22 The z-Transform

Recommended Problems

P22.1

An LTI system has an impulse response $h[n]$ for which the z-transform is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

(a) Plot the pole-zero pattern for $H(z)$.

(b) Using the fact that signals of the form z^n are eigenfunctions of LTI systems, determine the system output for all n if the input $x[n]$ is

$$x[n] = (\tfrac{3}{4})^n + 3(2)^n$$

P22.2

Consider the sequence $x[n] = 2^n u[n]$.

(a) Is $x[n]$ absolutely summable?

(b) Does the Fourier transform of $x[n]$ converge?

(c) For what range of values of r does the Fourier transform of the sequence $r^{-n}x[n]$ converge?

(d) Determine the z-transform $X(z)$ of $x[n]$, including a specification of the ROC.

(e) $X(z)$ for $z = 3e^{j\Omega}$ can be thought of as the Fourier transform of a sequence $x_1[n]$, i.e.,

$$2^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z),$$
$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X(3e^{j\Omega}) = X_1(e^{j\Omega})$$

Determine $x_1[n]$.

P22.3

Shown in Figure P22.3 is the pole-zero plot for the z-transform $X(z)$ of a sequence $x[n]$.

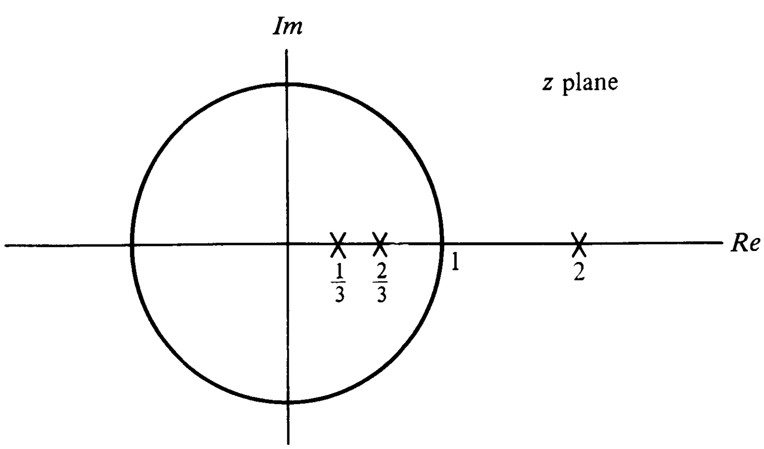

Figure P22.3

Determine what can be inferred about the associated region of convergence from each of the following statements.

(a) $x[n]$ is right-sided.

(b) The Fourier transform of $x[n]$ converges.

(c) The Fourier transform of $x[n]$ does not converge.

(d) $x[n]$ is left-sided.

P22.4

(a) Determine the z-transforms of the following two signals. Note that the z-transforms for both have the same algebraic expression and differ only in the ROC.

(i) $x_1[n] = (\frac{1}{2})^n u[n]$

(ii) $x_2[n] = -(\frac{1}{2})^n u[-n - 1]$

(b) Sketch the pole-zero plot and ROC for each signal in part (a).

(c) Repeat parts (a) and (b) for the following two signals:

(i) $x_3[n] = 2u[n]$

(ii) $x_4[n] = -(2)^n u[-n - 1]$

(d) For which of the four signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ in parts (a) and (c) does the Fourier transform converge?

P22.5

Consider the pole-zero plot of $H(z)$ given in Figure P22.5, where $H(a/2) = 1$.

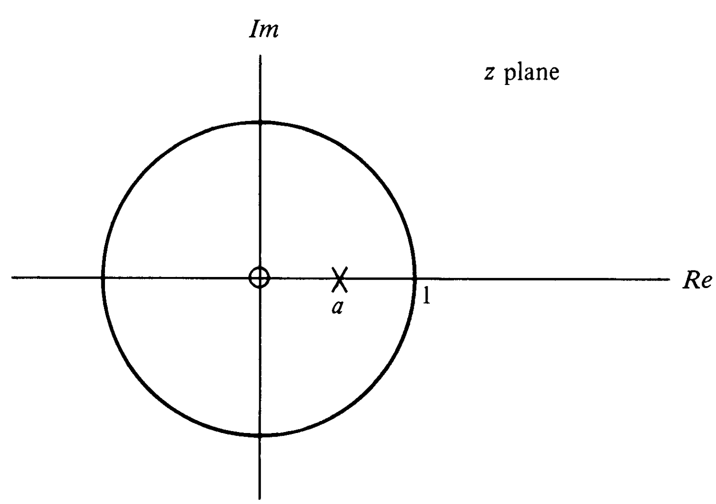

Figure P22.5

(a) Sketch $|H(e^{j\Omega})|$ as the number of zeros at $z = 0$ increases from 1 to 5.

(b) Does the number of zeros affect $\sphericalangle H(e^{j\Omega})$? If so, specifically in what way?

(c) Find the region of the z plane where $|H(z)| = 1$.

P22.6

Determine the z-transform (including the ROC) of the following sequences. Also sketch the pole-zero plots and indicate the ROC on your sketch.

(a) $(\frac{1}{3})^n u[n]$

(b) $\delta[n + 1]$

P22.7

For each of the following z-transforms determine the inverse z-transform.

(a) $X(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}, \qquad |z| > \dfrac{1}{2}$

(b) $X(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}}, \qquad |z| > \dfrac{1}{2}$

(c) $X(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}, \qquad |z| > \left|\dfrac{1}{a}\right|$

Optional Problems

P22.8

In this problem we study the relation between the z-transform, the Fourier transform, and the ROC.

(a) Consider the signal $x[n] = u[n]$. For which values of r does $r^{-n}x[n]$ have a converging Fourier transform?

(b) In the lecture, we discussed the relation between $X(z)$ and $\mathcal{F}\{r^{-n}x[n]\}$. For each of the following values of r, sketch where in the z plane $X(z)$ equals the Fourier transform of $r^{-n}x[n]$.

(i) $r = 1$

(ii) $r = \frac{1}{2}$

(iii) $r = 3$

(c) From your observations in parts (a) and (b), sketch the ROC of the z-transform of $u[n]$.

P22.9

(a) Suppose $X(z)$ on the circle $z = 2e^{j\Omega}$ is given by

$$X(2e^{j\Omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

Using the relation $X(re^{j\Omega}) = \mathcal{F}\{r^{-n}x[n]\}$, find $2^{-n}x[n]$ and then $x[n]$, the inverse z-transform of $X(z)$.

(b) Find $x[n]$ from $X(z)$ below using partial fraction expansion, where $x[n]$ is known to be causal, i.e., $x[n] = 0$ for $n < 0$.

$$X(z) = \frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}}$$

P22.10

A discrete-time system with the pole-zero pattern shown in Figure P22.10-1 is referred to as a first-order all-pass system because the magnitude of the frequency response is a constant, independent of frequency.

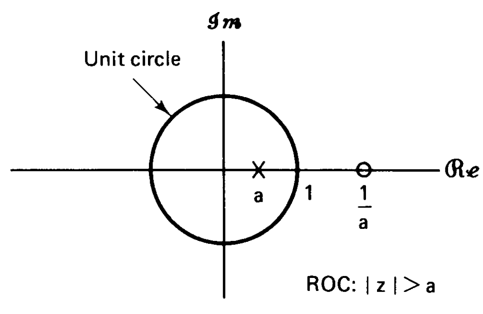

Figure P22.10-1

(a) Demonstrate algebraically that $|H(e^{j\Omega})|$ is constant.

(b) To demonstrate the same property geometrically, consider the vector diagram in Figure P22.10-2. Show that the length of v_2 is proportional to the length of v_1 independent of Ω by following these two steps:

(i) Express the length of v_1 using the law of cosines and the fact that it is one leg of a triangle for which the other two legs are the unit vector and a vector of length a.

(ii) In a manner similar to that in step (i), determine the length of v_2 and show that it is proportional in length to v_1 independent of Ω.

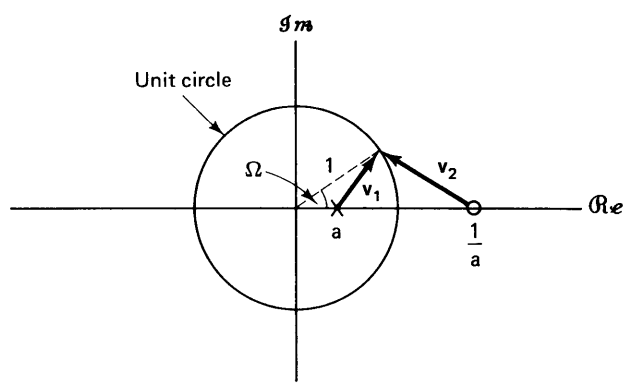

Figure P22.10-2

P22.11

Parts (a)–(e) (Figures P22.11-1 to P22.11-5) give pole-zero plots, and parts (i)–(iv) (Figures P22.11-6 to P22.11-9) give sketches of possible Fourier transform magnitudes. Assume that for all the pole-zero plots, the ROC includes the unit circle. For each pole-zero plot (a)–(e), specify which one *if any* of the sketches (i)–(iv) could represent the associated Fourier transform magnitude. More than one pole-zero plot may be associated with the same sketch.

(a)

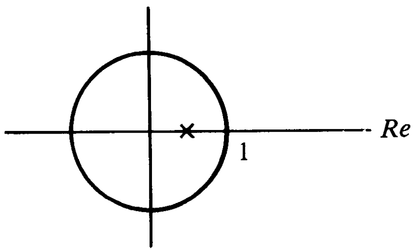

Figure P22.11-1

(b)

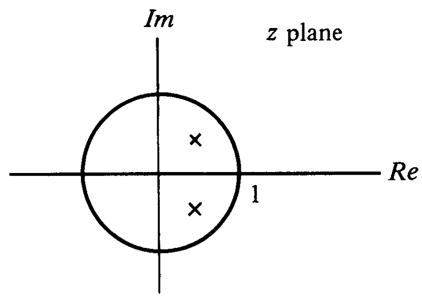

Figure P22.11-2

(c)

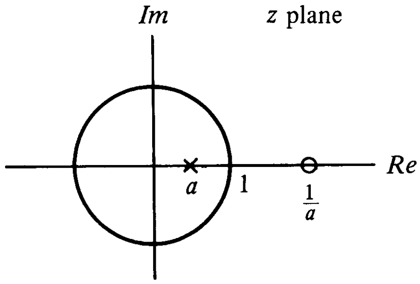

Figure P22.11-3

(d)

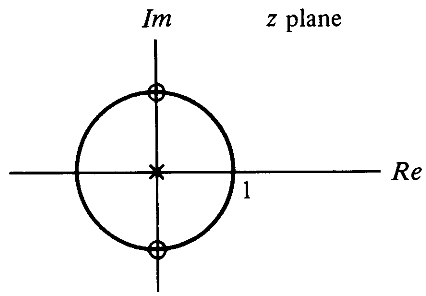

Figure P22.11-4

(e)

