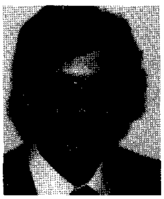
Formation of roll waves in laminar sheet flow

Formation d'un train d'ondes dans les écoulements superficiels laminaires



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SUMMARY
The formation of a series of roll waves in laminar sheet flows in a smooth channel is examined both theoretically and experimentally. Roll waves were observed in subcritical flows at a Froude number as low as 0.74. The recommended theoretical relationship for the celerity of roll waves is a function of the momentum correction factor. This relationship is in good agreement with measured celerities of roll waves. The period of roll waves remained fairly constant throughout these experiments. Previous derivations of the length required for the formation of roll wave were modified because experimental evidence demonstrates that the simplified relationship for the celerity of roll waves does not hold true for laminar sheet flows. Using the modified relationship, the dimensionless distance displays an hyperbolic variation with the Froude number and good agreement is obtained with experimental data. This analysis also demonstrates that for supercritical flows the distance is proportional to the ratio of flow depth and slope. Alternatively an equivalent function of Reynolds number and slope can be used.

RÉSUMÉ
La formation d'un train d'ondes dans les écoulements superficiels sur surface lisse est étudiée analytiquement et expérimentalement. Des trains d'ondes ont été observés dans des écoulements fluviaux à des nombre de Froude aussi faibles que 0,74. Une expression analytique fonction du coefficient de Boussinesq est recommandée pour décrire la vitesse de propagation des ondes. La période des ondes demeure constante sous les diverses conditions hydrauliques de cette étude. Les équations existantes décrivant la distance de développement des ondes ont dû être modifiées puisque cette étude expérimentale démontre que certaines hypothèses relatives à la vitesse de propagation des ondes ne sont pas valables pour les écoulements laminaires. Les modifications apportées aux équations donnent des résultats en accord avec les résultats expérimentaux. De plus, l'analyse démontre que pour les écoulements torrentiels, la distance est proportionelle au rapport de la profondeur d'écoulement sur la pente. Une expression équivalente en termes de pente et du nombre de Reynolds peut également être utilisée.

1 Introduction

Sheet flows in steep channels often exhibit surface instabilities which grow until a series of breaking or roll waves are formed. In this study the formation of roll waves under laminar conditions is discussed. Previous treatments of the laminar case have neglected the minimum channel length necessary for roll wave development. Theoretical derivations for turbulent flow [Montuori, 1963 and Liggett, 1975] indicate that simple criteria based on the Froude number are necessary though insufficient since the length required for the formation of roll waves is not considered. In the first part of this study, previous theories on the formation of roll waves are

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examined and modified in the light of laminar sheet flow characteristics. The experimental part of this study has been conducted in order to provide information regarding the validity of the derived theoretical relationships.

2 Steady uniform laminar sheet flow characteristics

The analysis of the free surface stability of laminar sheet flows assumes that steady uniform flow conditions exist prior to the occurrence of a small perturbation of the water surface. Two non-linear partial differential equations were derived by Saint-Venant to describe gradually varied unsteady flows. These are respectively the continuity and the momentum relationships. For steady uniform sheet flows, the continuity equation can be written as:

$$q = \bar{u}h \tag{1}$$

in which

q = unit discharge
$\bar{u}$ = mean velocity
h = flow depth

The momentum equation reduces to the so-called kinematic wave approximation for which the bed slope S is equal to the friction slope S_f. The friction slope is given by the Darcy-Weisbach equation:

$$S_f = \left(\frac{K\upsilon}{8g}\right)\frac{\bar{u}}{h^2} \tag{2}$$

in which

g = gravitational acceleration
υ = kinematic viscosity
K = the friction coefficient

After combining equations (1) and (2), the mean velocity and flow depth are:

$$\bar{u} = \left(\frac{8g}{K\upsilon}\right)^{1/3} S_f^{1/3} q^{2/3} \tag{3}$$

$$h = \left(\frac{K\upsilon}{8g}\right)^{1/3} S_f^{-1/3} q^{1/3} \tag{4}$$

The relationships are valid for uniform or gradually varied laminar sheet flows only. The velocity distribution with the distance y from the water surface is given by:

$$u = \frac{12g}{K\upsilon} S_f(h^2 - y^2) \tag{5}$$

This velocity profile decreases parabolically from a maximum of 1.5 times the mean velocity at the free surface to zero at the boundary.

