The J_2 Invariant *et al*

Erik Thompson
Colorado State University
Department of Civil Engineering
September 8, 2000

The second principal invariant of the stress deviator tensor, J_2, plays an important role in the mathematical theory of plasticity as well as other branches of nonlinear continuum mechanics. It is the purpose of this tutorial to show its relationship to other invariants in common use.

Basic Definition

The stress deviator is:

$$\begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_1 - \sigma_m & 0 \\ 0 & 0 & \sigma_1 - \sigma_m \end{bmatrix} \tag{1}$$

where

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Its second invariant is:

$$J_2 = -\left(S_1 S_2 + S_2 S_3 + S_3 S_1\right) \tag{2}$$

Note that its first invariant is:

$$J_1 = \left(S_1 + S_2 + S_3\right) \equiv 0 \tag{3}$$

Equation 3 can be used with Eq. 2 to obtain:

$$J_2 = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right) \tag{4}$$

which clearly shows that J_2 is positive for all states of stress.

J_2 can also be written in terms of the principal stresses as follows:

$$\begin{aligned}
J_2 &= -(\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m) \\
&\quad -(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m) \\
&\quad -(\sigma_3 - \sigma_m)(\sigma_1 - \sigma_m) \\
&= -\sigma_1\sigma_2 + \sigma_1\sigma_m + \sigma_2\sigma_m - \sigma_m^2 \\
&\quad -\sigma_2\sigma_3 + \sigma_2\sigma_m + \sigma_3\sigma_m - \sigma_m^2 \\
&\quad -\sigma_3\sigma_1 + \sigma_3\sigma_m + \sigma_1\sigma_m - \sigma_m^2 \\
&= -\sigma_1\sigma_2 - \sigma_1\sigma_2 - \sigma_3\sigma_1 \\
&\quad +3\sigma_m^2 + 3\sigma_m^2 - 3\sigma_m^2 \\
&= -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \\
&\quad +\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2 \\
&= -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \\
&\quad +\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1) \\
&= \frac{1}{3}\left(-\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)
\end{aligned}$$

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{5}$$

Effective Stress

J_2 has units of stress squared, hence its square root is often used in continuum mechanics. In addition, a constant is employed so that this new invariant has the value of the applied axial stress in one dimension. When this is done, it is called the effective stress:

$$\begin{aligned}
\sigma_{eff} &= \sqrt{3J_2} \\
&= \left\{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]\right)\right\}^{\frac{1}{2}}
\end{aligned} \tag{6}$$

The Octahedral Shear Stress

Another representation of the J_2 invariant is the shear stress on the planes whose normals make equal angles with the principal axes. There are eight such planes and they are referred to as the octahedral planes (see Fig. 1).

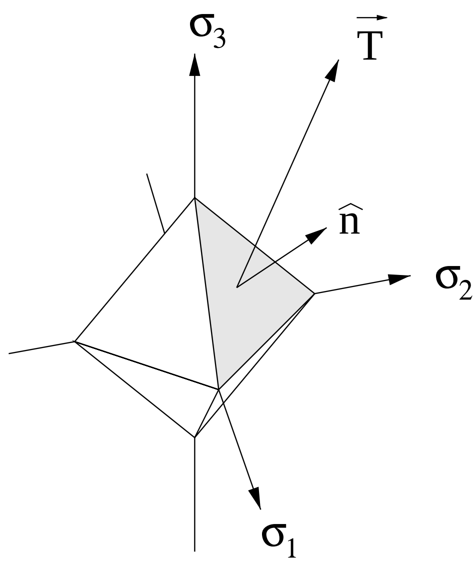

Figure 1: Octahedral Planes

In this figure, $\hat{n}$ is the unit normal vector to the plane shown, and it makes equal angles with the three principal directions. Thus its value is:

$$\hat{n} = \left(\frac{1}{\sqrt{3}}\right)\hat{\imath} + \left(\frac{1}{\sqrt{3}}\right)\hat{\jmath} + \left(\frac{1}{\sqrt{3}}\right)\hat{k} \tag{7}$$