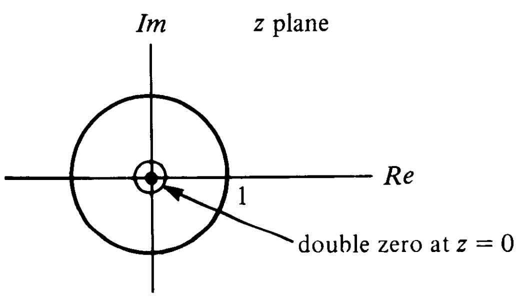

Figure P22.11-5

(i)

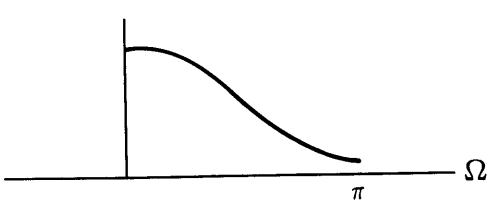

Figure P22.11-6

(ii)

Figure P22.11-7

(iii)

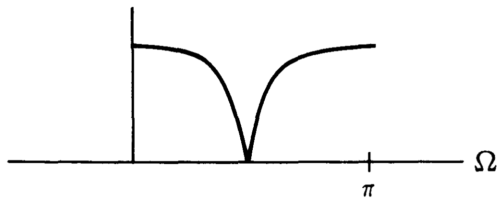

Figure P22.11-8

(iv)

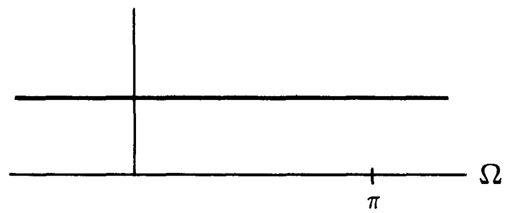

Figure P22.11-9

P22.12

Determine the z-transform for the following sequences. Express all sums in closed form. Sketch the pole-zero plot and indicate the ROC. Indicate whether the Fourier transform of the sequence exists.

(a) $(\frac{1}{2})^n\{u[n] - u[n - 10]\}$

(b) $(\frac{1}{2})^{|n|}$

(c) $7\left(\frac{1}{3}\right)^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[n]$

(d) $x[n] = \begin{cases} 0, & n < 0 \\ 1, & 0 \le n \le 9 \\ 0, & 9 < n \end{cases}$

P22.13

Using the power-series expansion

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \qquad |w| < 1,$$

determine the inverse of the following z-transforms.

(a) $X(z) = \log(1 - 2z), \qquad |z| < \frac{1}{2}$

(b) $X(z) = \log(1 - \frac{1}{2}z^{-1}), \qquad |z| > \frac{1}{2}$

23 Mapping Continuous-Time Filters to Discrete-Time Filters

Recommended Problems

P23.1

For each of the following sequences, determine the associated z-transform, including the ROC. Make use of properties of the z-transform wherever possible.

(a) $x_1[n] = (\frac{1}{2})^n u[n]$

(b) $x_2[n] = (-3)^n u[-n]$

(c) $x_3[n] = x_1[n] + x_2[n]$

(d) $x_4[n] = x_1[n - 5]$

(e) $x_5[n] = x_1[n + 5]$

(f) $x_6[n] = (\frac{1}{3})^n u[n]$

(g) $x_7[n] = x_1[n] * x_6[n]$

P23.2

A causal LTI system is described by the difference equation

$$y[n] - 3y[n - 1] + 2y[n - 2] = x[n]$$

(a) Find $H(z) = Y(z)/X(z)$. Plot the poles and zeros and indicate the ROC.

(b) Find the unit sample response. Is the system stable? Justify your answer.

(c) Find $y[n]$ if $x[n] = 3^n u[n]$.

(d) Determine the system function, associated ROC, and impulse response for all LTI systems that satisfy the preceding difference equation but are not causal. In each case, specify whether the corresponding system is stable.

P23.3

Carry out the proof for the following properties from Table 10.1 of the text (page 654).

(a) 10.5.2

(b) 10.5.3

(c) 10.5.6 [*Hint:* Consider

$$\frac{dX(z)}{dz} = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x[n]z^{-n}.\Big]$$

P23.4

Consider the second-order system with the pole-zero plot given in Figure P23.4. The poles are located at $z = re^{j\theta}$, $z = re^{-j\theta}$, and $H(1) = 1$.

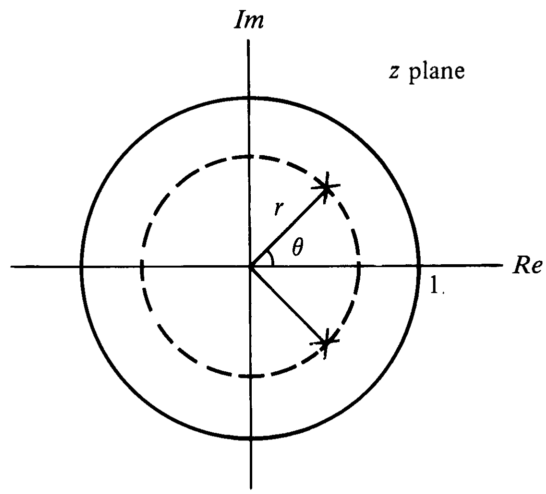

Figure P23.4

(a) Sketch $|H(e^{j\Omega})|$ as θ is kept constant at $\pi/4$ and with $r = 0.5, 0.75$, and 0.9.

(b) Sketch $H(e^{j\Omega})$ as r is kept constant at $r = 0.75$ and with $\theta = \pi/4, 2\pi/4$, and $3\pi/4$.

P23.5

Consider the system function

$$H(z) = \frac{z}{(z - \frac{1}{3})(z - 2)}$$

(a) Sketch the pole-zero locations.

(b) Sketch the ROC assuming the system is causal. Is the system stable?

(c) Sketch the ROC assuming the system is stable. Is the system causal?

(d) Sketch the remaining possible ROC. Is the corresponding system either stable or causal?

P23.6

Consider the continuous-time LTI system described by the following equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t) + 2\frac{dx(t)}{dt}$$

(a) Determine the system function $H_c(s)$ and the impulse response $h_c(t)$.

(b) Determine the system function $H_d(z)$ of a discrete-time LTI system obtained from $H_c(s)$ through impulse invariance.

(c) For $T = 0.01$ determine the impulse response associated with $H_d(z)$, $h_d[n]$.

(d) Verify that $h_d[n] = h_c(nT)$ for all n.

Optional Problems

P23.7

Consider a continuous-time LTI filter described by the following differential equation:

$$\frac{dy(t)}{dt} + 0.5y(t) = x(t)$$

A discrete-time filter is obtained by replacing the derivative by a first forward difference to obtain the difference equation

$$\frac{y[n+1] - y[n]}{T} + 0.5y[n] = x[n]$$

Assume that the resulting system is causal.

(a) Determine and sketch the magnitude of the frequency response of the continuous-time filter.

(b) Determine and sketch the magnitude of the frequency response of the discrete-time filter for $T = 2$.

(c) Determine the range of values of T (if any) for which the discrete-time filter is unstable.

P23.8

Consider an even sequence $x[n]$ (i.e., $x[n] = x[-n]$) with rational z-transform $X(z)$.

(a) From the definition of the z-transform show that

$$X(z) = X\left(\frac{1}{z}\right)$$

(b) From your result in part (a), show that if a pole (or a zero) of $X(z)$ occurs at $z = z_0$, then a pole (or a zero) must also occur at $z = 1/z_0$.

(c) Verify the result in part (b) for each of the following sequences:

(i) $\delta[n+1] + \delta[n-1]$

(ii) $\delta[n+1] - \frac{5}{2}\delta[n] + \delta[n-1]$

(d) Consider a real-valued sequence $y[n]$ with rational z-transform $Y(z)$.

(i) Show that $Y(z) = Y^*(z^*)$.

(ii) From part (i) show that if a pole (or a zero) of $Y(z)$ occurs at $z = z_0$, then a pole (or a zero) must also occur at $z = z_0^*$.

(e) By combining your result in part (b) with that in part (d), show that for a real, even sequence, if there is a pole (or a zero) of $H(z)$ at $z = \rho e^{j\theta}$, then there is also a pole (or a zero) of $H(z)$ at $z = \rho e^{-j\theta}$, at $z = (1/\rho)e^{j\theta}$, and at $z = (1/\rho)e^{-j\theta}$.

P23.9

In Section 10.5.5 of the text we stated the convolution property for the z-transform. To prove this property, we begin with the convolution sum expressed as

$$x_3[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{+\infty} x_1[k]x_2[n-k] \tag{P23.9-1}$$

(a) By taking the z-transform of eq. (P23.9-1) and using eq. (10.3) of the text (page 630), show that

$$X_3(z) = \sum_{k=-\infty}^{+\infty} x_1[k]\hat{X}_2(z),$$

where $\hat{X}_2(z) = Z\{x_2[n-k]\}$.

(b) Using the result in part (a) and the time-shifting property of z-transforms, show that

$$X_3(z) = X_2(z) \sum_{k=-\infty}^{+\infty} x_1[k]z^{-k}$$

(c) From part (b), show that

$$X_3(z) = X_1(z)X_2(z).$$

P23.10

Consider a signal $x[n]$ that is absolutely summable and its associated z-transform $X(z)$. Show that the z-transform of $y[n] = x[n]u[n]$ can have poles only at the poles of $X(z)$ that are inside the unit circle.

P23.11

Let $h_c(t)$, $s_c(t)$, and $H_c(s)$ denote the impulse response, step response, and system function, respectively, of a continuous-time, linear, time-invariant filter.

Let $h_d[n]$, $s_d[n]$, and $H_d(z)$ denote the unit sample response, step response, and system function, respectively, of a discrete-time, linear, time-invariant filter.

(a) If $h_d[n] = h_c(nT)$, does $s_d[n] = \sum_{k=-\infty}^{n} h_c(kT)$?

(b) If $s_d[n] = s_c(nT)$, does $h_d[n] = h_c(nT)$?

P23.12

Consider a continuous-time filter with input $x_c(t)$ and output $y_c(t)$ that is described by a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y_c(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x_c(t)}{dt^k} \tag{P23.12-1}$$

The filter is to be mapped to a discrete-time filter with input $x[n]$ and output $y[n]$ by replacing derivatives with central differences. Specifically, let $\nabla^{(k)}\{x[n]\}$ denote the kth central difference of $x[n]$, defined as follows:

$$\nabla^{(0)}\{x[n]\} = x[n]$$
$$\nabla^{(1)}\{x[n]\} = \left[\frac{x[n+1] - x[n-1]}{2}\right]$$
$$\nabla^{(k)}\{x[n]\} = \nabla^{(1)}\{\nabla^{(k-1)}\{x[n]\}\}$$

The difference equation for the digital filter obtained from the differential equation (P23.12-1) is then

$$\sum_{k=0}^{N} a_k \nabla^{(k)}\{y[n]\} = \sum_{k=0}^{M} b_k \nabla^{(k)}\{x[n]\}$$

(a) If the transfer function of the continuous-time filter is $H_c(s)$ and if the transfer function of the corresponding discrete-time filter is $H_d(z)$, determine how $H_d(z)$ is related to $H_c(s)$.

