The Log-law of the Wall in the Overlap from Dimensional Analysis

Junke Guo, M.ASCE*

September 19, 2022

Abstract

The log-law of the wall in the overlap for wall-bounded turbulent flows is re-derived from a pure dimensional analysis that is based on three experimental facts: (1) in the inner flow region, the mean velocity distribution is dominated by the viscous length scale; (2) in the outer flow region, the mean velocity distribution is dominated by the largest eddy size or the boundary layer thickness; and (3) there is an overlap between the inner and outer regions. These facts lead to a functional equation that yields a universal log-law in the overlap. Hence, the log-law is independent of any turbulence model, and it is an axiom accepted widely in fluid mechanics.

Keywords: Log-law; Pipe flow; Turbulent flow; Velocity distribution; Wall-bounded flow.

Introduction

The log-law of the wall was proposed by von Kármán (1930) from a similarity hypothesis and Prandtl (1932) from a mixing length hypothesis; it was experimentally confirmed by Nikuradse (1932, 1933) with pipe flow experiments. Later, it was shown by Millikan (1938) and others that it is universal in the overlap layer for wall-bounded turbulent flows, including pipe flow, channel flow, and boundary layer flow. The log-law is the foundation of the log-wake law (Coles 1956) and the second log-wake law (Guo 2020; Patel et al. 2021; Shan et al. 2022). Furthermore, it is a criterion for examining all turbulence models near a wall.

Nevertheless, with the publications of the second log-wake law, several readers wrote to the writer and asked: (1) Why is the log-law of the wall is an axiom in fluid mechanics and all turbulence models must meet it near a wall? (2) Why is the von Kármán constant κ a universal constant that does not vary with Reynolds number? (3) Why is the log-law invalid in the wake layer or why is a wake-law correction necessary in the outer flow region? The writer referred readers to Millikan (1938), but Millikan's derivation of the log-law in the overlap is slightly complicated and not straightforward. As a supplemental to the second log-wake law (Guo 2020; Patel et al. 2021; Shan et al. 2022), this technical note re-derives the log-law from a pure dimensional analysis, which does not need any turbulence model.

Derivation

To make this technical note as short as possible, a literature review on the log-law is referred to Guo (2017) and thus neglected here. For a pipe or boundary layer flow, physically, it contains many different eddy sizes or length scales. The smallest length scale is defined by the viscous length scale ν/u_*, where $\nu =$ kinematic

*Associate Professor, Dept. of Civil Eng., Univ. of Nebraska–Lincoln, Lincoln, NE 68588. E-mail: jguo2@unl.edu

fluid viscosity and $u_* =$ wall shear velocity; and the largest length scale is defined by the pipe radius or the boundary layer thickness h. Besides, for hydraulically-rough wall turbulence, there is a roughness length k_s. Therefore, the streamwise mean velocity distribution u is generally expressed by

$$u = F(y, \nu, u_*, k_s, h) \tag{1}$$

where F is a functional symbol; and $y =$ distance from the wall. Eq. (1) includes two fundamental dimensions: length L, and time T. Choosing y and u_* to represent the length dimension L and the time dimension T, respectively, the Buckingham Π-theorem reduces Eq. (1) to

$$\frac{u}{u_*} = F\left(\frac{yu_*}{\nu}, \frac{y}{k_s}, \frac{y}{h}\right) \tag{2}$$

where y is scaled by three different length scales: viscous length ν/u_*, roughness height k_s, and boundary layer thickness h. These three length scales make a wall-bounded turbulent flow very complicated. Fortunately, Eq. (2) can be simplified under different flow conditions and regions.

For a hydraulically smooth-wall turbulence ($k_s = 0$), Eq. (2) reduces to

$$\frac{u}{u_*} = F\left(\frac{yu_*}{\nu}, \frac{y}{h}\right) \tag{3}$$

where experiments (Nikuradse 1932, 1933; Hultmark et al. 2012) show that: (1) Near the wall, Eq. (3) is dominated by the viscous length scale ν/u_*; and the effect of the boundary layer thickness h is negligible. Thus, Eq. (3) reduces to

$$\frac{u}{u_*} = F\left(\frac{yu_*}{\nu}\right) \tag{4}$$

The region described by Eq. (4) is referred to as the law of the wall, including the viscous layer and the overlap in the inner flow region (Fig. 1), which is experimentally defined by $yu_*/\nu \geq 0$ and $y/h < 0.15$ (Hinze 1975). (2) Far from the wall, the flow is independent of the viscous length scale ν/u_* and dominated by the boundary layer thickness h. Eq. (3) then reduces to

$$\frac{u}{u_*} = F\left(\frac{y}{h}\right) \tag{5}$$

The region described by Eq. (5) is referred to as the outer flow region, including the overlap and the wake layer in Fig. 1. This region is experimentally defined by $yu_*/\nu > 800$ and $y/h \leq 1$, according to Hultmark et al. (2012) from high Reynolds number ($> 10^5$) flow experiments, as shown in Fig. 1. Note that conventionally, the lower limit of the outer region was defined by $yu_*/\nu = 70$ (Nikuradse 1932). Nevertheless, this difference does not matter for high Reynolds number flows because the viscous layer is negligible practically, particularly for river flows where Reynolds numbers are often greater than 10^6.