3 Theory on the stability of laminar sheet flow

In deriving a fundamental stability criteria for the water surface, several approaches were used by different researchers. Early investigations by Thomas (1939) and Stoker (1957) suggested that the flow is unstable when $S > 4g/C^2$ in which C is the Chézy coefficient. The foremost criterion for instability published in the Russian literature was derived by Vedernikov (1945, 1946). For laminar flows, the Vedernikov number Ve can be written as:

$$Ve = 2F\left(1 - R_h \frac{\partial P}{\partial A}\right) \tag{6}$$

in which

R_h = the hydraulic radius
P = the wetted perimeter
A = the cross-sectional area

The Froude number F defined as $\bar{u}/\sqrt{gh}$ represents the ratio of inertia to gravity forces. For an infinitely wide channel, the Vedernikov number is equal to twice the Froude number and the flow becomes unstable ($Ve > 1$) when the Froude number exceeds 0.5 for laminar flow as compared to 2.0 for turbulent flow. This critical Froude number was also reported by Robertson and Rouse (1941) and Powell (1948). Mayer (1961) observed roll waves in subcritical laminar sheet flows but mistakenly concluded that roll waves can form only when the slope is larger than 3%. Yih (1954, 1963, 1977) and Benjamin (1975) solved the problem of stability of sheet flows down an inclined plane using the Orr-Sommerfeld equation. For very long waves the flow is unstable when:

$$Re \geq \frac{5}{6S} \tag{7}$$

in which Re = the Reynolds number

This criterion was also suggested by Taylor and Kennedy (1961). If equation (2) is substituted into equation (7), uniform flow ($S = S_f$) and a K value of 24 are assumed, a critical Froude number of $F_c = 0.53$ results which is close to the Vedernikov criteria for wide rectangular channels. Ishihara et al. (1961) also suggested the critical value $F_c = 0.577$.
Unfortunately, these criteria based on the Froude number ignore the distance along the channel required for the formation of roll waves. This factor becomes particularly important for subcritical sheet flows since previous studies for turbulent flows [Montuori, 1963] demonstrate that the distance at which the waves are fully developed increases to infinity as the Froude number approaches the critical value.

3.1 *Formation and celerity of roll waves*

When the flow is unstable ($Ve > 1$) a minor perturbation of the water surface induces the formation of small waves. The amplitude of these waves increases gradually as they move downstream until a bore is formed and the wave breaks. The distance travelled between the point at which the perturbation is initiated and the breaking point of the wave defines the distance required for the formation of roll waves. This distance, ξ_c, is determined theoretically from the following procedure using the celerity of roll waves.

The total celerity, c, of a small gravity wave moving in a fluid with a uniform velocity distribution along the vertical is:

$$c = \bar{u} + \sqrt{gh} \tag{8}$$

In the more general case of a nonuniform vertical velocity distribution, the celerity can be theoretically derived from the momentum equation. After the momentum correction factor β_m is used instead of an empirical coefficient, the equation for celerity suggested by Arsenishvili (1965) can be written as:

$$c = \bar{u} + c_0 = \beta_m \bar{u} + \sqrt{gh + \beta_m(\beta_m - 1)\bar{u}^2} \tag{9}$$

in which c_0 is the celerity of the wave relative to the mean velocity $\bar{u}$; and

$$\beta_m = \frac{1}{\bar{u}^2 h} \int_0^h u^2 \, dy \tag{10}$$

When $\beta_m = 1$, equation (9) reduces to equation (8). For sheet flows, however, the momentum correction factor $\beta_m = 1.2$ is obtained from equations (5) and (10). The ratio of celerities $c/\sqrt{gh}$ is:

$$\frac{c}{\sqrt{gh}} = \beta_m F + \sqrt{1 + \beta_m(\beta_m - 1)F^2} \tag{11}$$

Equations (9) or (11) can be used to compute the celerity of roll waves in laminar sheet flows.