(b) For the continuous-time frequency response $H_c(j\omega)$, as indicated in Figure P23.12-1, sketch the discrete-time frequency response $H_d(e^{j\Omega})$ that would result from the mapping determined in part (a).

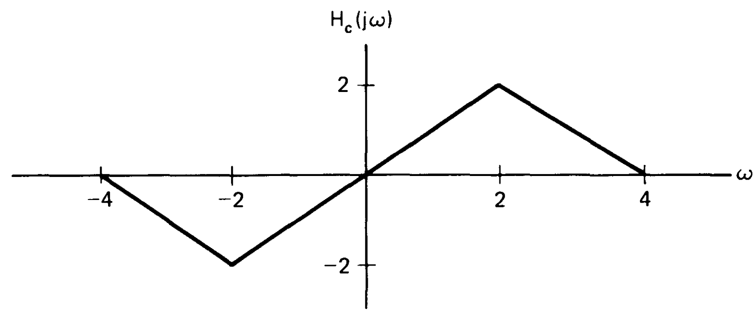

Figure P23.12

(c) Assume that $H_c(s)$ corresponds to a causal stable filter. If the region of convergence of $H_d(z)$ is specified to include the unit circle, will $H_d(z)$ necessarily correspond to a *causal* filter?

P23.13

In discussing impulse invariance in Section 10.8.1 of the text, we considered $H_c(s)$ to be of the form of eq. (10.84) of the text with only first-order poles. In this problem we consider how the presence of a second-order pole in eq. (10.84) would be reflected in eq. (10.87) of the text. Toward this end, consider $H_c(s)$ to be

$$H_c(s) = \frac{A}{(s - s_0)^2}$$

(a) By referring to Table 9.2 of the text (page 604), determine $h_c(t)$. (Assume causality.)

(b) Determine $h_d[n]$ defined as $h_d[n] = h_c(nT)$.

(c) By referring to Table 10.2 of the text (page 655), determine $H_d(z)$, the z-transform of $h_d[n]$.

(d) Determine the system function and pole-zero pattern for the discrete-time system obtained by applying impulse invariance to the following continuous-time system:

$$H_c(s) = \frac{1}{(s + 1)(s + 2)^2}$$

22 The z-Transform

Solutions to Recommended Problems

S22.1

(a) The z-transform $H(z)$ can be written as

$$H(z) = \frac{z}{z - \frac{1}{2}}$$

Setting the numerator equal to zero to obtain the zeros, we find a zero at $z = 0$. Setting the denominator equal to zero to get the poles, we find a pole at $z = \frac{1}{2}$. The pole-zero pattern is shown in Figure S22.1.

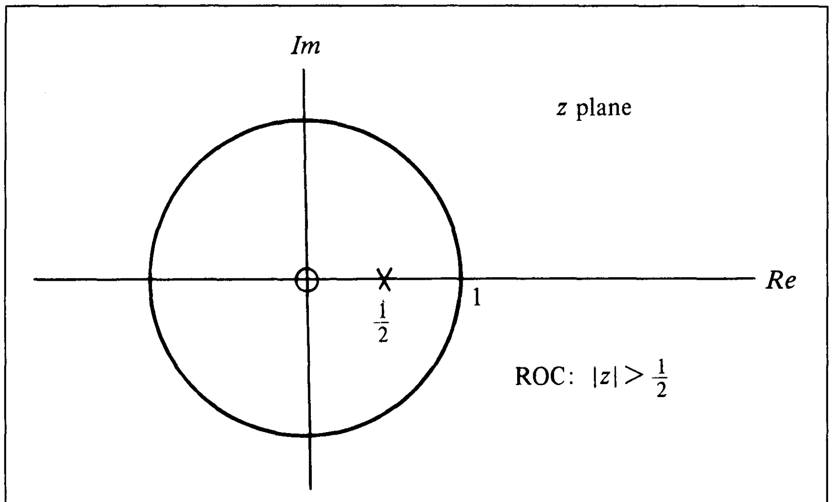

Figure S22.1

(b) Since $H(z)$ is the eigenvalue of the input z^n and the system is linear, the output is given by

$$\begin{aligned} y[n] &= \frac{1}{1 - \frac{1}{2}(\frac{4}{3})}\left(\frac{3}{4}\right)^n + 3\left[\frac{1}{1 - \frac{1}{2}(\frac{1}{2})}\right](2)^n \\ &= 3(\tfrac{3}{4})^n + 4(2)^n \end{aligned}$$

S22.2

(a) To see if $x[n]$ is absolutely summable, we form the sum

$$S_N = \sum_{n=0}^{N-1} |x[n]| = \sum_{n=0}^{N-1} 2^n = \frac{1 - 2^N}{1 - 2}$$

Since $\lim_{N\to\infty} S_N$ diverges, $x[n]$ is *not* absolutely summable.

(b) Since $x[n]$ is not absolutely summable, the Fourier transform of $x[n]$ does not converge.

(c) $S_N = \displaystyle\sum_{n=0}^{N-1} \left(\frac{2}{r}\right)^n = \frac{1 - \left(\frac{2}{r}\right)^N}{1 - \left(\frac{2}{r}\right)}$

$\lim_{N\to\infty} S_N$ is finite for $|r| > 2$. Therefore, the Fourier transform of $r^{-n}x[n]$ converges for $|r| > 2$.

(d) $$X(z) = \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} (2z^{-1})^n$$
$$= \frac{1}{1 - 2z^{-1}} \qquad \text{for } |2z^{-1}| < 1$$

Therefore, the ROC is $|z| > 2$.

(e) $$X_1(e^{j\Omega}) = \frac{1}{1 - \frac{2}{3}e^{-j\Omega}}$$

Therefore, $x_1[n] = (\frac{2}{3})^n u[n]$.

S22.3

(a) Since $x[n]$ is right-sided, the ROC is given by $|z| > \alpha$. Since the ROC cannot include poles, for this case the ROC is given by $|z| > 2$.

(b) The statement implies that the ROC includes the unit circle $|z| = 1$. Since the ROC is a connected region and bounded by poles, the ROC must be

$$\tfrac{2}{3} < |z| < 2$$

(c) For this situation there are three possibilities:

(i) $|z| < \frac{1}{3}$

(ii) $\frac{1}{3} < |z| < \frac{2}{3}$

(iii) $|z| > 2$

(d) This statement implies that the ROC is given by $|z| < \frac{1}{3}$.

S22.4

(a) (i) $$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{2}z^{-1})^n$$
$$= \frac{1}{1 - \frac{1}{2}z^{-1}},$$

with an ROC of $\left|\frac{1}{2z}\right| < 1$, or $|z| > \frac{1}{2}$.

(ii) $$X_2(z) = \sum_{n=-\infty}^{-1} (\tfrac{1}{2})^n z^{-n}$$

Letting $n = -m$, we have

$$X_2(z) = -\sum_{m=1}^{\infty} (\tfrac{1}{2})^{-m} z^m$$
$$= -\sum_{m=1}^{\infty} (2z)^m = -\frac{2z}{1 - 2z}$$
$$= \frac{1}{1 - \frac{1}{2}z^{-1}},$$

with an ROC of $|2z| < 1$, or $|z| < \frac{1}{2}$.

(b) (i)

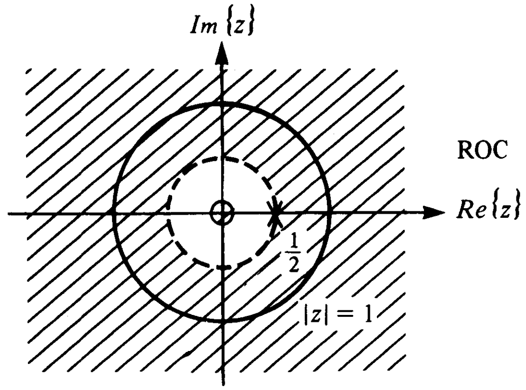

Figure S22.4-1

(ii)

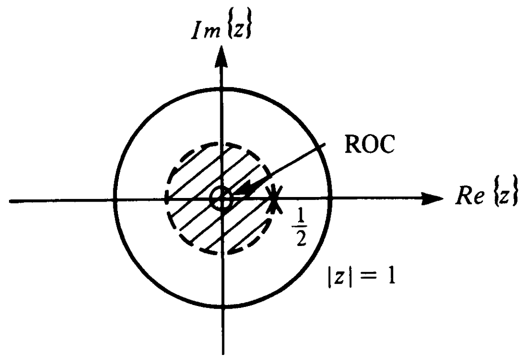

Figure S22.4-2

(c) (i) $X_3(z) = 2\sum_{n=0}^{\infty} z^{-n} = 2\left(\frac{1}{1-z^{-1}}\right) = \frac{2z}{z-1}$. The ROC is $|z| > 1$, as shown in Figure S22.4-3.

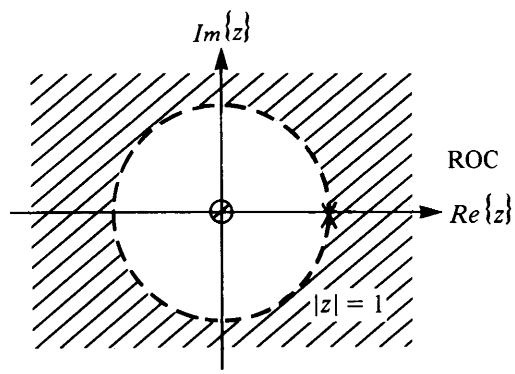

Figure S22.4-3

(ii)
$$\begin{aligned} X_4(z) &= -\sum_{n=-\infty}^{-1} 2^n z^{-n} = -\sum_{n=1}^{\infty} 2^{-n} z^n \\ &= -\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n = -\frac{z/2}{1-(z/2)} = \frac{z}{z-2}, \end{aligned}$$

with an ROC of $|z/2| < 1$, or $|z| < 2$, shown in Figure S22.4-4.