The overlap is the transition between the inner and outer flow regions, where $yu_*/\nu \to \infty$ in the inner region to make sure the flow is beyond the viscous layer (Fig. 1), and $y/h \to 0$ in the outer region to make sure the flow is beyond the wake layer (Fig. 1). In this overlap, the velocities from Eqs. (4) and (5) must be identical, namely

$$F\left(\frac{yu_*}{\nu}\right) = F\left(\frac{y}{h}\right) \tag{6}$$

Besides, the shear stresses and the energy dissipation rates from both Eqs. (4) and (5) in the overlap should be identical, which implies that the velocity gradients must be identical. Therefore, taking the derivative of Eq. (6) with respect to y on both sides gives

$$\frac{u_*}{\nu}F'\left(\frac{yu_*}{\nu}\right) = \frac{1}{h}F'\left(\frac{y}{h}\right) \tag{7}$$

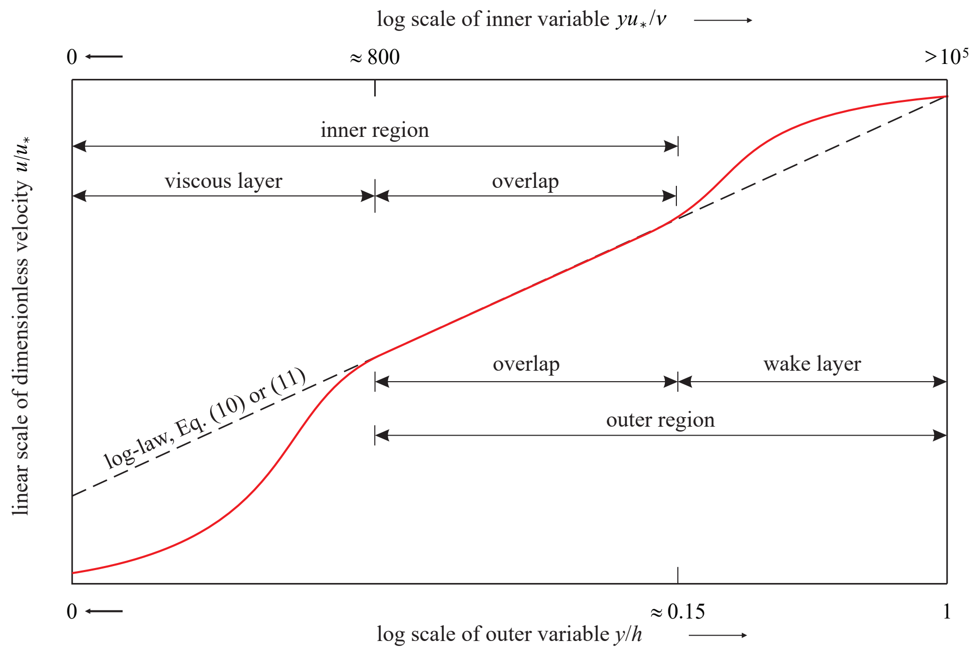

Figure 1: Dimensionless mean velocity distribution as a function of inner variable and outer variable, respectively.

Multiplying y on both sides of Eq. (7) yields

$$\frac{yu_*}{\nu}F'\left(\frac{yu_*}{\nu}\right)=\frac{y}{h}F'\left(\frac{y}{h}\right) \tag{8}$$

where the left-hand side is only a function of yu_*/ν while the right-hand side is only a function of y/h. If Eq. (8) holds true, it has to be a universal constant that, conventionally, is denoted as $1/\kappa$ with $\kappa =$ the von Kármán constant. Eq. (8) then becomes

$$\frac{yu_*}{\nu}F'\left(\frac{yu_*}{\nu}\right)=\frac{y}{h}F'\left(\frac{y}{h}\right)=\frac{1}{\kappa} \tag{9}$$

which answers the second question about the von Kármán constant.

Integrating Eq. (9), in terms of yu_*/ν and y/h, respectively, results in

$$F\left(\frac{yu_*}{\nu}\right)=\frac{1}{\kappa}\ln\frac{yu_*}{\nu}+C_1 \tag{10}$$

where $C_1 =$ an integration constant from the inner region, and

$$F\left(\frac{y}{h}\right)=\frac{1}{\kappa}\ln\frac{y}{h}+C_2 \tag{11}$$

where $C_2 =$ an integration constant from the outer region. It is noteworthy that κ, C_1 and C_2 should be universal constants though different researchers suggest slightly different values in the literature (Nikuradse 1932, 1933; Hinze 1975; Hult,ark et al. 2012).