3.2 *Perturbation analysis*

The following analysis of the channel length required for a small disturbance to become a breaking wave resembles the mathematical treatment used by Liggett (1975) and Dracos and Glenne (1967). First, time is removed from the momentum and continuity equations written in terms of x and t by introducing a moving coordinate system in ξ and η which travels at the constant wave speed, $\bar{u} + c_0$. The variable ξ replaces the original space coordinate x and defines position with respect to a fixed point, while the second coordinate, η, defines position relative to the moving coordinate system and is written:

$$\eta = (\bar{u} + c_0)t - \xi \tag{12}$$

A small perturbation moving at the wave speed $(\bar{u} + c_0)$ is imposed at $\xi = 0$ on an initially steady uniform flow. This is mathematically simulated by writing the equations for conservation of mass and momentum as follows:

$$B'(\bar{u} + c_0)\frac{\partial h'}{\partial \eta} + \bar{u}'B'\left(\frac{\partial h'}{\partial \xi} - \frac{\partial h'}{\partial \eta}\right) + A'\left(\frac{\partial u'}{\partial \xi} - \frac{\partial \bar{u}'}{\partial \eta}\right) = 0 \tag{13}$$

and

$$(\bar{u} + c_0)\frac{\partial \bar{u}'}{\partial \eta} + \bar{u}'\left(\frac{\partial \bar{u}'}{\partial \xi} - \frac{\partial \bar{u}'}{\partial \eta}\right) + g\left(\frac{\partial h'}{\partial \xi} - \frac{\partial h'}{\partial \eta}\right) = g(S - S_f') \tag{14}$$

in which the variables h', $\bar{u}'$, B', A' and S_f' represent perturbations in the neighborhood of the uniform flow values of flow depth h, velocity $\bar{u}$, top width B, cross-sectional area A and bed slope S. The perturbed variables, denoted by primes, are defined as follows:

$$h' = h + \frac{\partial h'}{\partial \eta}\eta + \frac{1}{2}\frac{\partial^2 h'}{\partial \eta^2}\eta^2 + 0(\eta^3) \tag{15}$$

$$\bar{u}' = \bar{u} + \frac{\partial \bar{u}'}{\partial \eta}\eta + \frac{1}{2}\frac{\partial^2 \bar{u}'}{\partial \eta^2}\eta^2 + 0(\eta^3) \tag{16}$$

$$B' = B + \frac{\partial B'}{\partial h'}\frac{\partial h'}{\partial \eta}\eta + 0(\eta^2) \tag{17}$$

$$A' = A + \frac{\partial A'}{\partial h'}\frac{\partial h'}{\partial \eta}\eta + 0(\eta^2) \tag{18}$$

$$S_f' = S + S\eta\left(\frac{1}{\bar{u}}\frac{\partial \bar{u}'}{\partial \eta} - \frac{2}{h}\frac{\partial h'}{\partial \eta}\right) + 0(\eta^2) \tag{19}$$

The truncated series are valid for small values of η and the solution is examined in the neighborhood of $\eta = 0$. An analysis of equations (13) and (14) using the perturbation defined by equations (15) through (19) as presented in Julien and Hartley (1985) results in the following relationship

$$\frac{\partial^2 h'}{\partial \xi \partial \eta} - \beta\left(\frac{\partial h'}{\partial \eta}\right)^2 + \gamma\frac{\partial h'}{\partial \eta} = 0 \tag{20}$$

in which for rectangular channels ($B = B'$ and $\partial B'/\partial h' = 0$), the coefficients β and γ are respectively:

$$\beta = \frac{3g}{c_0^2 + 2\bar{u}c_0 + gh} \tag{21}$$

and,

$$\gamma = \frac{gS}{\bar{u}^2}\left(1 - \frac{2c_0F^2}{\bar{u}}\right)\left[\frac{1}{2 + \frac{c_0}{\bar{u}} + \frac{\bar{u}}{c_0F^2}}\right] \tag{22}$$

This derivation improves the one given by Liggett (1975) since the wave celerity defined by equation (9) accounts for the velocity distribution present in laminar sheet flow. If a uniform velocity distribution is assumed (i.e. $c_0 = \sqrt{gh}$), equations (21) and (22) reduce to the relationship proposed by Liggett.

The solution of equation (20) is:

$$\frac{\partial h'}{\partial \eta} = \frac{\varepsilon}{\frac{\beta\varepsilon}{\gamma} + e^{\gamma\xi}} \tag{23}$$

in which ε is a constant of integration along the longitudinal distance ξ.

3.3 *Distance for the formation of roll waves*

The critical distance ξ_c at which the wave breaks is assumed to occur when the water surface is vertical. Mathematically, this condition is obtained when the denominator of equation (23) is set equal to zero, or when:

$$\xi_c = \frac{1}{\gamma} \ln \left(-\frac{\beta}{\gamma} \varepsilon \right) \tag{24}$$

Liggett (1975) derived a relationship between ε and the variables β, γ and the value of $\partial h/\partial \eta$ at the origin. Unfortunately this latter quantity is an unknown which is practically impossible to measure. The following approach which avoids the necessity of directly evaluating ε is recommended.