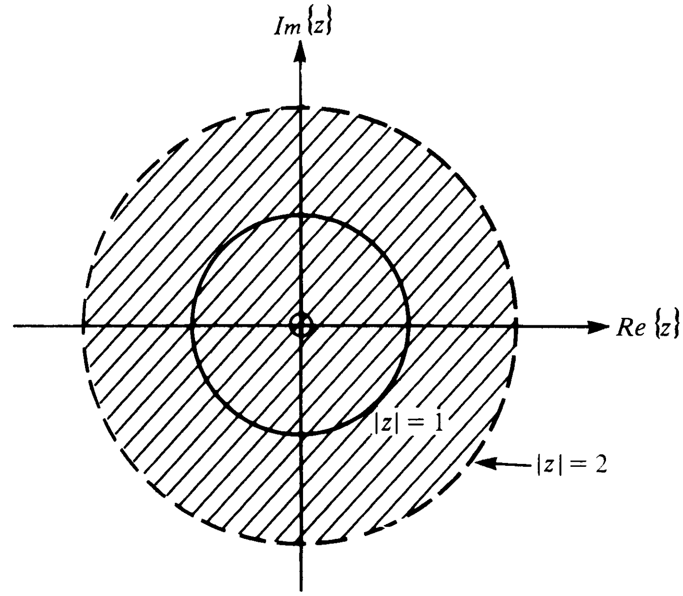

Figure S22.4-4

(d) For the Fourier transform to converge, the ROC of the z-transform must include the unit circle. Therefore, for $x_1[n]$ and $x_4[n]$, the corresponding Fourier transforms converge.

S22.5

Consider the pole-zero plot of $H(z)$ given in Figure S22.5-1, where $H(a/2) = 1$.

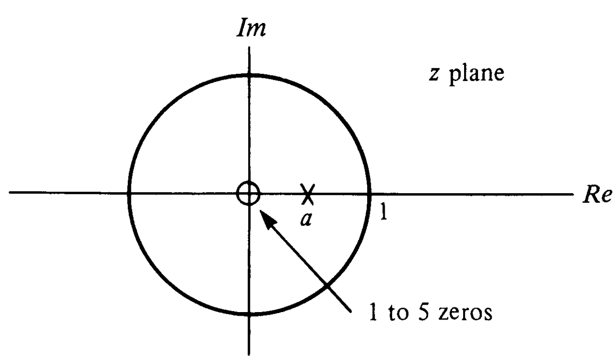

Figure S22.5-1

(a) When $H(z) = z/(z - a)$, i.e., the number of zeros is 1, we have

$$H(e^{j\Omega}) = \frac{\cos \Omega + j \sin \Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega},$$

and we can plot $|H(e^{j\Omega})|$ as in Figure S22.5-2.

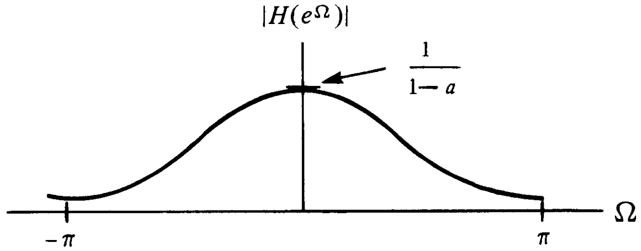

Figure S22.5-2

When $H(z) = z^2/(z - a)$, i.e., the number of zeros is 2, we have

$$H(e^{j\Omega}) = \frac{\cos 2\Omega + j \sin 2\Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega}$$

Hence, we see that the magnitude of $H(e^{j\Omega})$ does not change as the number of zeros increases.

(b) For one zero at $z = 0$, we have

$$H(z) = \frac{z}{z - a},$$
$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

We can calculate the phase of $H(e^{j\Omega})$ by $[\Omega - \sphericalangle \text{(denominator)}]$. For two zeros at 0, the phase of $H(e^{j\Omega})$ is $[2\Omega - \sphericalangle \text{(denominator)}]$. Hence, the phase changes by a linear factor with the number of zeros.

(c) The region of the z plane where $|H(z)| = 1$ is indicated in Figure S22.5-3.

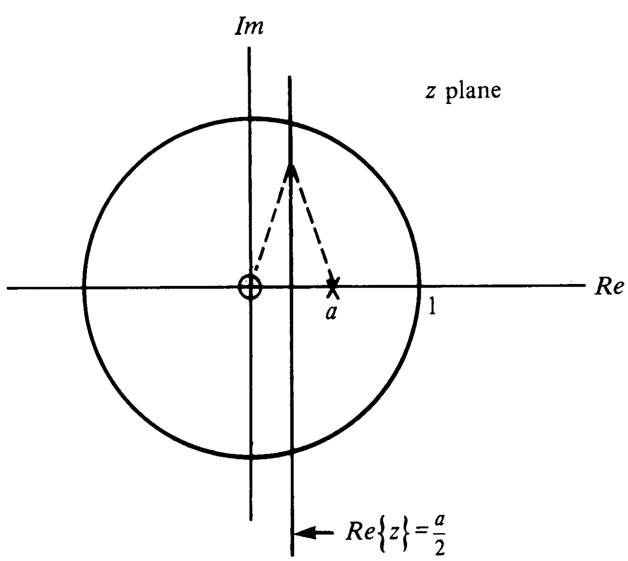

Figure S22.5-3

S22.6

(a) $(\frac{1}{3})^n u[n] \overset{Z}{\longleftrightarrow} \sum_{n=0}^{\infty} (\frac{1}{3})^n z^{-n}$

$$= \sum_{n=0}^{\infty} (3z)^{-n} = \frac{1}{1 - \frac{1}{3}z^{-1}} = \frac{z}{z - \frac{1}{3}}$$

Therefore, there is a zero at $z = 0$ and a pole at $z = \frac{1}{3}$, and the ROC is

$$\left|\frac{1}{3z}\right| < 1 \quad \text{or} \quad |z| > \frac{1}{3},$$

as shown in Figure S22.6-1.

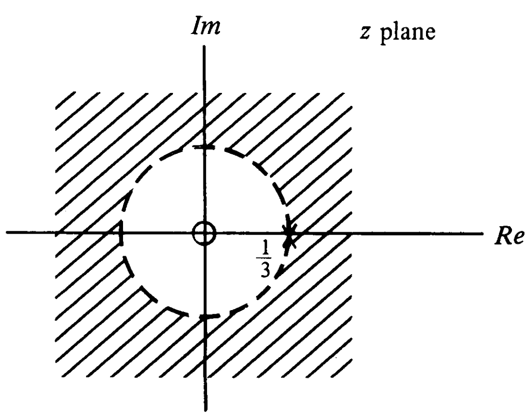

Figure S22.6-1

(b) $\delta[n+1] \overset{Z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n+1]z^{-n} = z,$

with the ROC comprising the entire z-plane, as shown in Figure S22.6-2.

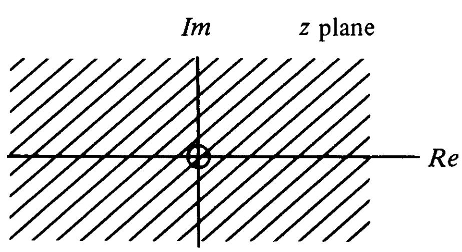

Figure S22.6-2

S22.7

(a) Using long division, we have

$$X(z) = 1 - \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} + \cdots$$

We recognize that

$$x[n] = (-\tfrac{1}{2})^n u[n]$$

(b) Proceeding as we did in part (a), we have

$$x[n] = (-\tfrac{1}{2})^n u[n]$$

(c) $$X(z) = \frac{z-a}{1-az}$$

$$= -\frac{1}{a} + \frac{\left(\frac{1-a^2}{a}\right)}{1-az}$$

$$= -\frac{1}{a} - \frac{\left(\frac{1-a^2}{a^2}\right)z^{-1}}{1-a^{-1}z^{-1}}$$

Therefore,

$$x[n] = -\frac{1}{a}\delta[n] - \frac{(1-a^2)}{a^{n+1}}u[n-1]$$

Solutions to Optional Problems

S22.8

(a) $$\mathcal{F}\{r^{-n}x[n]\} = \mathcal{F}\{r^{-n}u[n]\}$$

$$= \sum_{n=0}^{\infty} r^{-n}e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (re^{+j\Omega})^{-n}$$

For the sum to converge, we must have

$$\left|\frac{1}{re^{+j\Omega}}\right| < 1$$

Thus, $|r| > 1$.

(b) (i)

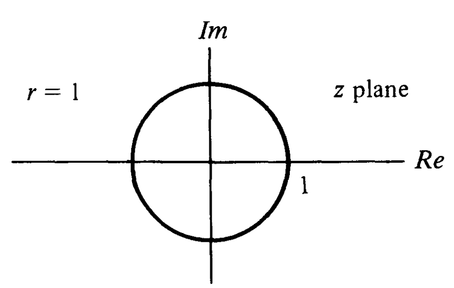

Figure S22.8-1

(ii)

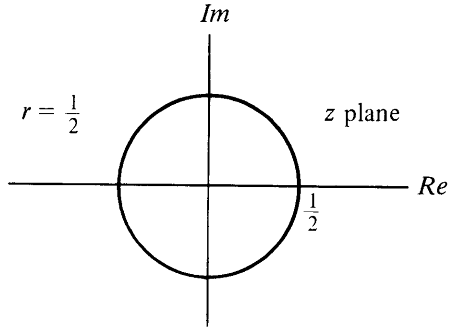

Figure S22.8-2

(iii)

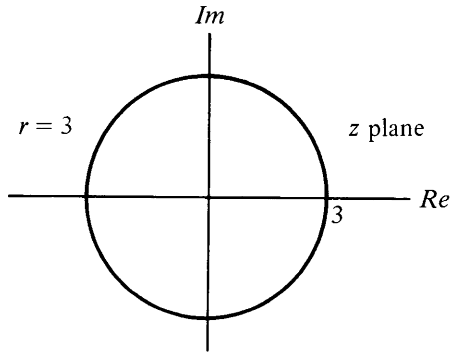

Figure S22.8-3

(c)

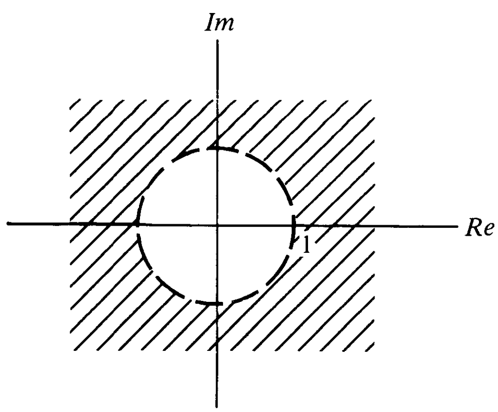

Figure S22.8-4

S22.9

(a) The inverse transform of

$$\frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

is $(\frac{1}{3})^n u[n]$. But from the given relation, we have

$$2^{-n}x[n] = (\tfrac{1}{3})^n u[n],$$

or

$$x[n] = (\tfrac{2}{3})^n u[n]$$

(b)
$$\begin{aligned}
\frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}} &= \frac{3 + 2z^{-1}}{(2 + z^{-1})(1 + z^{-1})} \\
&= \frac{1}{2 + z^{-1}} + \frac{1}{1 + z^{-1}}, \\
X(z) &= \frac{\frac{1}{2}}{1 + \frac{1}{2}z^{-1}} + \frac{1}{1 + z^{-1}} \\
&= \tfrac{1}{2}(-\tfrac{1}{2})^n u[n] + (-1)^n u[n]
\end{aligned}$$

