linear, and because the boundary conditions are likewise stated as linear functions of stress and/or displacements, the principle of superposition can be applied. That is, any two solutions (satisfying all governing equations and boundary conditions) can be added (subtracted) to obtain a third solution. If, therefore, we take a solution to a given problem and add it to itself, we obtain a third solution. This third solution will have displacements and stresses twice those of the original solution, and will have specified boundary conditions exactly twice those of the original problem. Hence, the displacement of all points vary linearly with the applied boundary displacements and tractions, i.e. if we double the applied boundary conditions, we double all displacements and stresses.

Now consider the principle of work-energy from mechanics. The total work performed by all forces acting on a system of particles is equal to the change in kinetic energy of the system of particles. If we now consider a solid under a prescribes system of surface tractions and displacements as having reached that position by a process of gradually increasing the the prescribed boundary conditions so slowly that the kinetic energy is zero at all times, then the work done by the internal forces plus the work done by the external forces must always equal zero. Because we know that the displacements are linear functions the applied forces, the total work they perform during loading must be

$$W_{\text{gradual}} = \frac{1}{2}\int_S (T_x u + T_y v + T_z w)dS + \frac{1}{2}\int_V (B_x u + B_y v + B_z w)dV$$

This must also be equal to the negative of the work performed by the internal forces, hence equal to the strain energy, U. Because this is equal to just half of the work that would be performed by the applied loads if they had remained constant (rather than being gradually increased) during the deformation, we have

$$U = W_{\text{gradual}} = \frac{1}{2}W$$

or

$$2U = \int_S (T_x u + T_y v + T_z w)dS + \int_V (B_x u + B_y v + B_z w)dV$$

thus proving Clapeyron's theorem.