The vector, $\vec{T}$, is the traction on this plane and its value, found using Cauchy's equation, is

$$\begin{aligned} \vec{T} &= \left(\frac{T_1}{\sqrt{3}}\right)\hat{\imath} + \left(\frac{T_2}{\sqrt{3}}\right)\hat{\jmath} + \left(\frac{T_3}{\sqrt{3}}\right)\hat{k} \\ &= \frac{1}{\sqrt{3}}\left(T_1\hat{\imath} + T_2\hat{\imath} + T_3\hat{\imath}\right) \end{aligned} \tag{8}$$

Its component normal to the plane is

$$\begin{aligned} \sigma_{oct} &= \vec{T} \cdot \hat{n} \\ &= \frac{1}{3}\left(T_1 + T_2 + T_3\right) \\ &= \sigma_m \end{aligned} \tag{9}$$

If the magnitude of the component of $\vec{T}$ tangent to the plane is designated as τ_{oct} (the octahedral shear stress) then by the Pythagorean theorem

$$\begin{aligned} \tau_{oct}^2 &= T^2 - \sigma_m^2 \\ &= \frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \sigma_m^2 \\ &= \frac{1}{9}\left(2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_3 - 2\sigma_3\sigma_1\right) \\ \tau_{oct}^2 &= \frac{2}{3} J_2 \end{aligned}$$

$$\tau_{oct} = \sqrt{\frac{2}{3} J_2} \tag{10}$$

Radius of deviation

The next interpretation we give to the J_2 invariant is the distance the stress deviates from a state of pure hydrostatic stress. In Fig. 2, the dashed line represents the locus of points in principal stress space that correspond to states of pure hydrostatic stress, i.e. all three principal stresses are equal. The vector $\vec{\sigma}$ is a generic state of stress. The vector $\vec{R}$, represents the perpendicular displacement of this state of stress from one of pure hydrostatic stress. That is, $\vec{R}$ is perpendicular to the hydrostatic line. We now show that the magnitude of this displacement vector is yet another way of expressing the J_2 invariant.

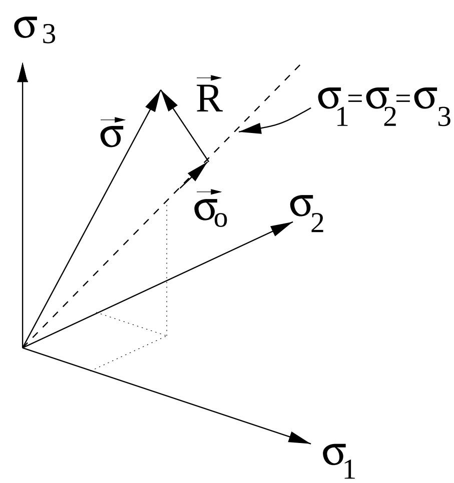

Figure 2: Radial distance from a state of pure hydrostatic stress

The unit vector in the direction of the hydrostatic line is

$$\hat{e} = \left(\frac{1}{\sqrt{3}}\right)\hat{i} + \left(\frac{1}{\sqrt{3}}\right)\hat{j} + \left(\frac{1}{\sqrt{3}}\right)\hat{k} \tag{11}$$

The projection of $\vec{\sigma}$ on the hydrostatic line is

$$\begin{aligned} \sigma_o &= \vec{\sigma} \cdot \hat{e} \\ &= \frac{1}{\sqrt{3}}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \end{aligned} \tag{12}$$

Thus the vector $\vec{\sigma}_o$ is

$$\begin{aligned} \vec{\sigma_o} &= \sigma_o \hat{e} \\ &= \frac{1}{\sqrt{3}}\left(\sigma_1 + \sigma_2 + \sigma_3\right)\frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right) \\ &= \sigma_m\left(\hat{i} + \hat{j} + \hat{k}\right) \end{aligned} \tag{13}$$

Note, the magnitude of this vector is not σ_m, but rather $\sqrt{3}\sigma_m$. By vector addition

$$\begin{aligned} \vec{R} &= \vec{\sigma} - \vec{\sigma_o} \\ &= (\sigma_1 - \sigma_m)\hat{i} + (\sigma_2 - \sigma_m)\hat{j} + (\sigma_3 - \sigma_m)\hat{k} \end{aligned} \tag{14}$$

Thus magnitude of $\vec{R}$ is

$$R = \left[(\sigma_1 - \sigma_m)^2 + (\sigma_2 - \sigma_m)^2 + (\sigma_3 - \sigma_m)^2\right]^{\frac{1}{2}} \tag{15}$$

which, by Eq. 4, gives us

$$R = \sqrt{2J_2} \tag{16}$$

Note that all states of stress which have equal R-values create a cylindrical surface about the hydrostatic line. This is shown in Fig. 3. The same is true, of course, of states of stress with constant J_2-values, constant σ_{eff}-values, or constant τ_{oct}-values.

