EINSTEIN INTEGRALS

PROCEDURE WE FOLLOWED FOR THE DISCUSSION

For the discussion, we have used the integration routines available in Mathematica 4.0 (there you can see the different integral schemes available). The mathematica file is attached to this email and named it as “ ROUSEfinal.nb“. There you will find a section “computation of integrals for different ZR and δ”. We have computed all the values to cover the different ranges of ZR and δ. Additionally, you will find a spreadsheet that contains all the data and graphs we have put based on these computations. In the excel file you will find that the computed values of integrals 1 and 2 are plotted (Figure 3 and 4) and they are in agreement with Einstein’s plots.

Then, once we had the data set, we performed the regression analysis and found those coefficients we mentioned in the discussion. We would say that there should be a better way of collapsing all the data and not just having different coefficients for every value of δ, but I did not have enough time to pursue this procedure.

Please if you have any concern or comment, just let us know.

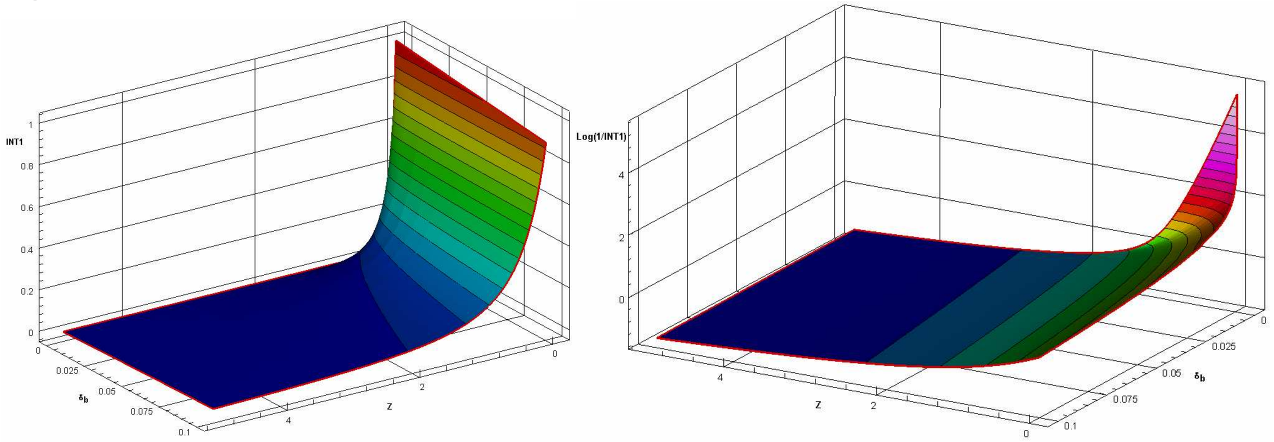
ANALYSIS OF INTEGRALS *INT₁* AND *INT₂*

$$INT_1 = \int_{\delta_b}^{1} \left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b} \right]^{Z_R} d\delta \tag{1}$$

$$INT_2 = \int_{\delta_b}^{1} \left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b} \right]^{Z_R} Ln(\delta) d\delta \tag{2}$$

INT_1

Fig. 1 shows a 3D view of how *INT_1* and Inverse of *INT_1* (1/*INT_1*) for different values of Z_R and δ_b.



(a) (b)

Fig. 1: (a) INT_1, (b) Log (1/INT_1)

- ***Analytical solution***

An analytical solution of INT_1 is presented as:

$$INT_1 = \frac{\pi Z_R}{Sin(\pi Z_R)} + \frac{(1-\delta_b)^{-Z_R}\,\delta_b \{Hypergeometric2F1(1-Z_R, -Z_R, 2-Z_R, \delta_b)\}}{Z_R - 1} \tag{3}$$

Where:

$$2F1 = Hypergeometric2F1(1-Z_R, -Z_R, 2-Z_R, \delta_b) = \sum_{K=0}^{\infty} \frac{(1-Z_R)_K (-Z_R)_K}{(2-Z_R)_K} \frac{\delta_b{}^K}{K!} \tag{4}$$

The solution stated by equation (3) sometimes requires a huge number of terms in order to have good accuracy.

As an example a series expansion of 2F1 around 0 with order 1 is performed.

$$INT_1 = \left(\frac{\delta_b}{1-\delta_b}\right)^{Z_R} \left[\frac{\pi Z_R}{Sin(\pi Z_R)} + \left(\frac{1}{\delta_b} - 1\right)^{Z_R} \left\{ \frac{\delta}{Z_R - 1} + \left(\frac{Z_R}{2-Z_R} + \frac{Z_R}{Z_R - 1}\right)\delta_b{}^2 + O[\delta_b]^3 \right\} \right] \tag{5}$$