After combining equations (21), (22) and (24), the distance ξ_c can be written as follows:

$$\xi_c = \frac{h}{S} \left\{ \Psi \left[\Phi + \ln \left(\frac{S}{3\varepsilon} \right) \right] \right\} \tag{25}$$

in which,

$$\Psi = \left[\frac{F^2}{\dfrac{2c_0F^2}{\bar{u}} - 1} \right] \left(2 + \frac{c_0}{\bar{u}} + \frac{\bar{u}}{c_0F^2} \right) \tag{26}$$

and,

$$\Phi = \ln \left(2\frac{c_0^2F^2}{\bar{u}^2} - \frac{c_0}{\bar{u}} \right) \tag{27}$$

From equation (11), $c_0/\bar{u}$ can be written as a function of the Froude number for a given value of β_m. Taking $\beta_m = 1.2$ for laminar sheet flows, the variable Ψ and Φ from equations (26) and

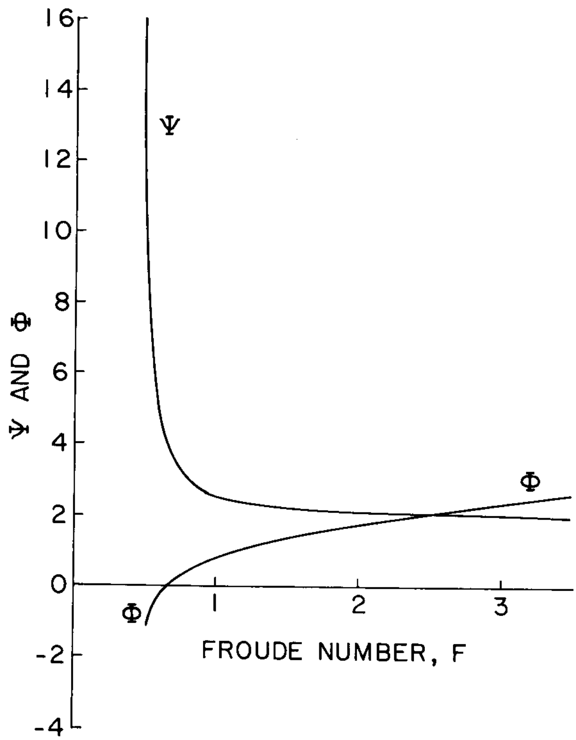

Fig. 1. Dimensionless variables Φ and Ψ as a function of Froude number.
Variables adimensionnelles Φ et Ψ en fonction du nombre de Froude.

(27) are dimensionless and unique functions of the Froude number as plotted in Fig. 1. For supercritical flows, Ψ has a nearly constant value of 2.0 while Φ increases gradually with the Froude number. It can also be demonstrated that over a fairly wide range of slopes the expression $\ln(S/3\varepsilon)$ will be substantially constant. If Φ is small compared to $\ln(S/3\varepsilon)$ then the following approximate relationship for ξ_c can be written:

$$\xi_c \cong D\frac{h}{S} \tag{28}$$

in which D is equivalent to the factor in braces in equation (25) and is approximately constant. Equations (24) is general while equation (28) represents a simplified expression applicable only to supercritical flows. The ability of equations (24) and (28) to predict the distance ξ_c is evaluated with laboratory data described in the following section.