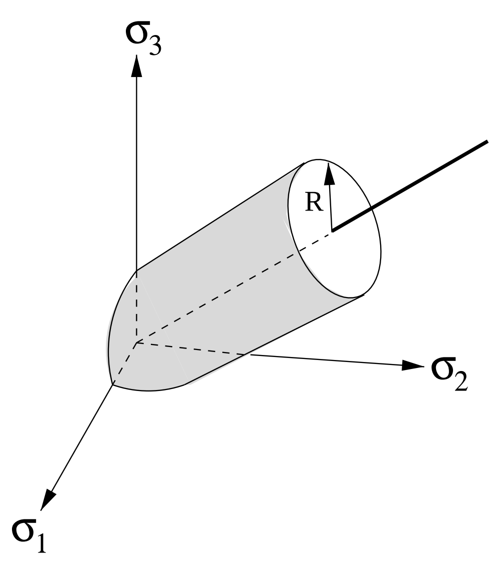

Figure 3: Cylinder formed by states of stress with equal R-values

Energy of Distortion

The final representation of the J_2 invariant is the energy of distortion. Strictly speaking, the energy of distortion is not an invariant of the stress tensor because its value is dependent on material properties. Never-the-less, it has the same form for all linear, isotropic materials and its value is directly proportional to J_2. Hence, for a given material, if one is known, so is the other.

For a linear elastic material, the relationship between the unit change in volume at a point and the mean stress is

$$(\epsilon_1 + \epsilon_2 + \epsilon_3) = \epsilon_v = 3K\sigma_m \tag{17}$$

Thus, if the hydrostatic component of stress is zero, there is no change in volume. It thus follows that the deviatoric stress tensor causes no unit change in volume because its hydrostatic component is zero. We conclude, therefore, that the deviatoric stress causes only distortion of the material (change in shape but not volume). The energy per unit volume necessary to cause this distortion is referred to as the energy of distortion. Its value can be determined by noting that the relationship between the deviatoric stress tensor and the deviatoric strain tensor is:

$$\begin{bmatrix} \sigma_1 - \sigma_m & 0 & 0 \\ 0 & \sigma_1 - \sigma_m & 0 \\ 0 & 0 & \sigma_1 - \sigma_m \end{bmatrix} = 2\mathrm{G} \begin{bmatrix} \epsilon_1 - \epsilon_v & 0 & 0 \\ 0 & \epsilon_1 - \epsilon_v & 0 \\ 0 & 0 & \epsilon_1 - \epsilon_v \end{bmatrix} \tag{18}$$

or more simply

$$[S] = 2\mathrm{G}[\epsilon'] \tag{19}$$

The work performed by the stress per unit volume to reach this state of distortion is

$$
\begin{aligned}
U_d &= \tfrac{1}{2}[S][\epsilon'] \\
&= \frac{1}{2}[S]\frac{1}{2\mathrm{G}}[S] \\
&= \frac{1}{4\mathrm{G}}(S_1S_1 + S_2S_2 + S_3S_3) \qquad (20) \\
&= \frac{1}{4\mathrm{G}}2J_2 \\
&= \frac{1}{2\mathrm{G}}J_2
\end{aligned}
$$

Thus, we have our final relationship.

SUMMARY

$$
\begin{aligned}
J_2 &= -(S_1S_2 + S_2S_3 + S_3S_1) \\
J_2 &= \tfrac{1}{2}(S_1^2 + S_2^2 + S_3^2) \\
J_2 &= \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]
\end{aligned}
$$

$$\sigma_{eff} = \sqrt{3J_2}$$

$$\tau_{oct} = \sqrt{\frac{2}{3}J_2}$$

$$R = \sqrt{2J_2}$$

$$U_d = \frac{1}{2\mathrm{G}}J_2$$