S22.10

(a) $H(z) = \dfrac{A(z^{-1} - a)}{(1 - az^{-1})}$, with A a constant

Therefore,

$$\begin{aligned}
H(e^{j\Omega}) &= \frac{A(e^{-j\Omega} - a)}{1 - ae^{-j\Omega}} \\
|H(e^{j\Omega})|^2 &= \frac{A^2(e^{-j\Omega} - a)(e^{j\Omega} - a)}{(1 - ae^{-j\Omega})(1 - ae^{j\Omega})} = A^2,
\end{aligned}$$

and thus,

$$|H(e^{j\Omega})| = |A|$$

(b) (i) $|v_1|^2 = 1 + a^2 - 2a\cos\Omega$

(ii)
$$\begin{aligned}
|v_2|^2 &= 1 + \frac{1}{a^2} - \frac{2}{a}\cos\Omega \\
&= \frac{1}{a^2}(a^2 + 1 - 2a\cos\Omega) \\
&= \frac{1}{a^2}|v_1|^2
\end{aligned}$$

S22.11

In all the parts of this problem, draw the vectors from the poles or zeros to the unit circle. Then estimate the frequency response from the magnitudes of these vectors, as was done in the lecture. The following rough association can be made:

(a) $\longleftrightarrow$ (i)
(b) $\longleftrightarrow$ (ii)
(c) $\longleftrightarrow$ (iv)
(d) $\longleftrightarrow$ (iii)
(e) $\longleftrightarrow$ (iv)

S22.12

(a) $x[n] = (\frac{1}{2})^n[u[n] - u[n - 10]]$. Therefore,

$$X(z) = \sum_{n=0}^{9} (\tfrac{1}{2})^n z^{-n}$$
$$= \sum_{n=0}^{9} (2z)^{-n} = \frac{1 - (2z)^{-10}}{1 - (2z)^{-1}}$$
$$= \frac{z^{10} - (\frac{1}{2})^{10}}{z^9(z - \frac{1}{2})}, \qquad |z| > 0,$$

shown in Figure S22.12-1. The Fourier transform exists.

(b) $x[n] = (\frac{1}{2})^{|n|} = (\frac{1}{2})^n u[n] + (\frac{1}{2})^{-n} u[-n - 1]$. But

$$\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z - \frac{1}{2}}, \qquad |z| > \frac{1}{2},$$

and

$$\left(\frac{1}{2}\right)^{-n} u[-n - 1] \overset{Z}{\longleftrightarrow} \frac{-z}{z - 2}, \qquad |z| < 2$$

Summing the two z-transforms, we have

$$X(z) = \frac{-\frac{3}{2}z}{(z - \frac{1}{2})(z - 2)}, \qquad \frac{1}{2} < |z| < 2$$

(See Figure S22.12-2.) The Fourier transform exists.

(c) $x[n] = 7(\frac{1}{3})^n \cos\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) u[n]$

Therefore,

$$X(z) = \sum_{n=0}^{\infty} 7 \left(\frac{1}{3}\right)^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] z^{-n}$$
$$= \frac{7}{2} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n [e^{j[(2\pi/6)n + (\pi/4)]} + e^{-j[(2\pi/6)n + (\pi/4)]}] z^{-n}$$
$$= \frac{7}{2}\left[e^{j\pi/4} \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{j(2\pi/6)} z^{-1}\right)^n + e^{-j\pi/4} \sum_{n=0}^{\infty} \left(\frac{1}{3} e^{-j(2\pi/6)} z^{-1}\right)^n \right]$$
$$= \frac{7}{2}\left[\frac{e^{j\pi/4}}{1 - \frac{1}{3}e^{j(2\pi/6)}z^{-1}} + \frac{e^{-j\pi/4}}{1 - \frac{1}{3}e^{-j(2\pi/6)}z^{-1}} \right]$$
$$= \frac{7z}{2}\left[\frac{e^{j\pi/4}}{z - \frac{1}{3}e^{j(2\pi/6)}} + \frac{e^{-j\pi/4}}{z - \frac{1}{3}e^{-j(2\pi/6)}} \right]$$
$$= \frac{7z}{2}\left[\frac{2z \cos\left(\frac{\pi}{4}\right) - \frac{2}{3}\cos\left(\frac{2\pi}{6} - \frac{\pi}{4}\right)}{(z - \frac{1}{3}e^{j(2\pi/6)})(z - \frac{1}{3}e^{-j(2\pi/6)})} \right], \qquad \text{where } |z| > \frac{1}{3}$$

The pole-zero plot and ROC are shown in Figure S22.12-3. Clearly, the Fourier transform exists.

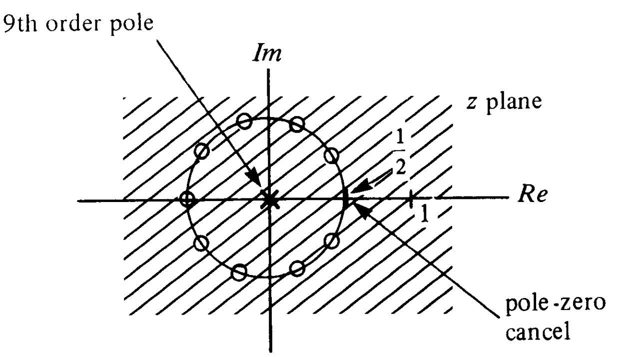

Figure S22.12-1

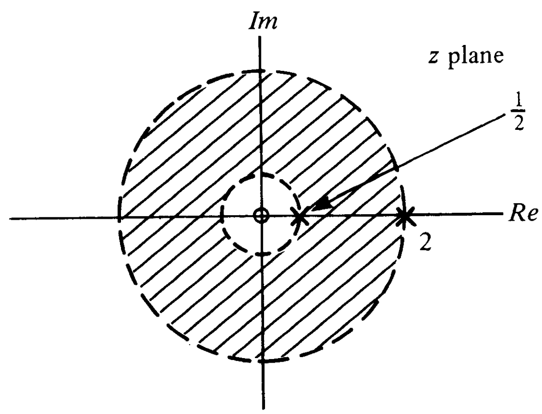

Figure S22.12-2

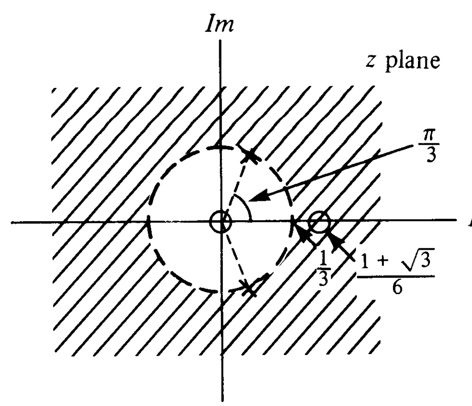

Figure S22.12-3

(d) $X(z) = \sum_{n=0}^{9} z^{-n} = \frac{1 - z^{10}}{1 - z^{-1}} = \frac{z^{10} - 1}{z^9(z - 1)}$

The ROC is all z except $z = 0$, shown in Figure S22.12-4. The Fourier transform exists.

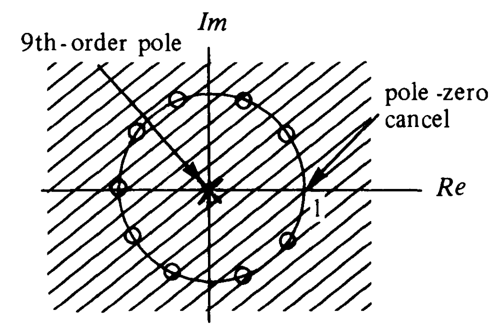

Figure S22.12-4

S22.13

(a) From

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \qquad |w| < 1,$$

we find

$$\log(1 - 2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i}, \qquad |2z| < 1$$

$$= -\sum_{i=1}^{\infty} \frac{2^i}{i} z^i, \qquad |z| < \frac{1}{2},$$

$$x[n] = \begin{cases} \frac{2^{-n}}{n}, & n < 0, \\ 0, & n \geq 0 \end{cases}$$

(b) We solve this similarly to the way we solved part (a).

$$\log\left(1 - \frac{1}{2}z^{-1}\right) = -\sum_{i=1}^{\infty} \frac{(\frac{1}{2}z^{-1})^i}{i}, \qquad \left|\frac{1}{2}z^{-1}\right| < 1$$

$$= -\sum_{i=1}^{\infty} \frac{(\frac{1}{2})^i}{i} z^{-i}, \qquad \frac{1}{2} < |z|,$$

$$x[n] = \begin{cases} -\frac{1}{n}\left(\frac{1}{2}\right)^n, & n > 0, \\ 0, & n \leq 0 \end{cases}$$

23 Mapping Continuous-Time Filters to Discrete-Time Filters

Solutions to Recommended Problems

S23.1

(a) $X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad \left|\frac{1}{2}z^{-1}\right| < 1,$

so the ROC is $|z| > \frac{1}{2}$.

(b) $X_2(z) = \sum_{n=-\infty}^{0} (-3)^n z^{-n}$

$= \sum_{n=0}^{\infty} (-3)^{-n} z^n = \frac{1}{1 + \frac{1}{3}z}, \qquad |-3^{-1}z| < 1,$

so the ROC is $|z| < 3$.

We can also show this by using the property that

$$x[-n] \overset{Z}{\longleftrightarrow} X(z^{-1})$$

Letting $x[-n] = x_2[n]$, we have

$$x[-n] = (-3)^n u[-n],$$
$$x[+n] = (-\tfrac{1}{3})^n u[n],$$
$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}}$$

Therefore,

$$X_2(z) = \frac{1}{1 + \frac{1}{3}z},$$

and the ROC is $|z| < 3$.

(c) Using linearity we see that

$$X_3(z) = X_1(z) + X_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z}$$

The ROC is the common ROC for $X_1(z)$ and $X_2(z)$, which is $\frac{1}{2} < |z| < 3$, as shown in Figure S23.1.

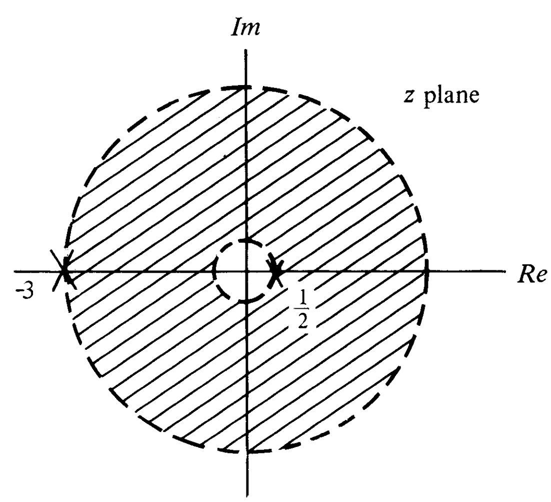

Figure S23.1

(d) Using the time-shifting property

$$x[n - n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z),$$

we have

$$X_4(z) = z^{-5}X_1(z) = \frac{z^{-5}}{1 - \frac{1}{2}z^{-1}}$$

Delaying the sequence does not affect the ROC of the corresponding z-transform, so the ROC is $|z| > \frac{1}{2}$.