Table 1. Data summary

S (1)	Re (2)	c m/s (3)	T s (4)	ξ_c m (5)	F (6)	$\frac{gS\xi_c}{\bar{u}^2}$ (7)	D (8)	E mm (9)
0.040	335	0.46	1.33	0.91	2.11	6.03	26.6	1.80
0.040	400	0.50	1.61	0.91	2.31	4.74	25.4	1.71
0.040	500	0.57	1.96	0.91	2.58	3.53	23.5	1.59
0.035	68	0.22	1.27	2.74	0.89	147.30	-	-
0.035	95	0.26	1.28	1.52	1.05	51.20	56.6	3.81
0.035	141	0.34	1.45	1.52	1.28	30.50	49.7	3.35
0.035	188	0.34	1.32	2.13	1.48	28.70	62.6	4.26
0.035	265	0.42	1.37	2.13	1.76	18.40	56.9	3.81
0.035	380	0.43	1.19	2.74	2.11	14.40	64.4	4.33
0.030	90	0.24	1.35	2.13	0.95	72.70	-	-
0.030	122	0.25	1.43	2.13	1.10	49.60	59.7	4.02
0.030	200	0.36	1.47	2.74	1.41	33.10	65.8	4.36
0.030	260	0.41	1.25	2.13	1.61	23.10	46.6	3.99
0.030	340	0.42	1.37	2.13	1.84	16.30	42.9	3.66
0.030	360	0.48	1.35	1.52	2.15	5.90	27.1	1.83
0.030	550	-	-	1.52	2.35	4.68	25.8	1.74
0.025	65	-	-	7.62	0.74	380.00	-	-
0.025	71	-	-	7.62	0.77	340.00	-	-
0.025	85	0.22	1.52	3.35	0.84	118.00	-	-
0.025	104	0.24	1.52	3.35	0.93	91.00	-	-
0.025	130	0.33	1.54	2.74	1.04	54.70	59.1	3.96
0.025	200	0.34	1.61	2.74	1.29	30.70	51.1	3.44
0.025	246	0.38	1.75	2.13	1.43	18.20	37.2	2.50
0.025	320	0.44	1.52	1.52	1.63	9.20	24.5	1.62
0.025	420	0.47	1.19	0.91	1.87	3.83	13.3	0.88
0.025	530	0.50	1.15	1.52	2.10	4.65	20.4	1.37
0.015	140	0.26	1.72	2.74	0.84	41.60	-	-
0.015	173	0.27	1.33	2.13	0.93	24.30	-	-
0.015	260	0.33	1.43	2.13	1.14	14.20	18.4	1.25
0.015	320	0.40	1.23	2.13	1.26	10.80	17.2	1.16
0.015	450	0.43	1.08	1.52	1.50	5.90	11.0	0.88
mean			1.41				38.5	2.67
standard deviation			0.20				18.5	1.28

4 Experimental investigation

Laboratory experiments were conducted to determine laminar flow conditions which produced roll waves. The measurements of roll waves included the length required for their formation, wave frequency and wave celerity.

A 0.21 m wide by 9.75 m long rectangular flume was utilized for these experiments. The slope was adjusted between 1.5 and 4% and the unit discharge varied between 6.5×10^{-5} to 5.5×10^{-4} m^2/s. The water temperature was measured and the theoretical friction factor $K = 24$ was verified. Inlet conditions were shown by Brock (1965) to be important in the evaluation of the distance ξ_c. In our experiments a honeycomb flow straightener was placed at the entrance of the channel to break up any large eddies. The initial perturbation triggering the formation of roll waves was assumed to occur immediately downstream of the straightener. The distance ξ_c was measured from this point to the location where a well-defined breaking wave was observed. All the data collected in this experimental investigation were reported by Julien and Hartley (1985). A summary of the experimental data is presented in Table 1. The first five columns read as follows: slope, Reynolds number, wave celerity, wave period and distance for roll wave formation. In column six the Froude number was computed from the Reynolds number and slope.

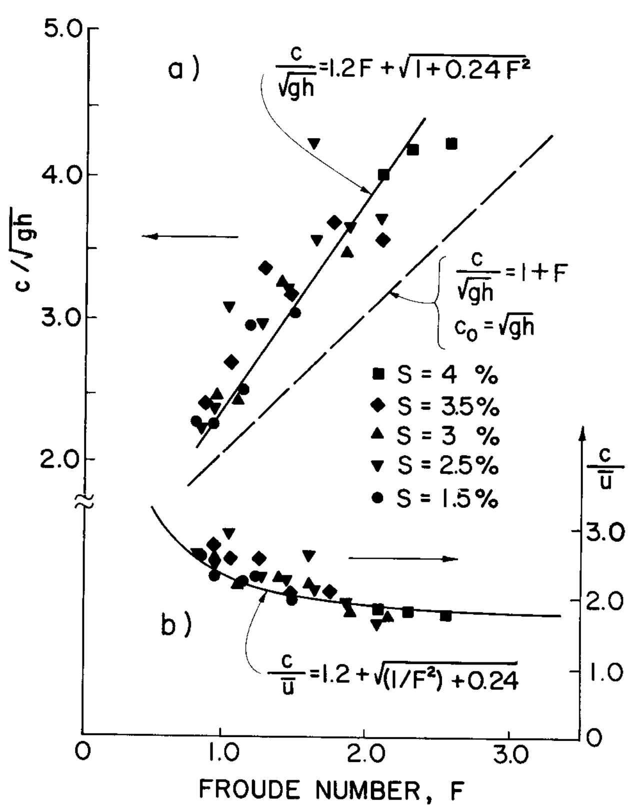

Fig. 2. Dimensionless celerity as a function of Froude number.
Vitesse de propagation adimensionnelle en fonction du nombre de Froude.