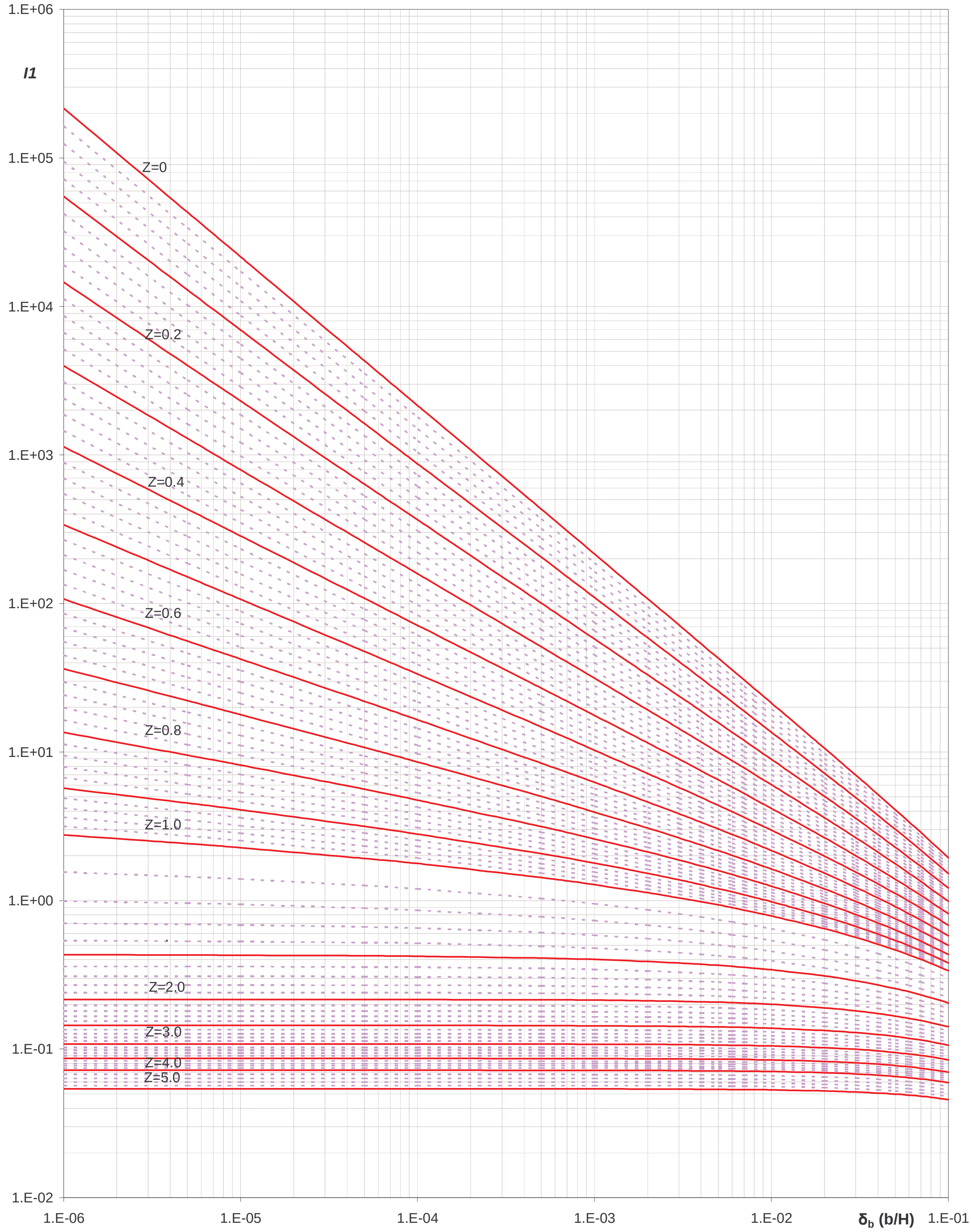
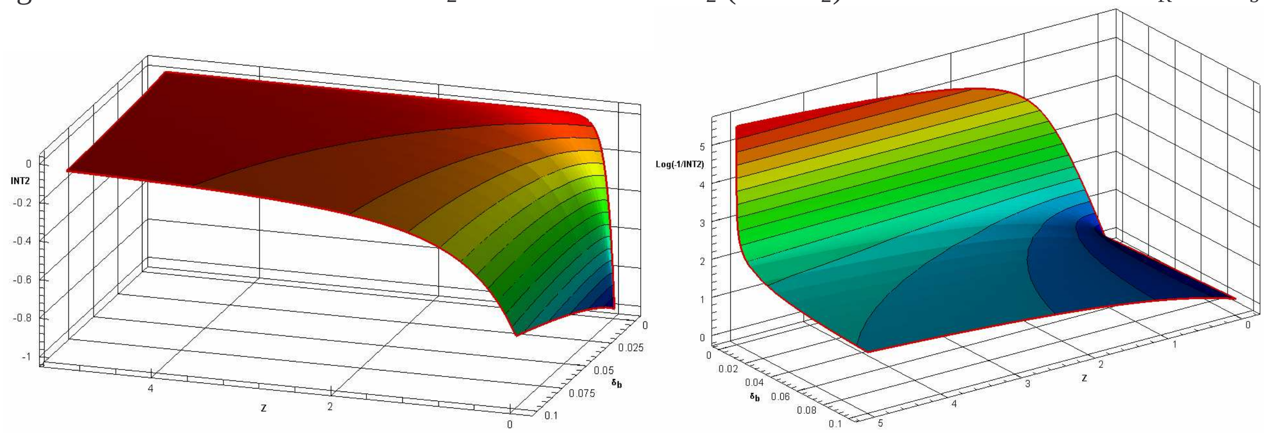