(e) Using the time-shifting property, we have

$$X_5(z) = \frac{z^5}{1 - \frac{1}{2}z^{-1}},$$

and the ROC is $|z| > \frac{1}{2}$.

(f) $X_6(z) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{3}z^{-1}},$

and the ROC is $|\frac{1}{3}z^{-1}| < 1$, or $|z| > \frac{1}{3}$.

(g) Using the convolution property, we have

$$X_7(z) = X_1(z)X_6(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right),$$

and the ROC is $|z| > \frac{1}{2}$, corresponding to $\mathrm{ROC}_1 \cap \mathrm{ROC}_6$.

S23.2

(a) We have

$$y[n] - 3y[n-1] + 2y[n-2] = x[n]$$

Taking the z-transform of both sides, we obtain

$$Y(z)[1 - 3z^{-1} + 2z^{-2}] = X(z),$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z-2)(z-1)},$$

and the ROC is outside the outermost pole for the causal (and therefore right-sided) system, as shown in Figure S23.2.

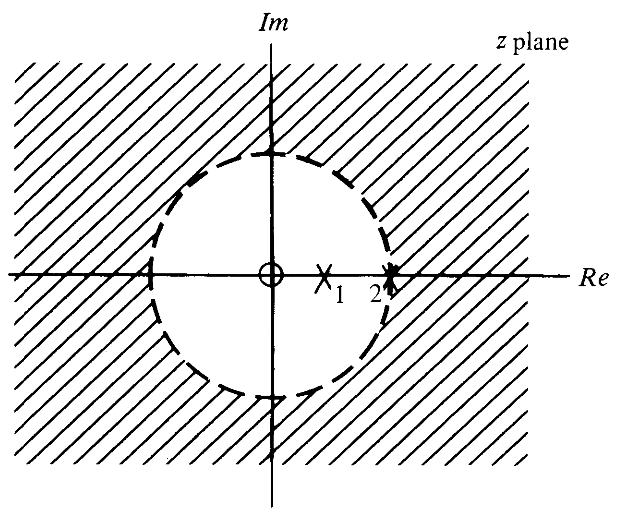

Figure S23.2

(b) Using partial fractions, we have

$$H(z) = \frac{1}{(1 - 2z^{-1})(1 - z^{-1})} = \frac{2}{1 - 2z^{-1}} + \frac{-1}{1 - z^{-1}}$$

By inspection we recognize that the corresponding causal $h[n]$ is the sum of two terms:

$$\begin{aligned} h[n] &= (2)2^n u[n] + (-1)1^n u[n] \\ &= 2^{n+1} u[n] - u[n] \\ &= (2^{n+1} - 1)u[n]. \end{aligned}$$

The system is not stable because the ROC does not include the unit circle. We can also conclude this from the fact that

$$\sum_{n=0}^{\infty} |2^{n+1}| = \infty$$

(c) Since $x[n] = 3^n u[n]$,

$$X(z) = \frac{1}{1 - 3z^{-1}}, \qquad |z| > 3,$$

so

$$Y(z) = H(z)X(z) = \frac{1}{(1 - 3z^{-1})} \frac{1}{(1 - 2z^{-1})(1 - z^{-1})}$$

Using partial fractions, we have

$$Y(z) = \frac{\frac{9}{2}}{1 - 3z^{-1}} + \frac{-4}{1 - 2z^{-1}} + \frac{\frac{1}{2}}{1 - z^{-1}}, \qquad |z| > 3$$

since the output is also causal. Therefore,

$$y[n] = (\tfrac{9}{2})3^n u[n] - (4)2^n u[n] + \tfrac{1}{2} u[n]$$

(d) There are two other possible impulse responses for the same

$$H(z) = \frac{1}{1 - 3z^{-1} + 2z^{-2}}$$

corresponding to different ROCs. For the ROC $|z| < 1$ the system impulse response is left-sided. Therefore, since

$$H(z) = \frac{2}{1 - 2z^{-1}} + \frac{-1}{1 - z^{-1}},$$

then

$$\begin{aligned} h[n] &= -(2)2^n u[-n-1] + (1)u[-n-1] \\ &= -2^{n+1}u[-n-1] + u[-n-1] \end{aligned}$$

For the ROC $1 < |z| < 2$, which yields a two-sided impulse response, we have

$$h[n] = -2^{n+1}u[-n-1] - u[n]$$

since the second term corresponding to $-1/(1 - z^{-1})$ has the ROC $1 < |z|$. Neither system is stable since the ROCs do not include the unit circle.

S23.3

(a) Consider

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n}$$

Letting $m = n - n_0$, we have

$$\begin{aligned} X_1(z) &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m]z^{-m} \\ &= z^{-n_0}X(z) \end{aligned}$$

It is clear that the ROC of $X_1(z)$ is identical to that of $X(z)$ since both require that $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converge in the ROC.

(b) Property 10.5.3 corresponds to multiplication of $x[n]$ by a real or complex exponential. There are three cases listed in the text, which we consider separately here.

(i)
$$\begin{aligned} X_1(z) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\,(ze^{-j\Omega_0})^{-n} \\ &= X(ze^{-j\Omega_0}), \end{aligned}$$

with the same ROC as for $X(z)$.

(ii) Now suppose that

$$\begin{aligned} X_2(z) &= \sum_{n=-\infty}^{\infty} z_0^n x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} \\ &= X\left(\frac{z}{z_0}\right) \end{aligned}$$

Letting $z' = z/z_0$, we see that the ROC for $X_2(z)$ are those values of z such that z' is in the ROC of $X(z')$. If the ROC of $X(z)$ is $R_0 < |z| < R_1$, then the ROC of $X_2(z)$ is $R_0|z_0| < |z| < R_1|z_0|$.

(iii) This proof is the same as that for part (ii), with $a = z_0$.

(c) We want to show that

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

Consider

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Then

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} -nx[n]z^{-n-1}$$
$$= -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n},$$

so

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n},$$

which is what we wanted to show. The ROC is the same as for $X(z)$ except for possible trouble due to the presence of the z^{-1} term.

S23.4

(a)

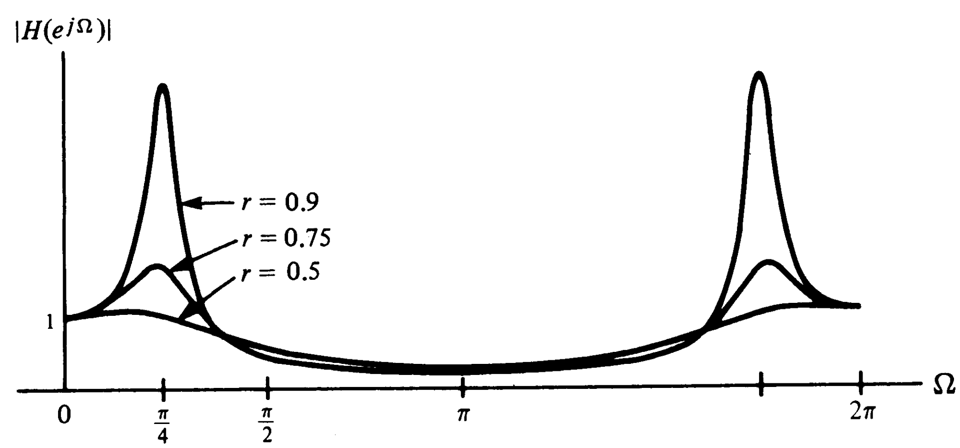

Figure S23.4-1

(b)

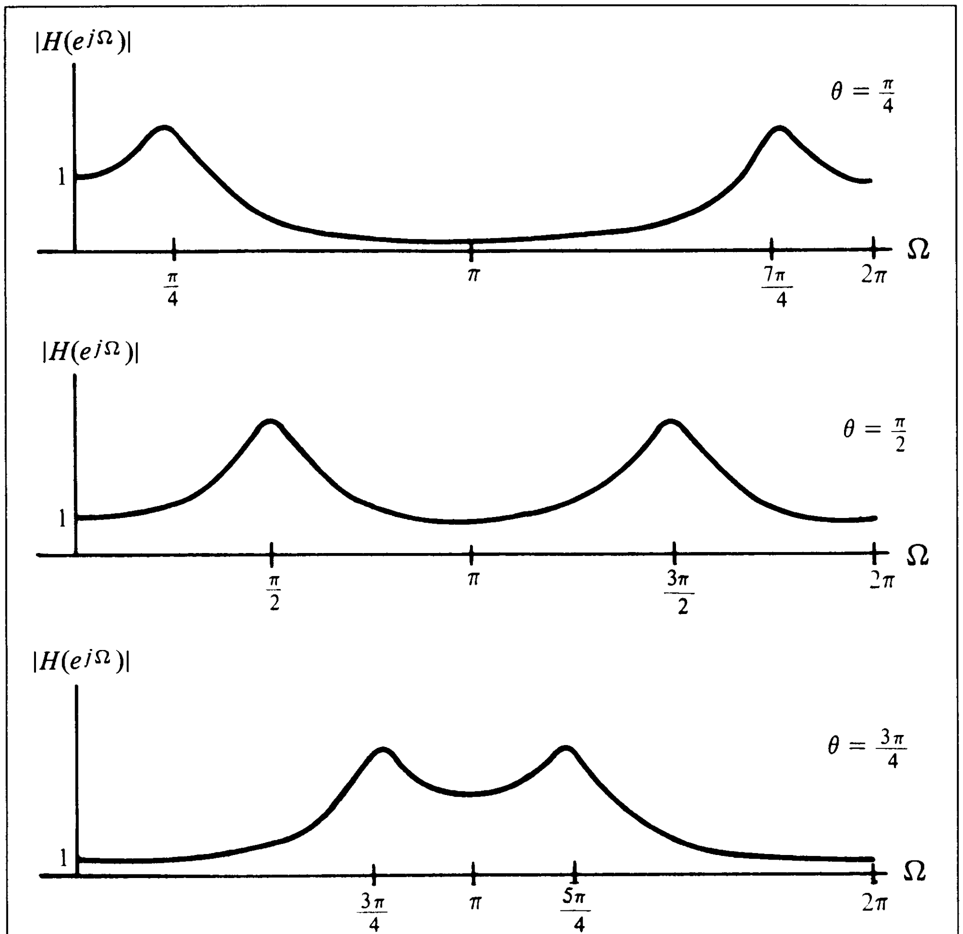

Figure S23.4-2

S23.5

(a)

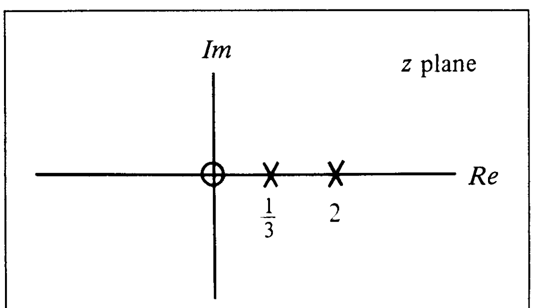

Figure S23.5-1

(b)

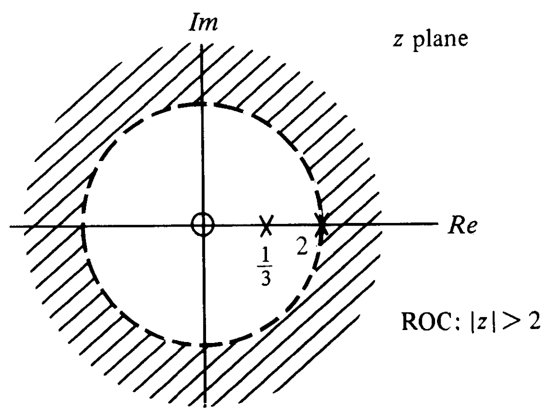

Figure S23.5-2

The ROC is $|z| > 2$. The system is not stable because the ROC does not include the unit circle.