Eqs. (10) and (11) reproduce the classic log-law in the overlap, without any turbulence model. Therefore, the log-law is universal in the overlap layer and considered an axiom for wall-bounded turbulence near a wall; it then is a criterion for all turbulence models. This answers the first question why the log-law of the wall is an axiom so that all turbulence models must meet it near a wall.

For hydraulically rough-wall turbulence, the roughness scale k_s dominates the near-wall flow so the viscous length scale ν/u_* is replaced by k_s. Hence, Eq. (10) becomes

$$\frac{u}{u_*} = \frac{1}{\kappa} \ln \frac{y}{k_s} + C_1 \tag{12}$$

Finally, in the wake layer in Fig. 1, according to Millikan (1938), a correction function $W(y/h)$ must be added to the log-law near the pipe axis or the boundary layer edge. Coles (1956) found that the shape of this correction function is similar to that of a wake flow. Therefore, this correction function is referred to as the wake law. Consequently, Hinze (1975) found that the Coles wake function can be approximated by a sine-squared function, which specifies the Coles log-wake law in the outer flow region as:

$$\frac{u}{u_*} = (\text{the log-law}) + \frac{2\Pi}{\kappa} \sin^2 \frac{\pi y}{2h} \tag{13}$$

where the log-law is one of Eqs. (10), (11) and (12); and $\Pi =$ wake strength that is independent of Reynolds number but varies with flow type. This answers the third question why the log-law is invalid in the wake layer and why a wake-law correction is necessary in the outer region. Note that in practice, the viscous layer is often negligible (Fig. 1) so that the log-wake law is approximated in the entire flow domain.

Furthermore, if a wall-bounded turbulent flow is restricted by two parallel walls or boundary shear stresses, the log-wake law of Eq. (13) must be modified to be the second log-wake law (Guo 2020; Patel et al. 2021; Shan et al. 2022), which accounts for the interaction between the two walls or boundary conditions.

Conclusions

This technical note re-derived the log-law of the wall in the overlap from a pure dimensional analysis. It shows that: (1) the log-law of the wall in the overlap is a pure mathematical law that is independent of any turbulence model; (2) the von Kármán constant is a pure mathematical constant that is independent of Reynolds number; and (3) a wake-law correction is necessary in the outer flow region.

Data Availability Statement

No data, models, or code were generated or used during this study.

References

Coles, D. E. (1956). “The law of the wake in the turbulent boundary layer.” *J. Fluid Mech.* 1, 191–226.

Guo, J. 2017. “Eddy viscosity and complete log-law for turbulent pipe flow at high Reynolds numbers.” *J. Hydraul. Res.* 55(1), 27-39.

Guo, J. 2020. “Second log-wake law from pipe symmetry and its applications in symmetric and antisymmetric channel flows.” *J. Hydraul. Eng.* 146(11): 06020014.

Hinze, J. O. 1975. *Turbulence.* McGraw-Hill, New York.

Hultmark, M., M. Vallikivi, S. C. C. Bailey, and A. J. Smits. 2012. “Turbulent pipe flow at extreme Reynolds numbers.” *Physical Review Letters*. 108(9), 094501.

Millikan, C. B. 1938. “A critical discussion of turbulent flows in channels and circular tubes.” *Proc. 5th international congress on applied mechanics* (pp. 386–392). Wiley, New York.

Nikuradse, J. 1932. *Laws of turbulent flow in smooth pipes*. NASA TT F-10, 359 (Translation of VDI Fosch. 356).

Nikuradse, J. 1933. *Laws of flow in rough pipes*. NASA Technical Memorandum 1292 (Translation of VDI Fosch. 361).

Patel, N., J. Shahi, and J. Guo. 2021. “Applications of second log-wake law for turbulent velocity distributions in laboratory flumes and natural rivers.” *J. Hydraul. Eng.* 147(9): 06021010.

Prandtl, L. 1932. Zur turbulenten Stromung in Rohren und langs Platten. *Ergeb. Aerod. Versuch. Goettingen, IV Lieferung, p18.*

Shan, H., K. Kereny, N. Patel, N., and J. Guo. 2022. “Laboratory test of second log-wake law for Effects of Ice-Cover and Wind Shear Stress on River Velocity Distributions.” *J. Cold Regions Eng.* 36(2): 04022001.

von Kármán, T. 1930. ”Mechanische Ähnlichkeit und Turbulenz”, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Fachgruppe 1 (Mathematik), 5: 58–76.