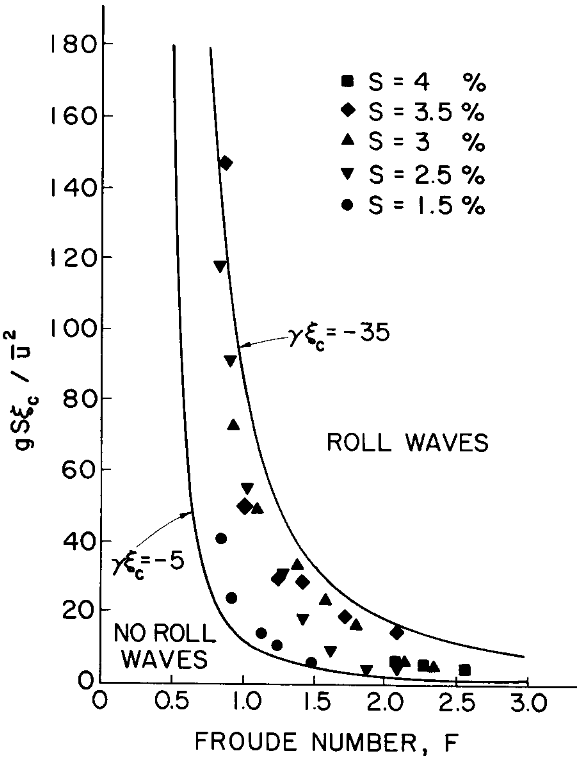

Fig. 3. Dimensionless critical distance as a function of Froude number.
Distance critique adimensionnelle en fonction du nombre de Froude.

4.1 *Analysis of wavelength, period and celerity*

The wavelength can be evaluated from the wave celerity and the period. The observed values of the ratio $c/\sqrt{gh}$ have been plotted against the Froude number on Fig. 2a along with solid and dashed lines representing equation (11) with $\beta_m = 1.2$ and equation (8) for which $\beta_m = 1.0$. The superiority of equation (11) is well supported by the data. Equation (11) can also be written as the ratio of the wave celerity to the mean flow velocity $\bar{u}$ as follows:

$$\frac{c}{\bar{u}} = \beta_m + \sqrt{\frac{1}{F^2} + \beta_m(\beta_m - 1)} \tag{29}$$

For unstable flows ($F > 0.5$), the ratio $c/\bar{u}$ calculated from equation (29) ($\beta_m = 1.2$) decreases from 3.26 to a minimum of 1.69 as shown in Fig. 2b.
The measured wave periods T shown in Table 1 (Col. 4) were fairly constant with a mean value of $\bar{T} = 1.41$ seconds and a standard deviation of 0.20 seconds.
The wavelength L can be obtained from equation (29) in which c is replaced by L/T:

$$\frac{L}{\bar{u}T} = 1.20 + \sqrt{\frac{1}{F^2} + 0.24} \tag{30}$$

For a given mean velocity, the wavelength increases with decreasing Froude number and a first approximation of the wavelength is obtained from the mean value of wave period $\bar{T} = 1.41s$.

4.2 *Critical distance for the formation of roll waves*

Two relationships (equations (24) and (28)) for the distance required for the formation of roll waves are examined in the light of experimental data for laminar sheet flows. In equation (24), the distance ξ_c is a function of β, γ, and ε. The parameters β and γ are computed from equations (21) and (22). The values of $\ln \varepsilon$ calculated from equation (24) using measured values of ξ_c ranged from -61 to -9.4 with a mean value of -25.7. As suggested by Montuori (1963) and Liggett (1975), the values of $\ln(-\beta\varepsilon/\gamma)$ or $\gamma\xi_c$ were computed. Measured values of ξ_c were converted to the dimensionless parameter $gS\xi_c/\bar{u}^2$ in Table 1 (Col. 7) and plotted againt the Froude number in Fig. 3. This figure clearly defines a region where roll waves were observed ($\xi > -35/\gamma$) and a region where roll waves were not completely developed ($\xi < -5/\gamma$). Between these limits exists a zone of uncertainty defined by $-35/\gamma > \xi_c > -5/\gamma$. This figure can be used to estimate the distance for the formation of roll waves from the parameter γ. The evaluation of γ from equation (22) is possible provided the variables S, $\bar{u}$, c_0 and F are known.