Fig. 2: Einstein integral I_1

INT_2

Fig. 3 shows a 3D view of how *INT_2* and Inverse of *INT_2* (-1/*INT_2*) for different values of Z_R and δ_b.



(a) (b)

Fig. 3: (a) INT_2, (b) Log (-1/INT_2)

- ***Analytical solution***

Guo and Julien (2004) have presented this formulation.

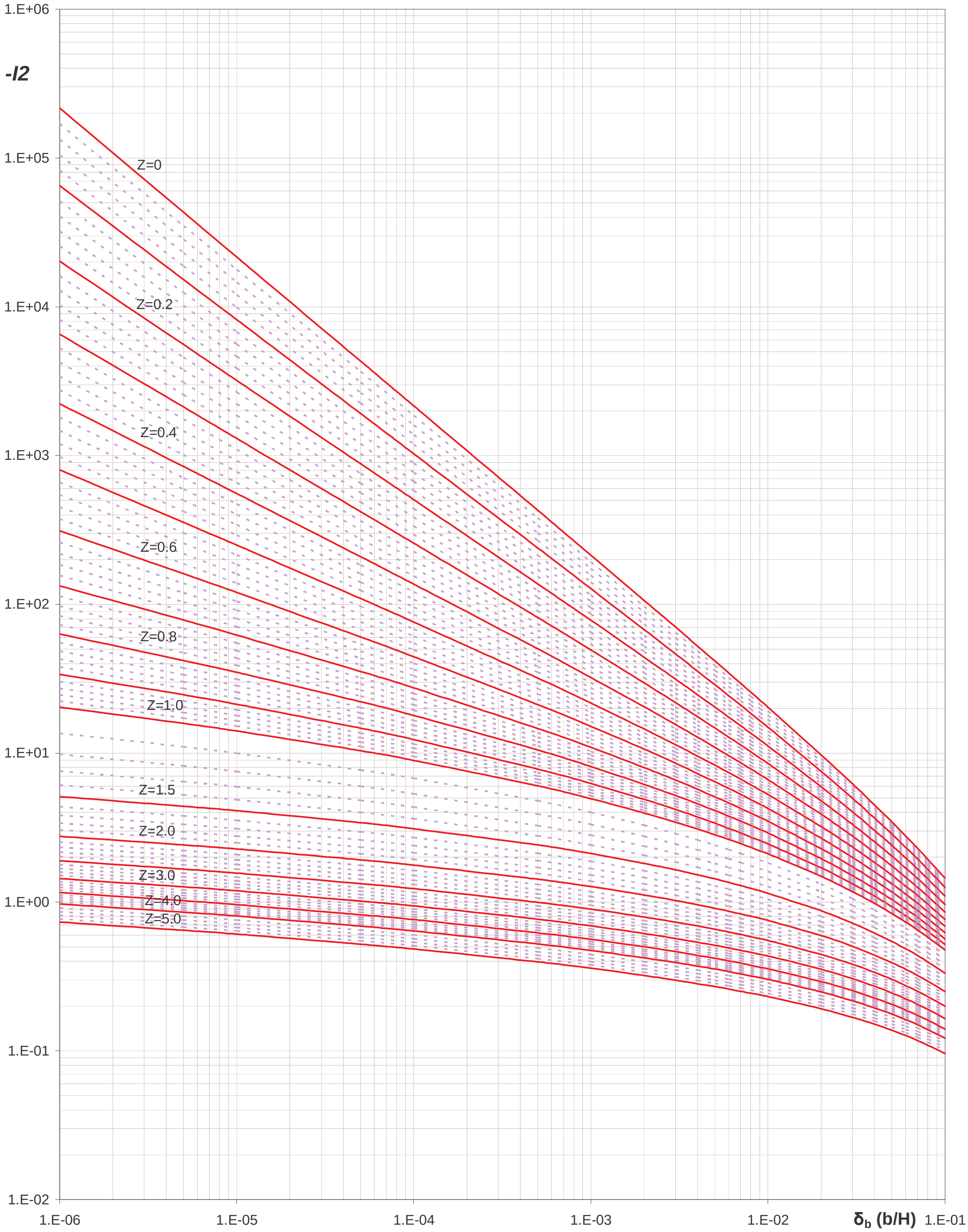


Fig. 3: Einstein integral –I_2