(c)

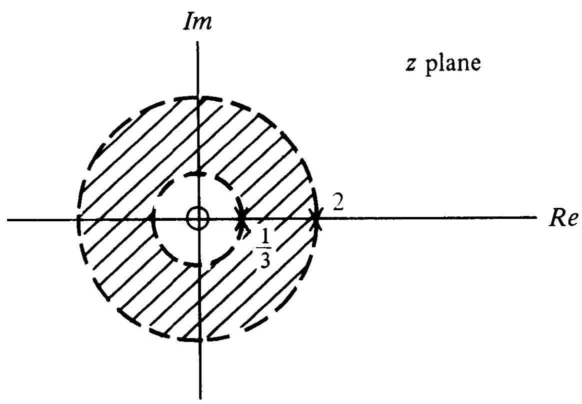

Figure S23.5-3

The ROC is $2 > |z| > \frac{1}{3}$, which for this case includes the unit circle. The corresponding impulse response is two-sided because the ROC is annular. Therefore, the system is not causal.

(d)

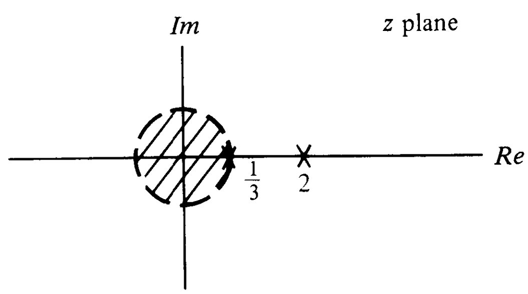

Figure S23.5-4

The remaining ROC does not include the unit circle and is not outside the outermost pole. Therefore, the system is not stable and not causal.

S23.6

(a) $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t) + 2\frac{dx(t)}{dt}$$

is the system differential equation. Taking Laplace transforms of both sides, we have

$$Y(s)(s^2 + 5s + 6) = X(s)(1 + 2s),$$

so

$$H_c(s) = \frac{Y(s)}{X(s)} = \frac{1 + 2s}{s^2 + 5s + 6}$$
$$= \frac{1 + 2s}{(s + 3)(s + 2)} = \frac{5}{s + 3} + \frac{-3}{s + 2}$$

Assuming the system is causal, we obtain by inspection

$$h_c(t) = 5e^{-3t}u(t) - 3e^{-2t}u(t)$$

(b) Using the fact that the continuous-time system function $A_k/(s - s_k)$ maps to the discrete-time system function $A_k/(1 - e^{s_kT}z^{-1})$ (see page 662 of the text), we have

$$H_d(z) = \frac{5}{1 - e^{-3T}z^{-1}} - \frac{3}{1 - e^{-2T}z^{-1}}$$

(c) Suppose $T = 0.01$. Then

$$H_d(z) = \frac{5}{1 - e^{-0.03}z^{-1}} - \frac{3}{1 - e^{-0.02}z^{-1}}$$

Letting $a = e^{-0.03}$, $b = e^{-0.02}$, we have

$$H_d(z) = \frac{5}{1 - az^{-1}} - \frac{3}{1 - bz^{-1}}$$

So by inspection, assuming causality,

$$h_d[n] = 5a^nu[n] - 3b^nu[n]$$

(d) From part (a), we have

$$h_c(t) = 5e^{-3t}u(t) - 3e^{-2t}u(t)$$

Replacing t by $nT = 0.01n$, we have

$$h_c(nT) = 5e^{-0.03n}u(0.01n) - 3e^{-0.02n}u(0.02n)$$

Letting $a = e^{-0.03}$ and $b = e^{-0.02}$ yields

$$h_c(nT) = 5a^nu[n] - 3b^nu[n],$$

which agrees with the result in part (c).

Solutions to Optional Problems

S23.7

(a) The differential equation is

$$\frac{dy(t)}{dt} + 0.5y(t) = x(t)$$

Taking the Laplace transform yields

$$Y(s)[s + 0.5] = X(s),$$

so

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s + 0.5},$$

$$H(\omega) = \frac{1}{j\omega + 0.5},$$

which is sketched in Figure S23.7-1.

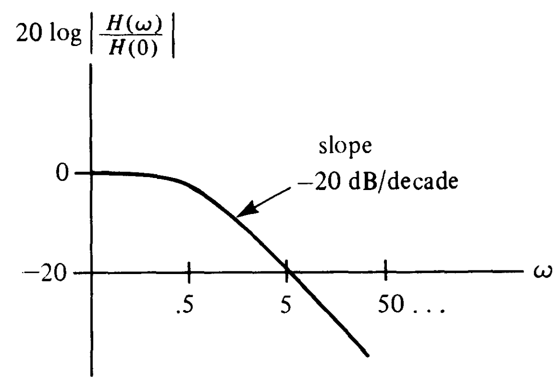

Figure S23.7-1

(b) $\dfrac{y[n+1] - y[n]}{T} + 0.5y[n] = x[n]$

Taking the z-transform of both sides yields

$$\frac{1}{T}(z - 1)Y(z) + 0.5Y(z) = X(z),$$

$$Y(z)\left(0.5 + \frac{z - 1}{T}\right) = X(z)$$

Letting $T = 2$ yields

$$\frac{Y(z)}{Z(z)} = H_d(z) = \frac{2}{z}$$

Now since

$$|H_d(e^{j\Omega})| = |H_d(z)|\Big|_{z=e^{j\Omega}}$$

we have

$$|H_d(e^{j\Omega})| = 2, \qquad \text{for all } \Omega,$$

which is an all-pass filter and is sketched in Figure S23.7-2.

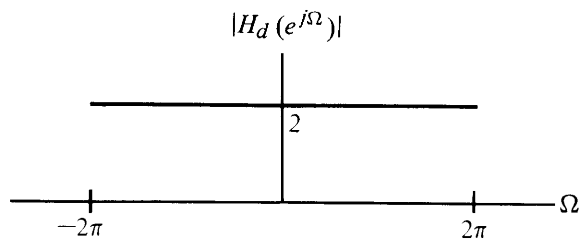

Figure S23.7-2

(c)
$$H_d(z) = \frac{1}{0.5 - \frac{1}{T} + \frac{1}{T}z} = \frac{T}{(0.5T - 1) + z}$$

The pole is located at $z_0 = -(0.5T - 1)$ and, since we assume causality, we require that the ROC be outside this pole. When the pole moves onto or outside the unit circle, stability does not exist. The filter is unstable for

$$|z_0| \geq 1 \qquad \text{or} \qquad |-(0.5T - 1)| \geq 1,$$
$$|0.5T - 1| \geq 1,$$
$$T \geq 4$$

Therefore, for $T \geq 4$, the system is not stable.

S23.8

(a)
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$X(z^{-1}) = \sum_{n=-\infty}^{\infty} x[n]z^{n}$$

Letting $m = -n$, we have

$$X(z^{-1}) = \sum_{m=-\infty}^{\infty} x[-m]z^{-m} = \sum_{m=-\infty}^{\infty} x[m]z^{-m} = X(z)$$

(b)
$$X(z) = A\frac{\prod_k (z - a_k)}{\prod_k (z - b_k)}$$

from the definition of a rational z-transform. Now

$$X(z^{-1}) = A\frac{\prod_k (z^{-1} - a_k)}{\prod_k (z^{-1} - b_k)}$$

Each pole (or zero) at z_0 in $X(z)$ goes to a pole (or zero) z_0^{-1} in $X(z^{-1})$. This implies that $z_0 = 1$ or that $X(z)$ must have another pole (or zero) at z_0^{-1}.

(c) (i) $x[n] = \delta[n+1] + \delta[n-1]$,

$$X(z) = z + z^{-1} = \frac{z^2+1}{z} = \frac{(z+j)(z-j)}{z}$$

The zeros are at $z = j,\ 1/j$, and the poles are at $z = 0$, $z = \infty$.

(ii) $x[n] = \delta[n+1] - \frac{5}{2}\delta[n] + \delta[n-1]$,

$$X(z) = z - \tfrac{5}{2} + z^{-1} = \frac{z^2 - \frac{5}{2}z + 1}{z} = \frac{(z-\frac{1}{2})(z-2)}{z}$$

The zeros are at $z = \frac{1}{2}$, 2, and the poles are at 0, ∞.