If the flow is supercritical, the evaluation of ξ_c from equation (28) involves only the flow depth, slope and the coefficient D. From the experimental data, it was demonstrated that Φ is small compared to $\ln(S/3\varepsilon)$ and therefore D is expected to be substantially constant. The values of D presented in Table 1 (Col. 8) were computed from the experimental values of ξ_c, S and h using equation (28). The mean value for D is 38.5 with a standard deviation equal to 18.5. Equation (28) is therefore recommended to estimate ξ_c for supercritical flows, when depth and slope are known. The flow depth in equation (28) can also be replaced by a function of the slope ($S \cong S_f$) and the Reynolds number from equation (4):

$$\xi_c \cong E\,\frac{Re^{1/3}}{S^{4/3}} \tag{31}$$

in which,

$$E = D\left(\frac{K\upsilon^2}{8g}\right)^{1/3} \tag{32}$$

These relationships indicate that for the same slope and Reynolds number, the constant E, and therefore the critical distance ξ_c increases with increasing viscosity and surface roughness K. The parameter E has dimensions of length. Values of E from the experiments are tabulated in Table 1 (Col. 9). This parameter has a mean value of 2.67 mm and a coefficient of variation of 48%. Equation (31) is recommended for supercritical laminar sheet flows over smooth surfaces. It should be noted that the mean values of the coefficients $D = 38.5$ and $E = 2.67$ mm apply to the range of conditions used in this experimental study. These values may not be applicable beyond this range.

5 Applications

It is recognized that this study is mainly of theoretical interest. However, roll waves in highly viscous mudflows have been observed in steep mountain channels. Investigations of these flows should benefit from the analysis presented in this paper.

6 Summary and conclusions

The formation of roll waves in laminar sheet flows is examined using a theoretical analysis supported by experimental data. Previous investigations indicate that roll waves are theoretically possible in laminar sheet flows at Froude numbers as low as 0.50 as compared with 2.0 in turbulent flow. The existence of roll waves at Froude numbers near the lower limit is difficult to verify experimentally because of the extreme channel lengths required. However, in this study, roll waves were observed in laminar, subcritical flow at a Froude number as low as 0.74.

The parabolic velocity distribution in laminar sheet flows implies that the momentum correction factor is larger than unity ($\beta_m = 1.2$). This suggests that the relationship $c = \bar{u} + \sqrt{gh}$ used in previous studies is not applicable to laminar sheet flows and should be replaced by one which uses β_m. The proposed relationship (equation (11)) reduces to $c_0 = \sqrt{gh}$ when $\beta_m = 1$ and is in good agreement with the measured celerities of roll waves when $\beta_m = 1.2$ as shown in Fig. 2. The measured periods of roll waves remained fairly constant in the experimental study at $\bar{T} = 1.41$ second. The wavelength is shown to vary between $1.69\overline{uT} < L < 3.26\overline{uT}$.

The linearized derivation by Liggett (1975) of the length, ξ_c, required for the formation of roll waves has been modified to account for the parabolic velocity distribution of laminar sheet flows. The modified derivation gives more general expressions for the coefficients β and γ which reduce to those proposed by Liggett when $\beta_m = 1$. The results indicate that the length ξ_c is a function of several flow variables and a constant of integration ε which could be calculated from experiments. Though the parameter $\ln \varepsilon$ varies widely $(-61 < \ln \varepsilon < -9.4)$, the dimensionless distance shown in Fig. 3 displays a similar relationship to the Froude number as found by Montuori (1963) for turbulent flows. For supercritical flows, ξ_c is proportional to the ratio of flow depth and slope. Alternatively, an equivalent function of Reynolds number and slope may be used.

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Notations

A	cross-sectional area for uniform flow
A'	cross-sectional area for flow with a small perturbation
B	top channel width for uniform flow
B'	top channel width for flow with a small perturbation
c	wave celerity
c_0	velocity of the wave relative to the mean velocity $\bar{u}$
C	Chézy coefficient
D, E	empirical constants in equations (28) and (32)
F	Froude number
F_c	critical Froude number
g	gravitational acceleration
h	uniform flow depth
h'	flow depth for flow with a small perturbation
K	friction parameter
L	wavelength

P	wetted perimeter
q	unit discharge
R	ratio of two functions of the Froude number
Re	Reynolds number
R_h	hydraulic radius
S	channel slope
S_f	friction slope
S_f'	friction slope for flow with a small perturbation
t	time
T	wave period
$\bar{T}$	mean wave period
u	velocity
$\bar{u}$	mean velocity for uniform flow
$\bar{u}'$	mean velocity for flow with a small perturbation
Ve	Vedernikov number
y	distance from the water surface
β, γ	functions of flow variables in equations (21) and (22)
β_m	momentum correction factor
ε	constant of integration
η	transformed distance moving with the wave
υ	kinematic viscosity of water
ξ	distance downslope
ξ_c	critical distance at which roll waves are formed
Φ, Ψ	functions of the Froude number

References / Bibliographie

ARSENISHVILI, K. I., *Effect of wave formation on hydro-engineering structures,* Translated from Russian, 1965, 134 p.

BENJAMIN, T. B., Wave formation in laminar flow down on inclined plane, Journal of Fluid Mechanics, Vol. 2, 1975, pp. 554-574.

BROCK, R. R., Development of roll-wave trains in open channels, Journal of the Hydraulics Division, ASCE, Vol. 95, No. HY4, 1969, pp. 1401-1427.

CHOW, V. T., *Open channel hydraulics,* McGraw-Hill, 1959, 680 p.

DRACOS, T. A. and GLENNE B., Stability criteria for open-channel flow, Journal of the Hydraulics Division, Vol. 93, No. HY6, November 1967, pp. 79-101.

ISHIHARA, T., IWAGAKI, Y. and IWASA, Y., Discussion on "Roll waves and slug flows in inclined open channels", Transactions of the ASCE, 1961, pp. 548-563.

JULIEN, P. Y. and HARTLEY, D. M., Formation of roll waves in laminar sheet flow, Report CER84-85PYJ-DMH18, Department of Civil Engineering, Colorado State University, 1985, 34 p.

LIGGETT, J. A. Stability, Chapter 6 in *Unsteady Flow in Open Channels,* WRP, 1975, pp. 259-282.

MAYER, P. G., Roll waves and slug flows in inclined open channels, Transactions of the ASCE, 1961, pp. 505-535.

MONTUORI, C., Discussion on "Stability aspects of flow in open channels", (by F. F. Escoffier and M. B. Boyd) Journal of the Hydraulics Division, ASCE, Vol. 89, No. HY4, July 1963, pp. 264-273.

POWELL, R. W., Vedernikov's criterion for ultra-rapid flow, Transactions of AGU, Vol. 29, No. 6, 1948, pp. 882-886.

ROBERTSON, J. M. and ROUSE, H., On the four regions of open channel flow, Civ. Eng., Vol. 11, 1941, pp. 169-171.

STOKER, J. J., Mathematical hydraulics, in *Water waves,* Wiley Interscience, 1957, pp. 451-509.

TAYLOR, R. H. and KENNEDY, J. F., Discussion on roll waves and slug flows in inclined open channels by P. G. Mayer, Transactions of the ASCE, 1961, pp. 541–548.

THOMAS, H. A., The propagation of waves in steep prismatic conduits, Proc. Hydraulic Conference, Univ. of Iowa, 1939, pp. 214–229.

VEDERNIKOV, V. V., Conditions at the front of a translation wave disturbing a steady motion of a real fluid, USSR Academy of Sciences, Vol. 48, No. 4, 1945, pp. 239–242.

VEDERNIKOV, V. V., Characteristic features of a liquid flow in an open channel, USSR Academy of Sciences, Vol. 52, 1946, pp. 207–210.

YIH, C. S., Stability of parallel laminar flow with a free surface, Proc. 2nd U.S. Congress in Applied Mechanics, 1954, pp. 623–628.

YIH, C. S., Stability of liquid flow down an inclined plane, Physics Fluids, Vol. 6, 1963, pp. 321–334.

YIH, C. S., *Fluid mechanics,* West River Press, 1977, pp. 502–507.