(d) (i)

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n},$$

$$Y^*(z) = \left(\sum_{n=-\infty}^{\infty} y[n]z^{-n}\right)^* = \sum_{n=-\infty}^{\infty} y[n](z^*)^{-n} = Y(z^*),$$

so

$$Y^*(z^*) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = Y(z)$$

(ii) Since $Y(z)$ is rational,

$$Y(z) = A\frac{\prod_k (z-a_k)}{\prod_k (z-b_k)}$$

Now if a term such as $(z - a_k)$ appears in $Y(z)$, a term such as $(z^* - a_k)^*$ must also appear in $Y(z)$. For example,

$$\begin{aligned} Y(z) &= (z-a_k)(z^*-a_k)^*, \\ Y^*(z^*) &= [(z^*-a_k)(z-a_k)^*]^* \\ &= (z-a_k^*)(z-a_k) = Y(z) \end{aligned}$$

So if a pole (or zero) appears at $z = a_k$, a pole (or zero) must also appear at $z = a_k^*$ because

$$(z - a_k^*) = 0 \Rightarrow z = a_k^*$$

(e) Both conditions discussed in parts (b) and (d) hold, i.e., a real, even sequence is considered. A pole at $z = z_p$ implies a pole at $1/z_p$ from part (b). The poles at $z = z_p$ and $z = 1/z_p$ imply poles at $z = z_p^*$ and $z = (1/z_p)^*$ from part (d). Therefore, if $z_p = \rho e^{j\theta}$, poles exist at

$$\frac{1}{\rho e^{j\theta}} = \frac{1}{\rho}e^{-j\theta}, \qquad (\rho e^{j\theta})^* = \rho e^{-j\theta}, \qquad \left(\frac{1}{\rho e^{j\theta}}\right)^* = \frac{1}{\rho}e^{j\theta}$$

S23.9

(a)

$$\begin{aligned} X_3(z) &= \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k]\hat{X}_2(z), \end{aligned}$$

where

$$\hat{X}_2(z) = \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$
$$= Z\{x_2[n-k]\}$$

(b) $Z\{x_2[n-k]\} = z^{-k}X_2(z)$ from the time-shifting property of the z-transform, so

$$X_3(z) = \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}X_2(z)$$

(c) $X_3(z) = X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}$
$= X_1(z)X_2(z)$

S23.10

Consider $x[n]$ to be composed of a causal and an anticausal part:

$$x[n] = x[n]u[-n-1] + x[n]u[n]$$

Let

$$x_1[n] = x[n]u[-n-1],$$
$$x_2[n] = x[n]u[n],$$

so that

$$x[n] = x_1[n] + x_2[n]$$

and

$$X(z) = X_1(z) + X_2(z)$$

It is clear that every pole of $X_2(z)$ is also a pole of $X(z)$. The only way for this not to be true is by pole cancellation from $X_1(z)$. But pole cancellation cannot happen because a pole a_k that appears in $X_2(z)$ yields a contribution $(a_k)^n u[n]$, which cannot be canceled by terms of $x_1[n]$ that are of the form $(b_k)^n u[-n-1]$.

From the linearity property of z-transforms, if

$$y[n] = y_1[n] + y_2[n]$$

then

$$Y(z) = Y_1(z) + Y_2(z),$$

with the ROC of $Y(z)$ being at least the intersection of the ROC of $Y_1(z)$ and the ROC of $Y_2(z)$. The "at least" specification is required because of possible pole cancellation. In our case, pole cancellation cannot occur, so the ROC of $X(z)$ is exactly the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$.

Now suppose $X_2(z)$ has a pole outside the unit circle. Since $x_2[n]$ is causal, the ROC of $X_2(z)$ must be outside the unit circle, which implies that the ROC of $X(z)$ must be outside the unit circle. This is a contradiction, however, because $x[n]$ is assumed to be absolutely summable, which implies that $X(z)$ has an ROC that includes the unit circle.

Therefore, all poles of the z-transform of $x[n]u[n]$ must be within the unit circle.

S23.11

(a) If $h[n] = h_c(nT)$, then

$$s_d[n] = \sum_{k=-\infty}^{n} h_c(kT)$$

The proof follows.

$$s_d[n] = \sum_{k=-\infty}^{\infty} u[n-k]h_d[k]$$
$$= \sum_{k=-\infty}^{n} h_d[k],$$

but $h_d[k] = h_c(kT)$, so

$$s_d[n] = \sum_{k=-\infty}^{n} h_c(kT)$$

(b) If $s_d[n] = s_c(nT)$, then $h_d[n]$ does not necessarily equal $h_c(nT)$. For example,

$$h_c(t) = e^{-at}u(t),$$
$$s_c(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)\,d\tau$$
$$= \int_0^t e^{-a\tau}d\tau = \frac{1}{a}(1-e^{-at}), \qquad t \geq 0$$
$$s_d[n] = s_c(nT) = \frac{1}{a}(1-e^{-anT}), \qquad n \geq 0$$

However,

$$s_d[n] = \sum_{k=-\infty}^{n} h_d[k],$$

so

$$s_d[n] - s_d[n-1] = h_d[n]$$

and, in our case, for $n \geq 0$,

$$h_d[n] = \frac{1}{a}(1-e^{-anT}) - \frac{1}{a}(1-e^{-a(n-1)T})$$
$$= \frac{1}{a}e^{-anT}(e^{aT}-1)$$

But, for $n \geq 0$,

$$h_c(nT) = e^{-anT}u(nT)$$
$$\neq \frac{1}{a}e^{-anT}(e^{aT}-1)$$

S23.12

(a) From the differential equation

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s),$$

we have

$$\frac{Y(s)}{X(s)} = H_c(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Now consider

$$y_1[n] \equiv \nabla^{(1)}\{x[n]\} \equiv \frac{x[n+1] - x[n-1]}{2},$$

$$Y_1(z) = Z\{y_1[n]\} = Z\{\nabla^{(1)}\{x[n]\}\} = \left(\frac{z - z^{-1}}{2}\right) X(z),$$

$$y_2[n] = \nabla^{(2)}\{x[n]\} = \nabla^{(1)}\{y_1[n]\} = \frac{y_1[n+1] - y_1[n-1]}{2},$$

$$Y_2(z) = \frac{z - z^{-1}}{2} Y_1(z) = \left(\frac{z - z^{-1}}{2}\right)^2 X(z)$$

By induction,

$$Z\{\nabla^{(k)}\{x[n]\}\} = \left(\frac{z - z^{-1}}{2}\right)^k X(z)$$

Therefore,

$$\sum_{k=0}^{N} a_k \left(\frac{z - z^{-1}}{2}\right)^k Y(z) = \sum_{k=0}^{M} b_k \left(\frac{z - z^{-1}}{2}\right)^k X(z),$$

$$H_d(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \left(\frac{z - z^{-1}}{2}\right)^k}{\sum_{k=0}^{N} a_k \left(\frac{z - z^{-1}}{2}\right)^k}$$

$$= H_c(s)\Big|_{s=(z-z^{-1})/2}$$

(b) $H_c(s)\Big|_{s=(z-z^{-1})/2} = H_d(z)$

from part (a). Consider $s = j\omega$, $z = e^{j\Omega}$. So

$$j\omega = \frac{e^{j\Omega} - e^{-j\Omega}}{2},$$

and, thus, $\omega = \sin \Omega$ is the mapping between discrete-time and continuous-time frequencies. Since $H(\omega) = \omega$ for $|\omega| < 1$, $H_d(e^{j\Omega})$ is as indicated in Figure S23.12.

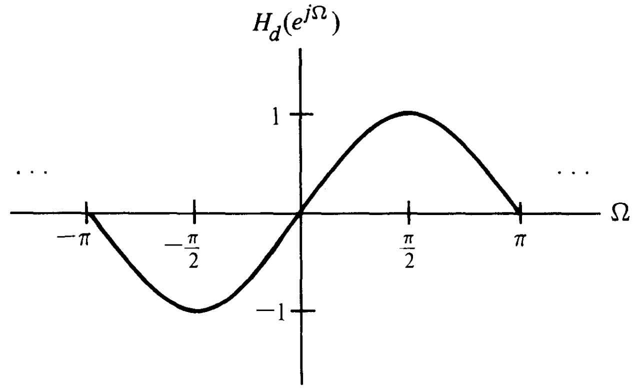

Figure S23.12

(c) From part (a) we see that

$$H_d(z) = H_d(z^{-1})$$

and that $H_d(z)$ is a rational z-transform.

$$H_d(z) = \frac{A\prod_{i=1}^{P} (z - z_{0_i})^{M_i}}{\prod_{i=1}^{Q} N_i(z - z_{p_i})^{N_i}}$$

Therefore, if a term such as $(z - z_{0_i})$ appears, $(z^{-1} - z_{0_i})$ must also appear. If $H_d(z)$ has a pole within the unit circle, it must also have a pole outside the unit circle. If the ROC includes the unit circle, it is therefore not outside the outermost pole (which lies outside the unit circle) and, therefore, $H_d(z)$ does not correspond to a causal filter.

Consider

$$H_c(s) = \frac{1}{s + \frac{1}{2}}$$

corresponding to a stable, causal $h_c(t)$.

$$H_d(z) = H_c(s)\Big|_{s=(z-z^{-1})/2} = \frac{1}{\dfrac{z - z^{-1}}{2} + \dfrac{1}{2}}$$

$$= \frac{2z}{z^2 + z - 1} = \frac{2z}{\left[z - \dfrac{(-1 + \sqrt{5})}{2}\right]\left[z - \dfrac{(-1 - \sqrt{5})}{2}\right]},$$

so poles of z are at 0.618, −1.618. Therefore, $H_d(z)$ is not causal if it is assumed stable because stability and causality require that all poles be inside the unit circle.

S23.13

(a) We are given that

$$H_c(s) = \frac{A}{(s - s_0)^2}$$

From Table 9.2 of the text (page 604), we see that $h_c(t) = Ate^{s_0 t}u(t)$. To verify, consider

$$\frac{1}{s - s_0} = \int_{-\infty}^{\infty} e^{s_0 t}u(t)e^{-st}dt,$$

$$\frac{d}{ds}\left(\frac{1}{s - s_0}\right) = \frac{d}{ds}\left[\int_{-\infty}^{\infty} e^{s_0 t}u(t)e^{-st}dt\right],$$

$$\frac{1}{(s - s_0)^2} = \int_{-\infty}^{\infty} te^{s_0 t}u(t)e^{-st}dt$$

Therefore,

$$te^{s_0 t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s - s_0)^2}$$

(b) $h_d[n] = h_c(nT) = AnTe^{s_0 nT}u[n]$

(c) $H_d(z) = \sum_{n=-\infty}^{\infty} h_d[n]z^{-n} = AT\sum_{n=0}^{\infty} ne^{s_0 nT}z^{-n}$

From Table 10.2 of the text (page 655),

$$n\alpha^n u[n] \overset{Z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$$

This can be verified:

$$\frac{1}{1 - \alpha z^{-1}} = \sum_{n=-\infty}^{\infty} \alpha^n u[n]z^{-n}$$

$$\frac{d}{dz}\left(\frac{1}{1 - \alpha z^{-1}}\right) = \frac{d}{dz}\sum_{n=-\infty}^{\infty} \alpha^n u[n]z^{-n}$$

$$\frac{-\alpha z^{-2}}{(1 - \alpha z^{-1})^2} = \sum_{n=-\infty}^{\infty} (-n)\alpha^n u[n]z^{-n-1}$$

$$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2} = \sum_{n=-\infty}^{\infty} n\alpha^n u[n]z^{n}$$

In our case, $\alpha = e^{s_0 T}$, so

$$H_d(z) = \frac{ATe^{s_0 T}z^{-1}}{(1 - e^{s_0 T}z^{-1})^2}$$

(d) $H_c(s) = \dfrac{1}{(s + 1)(s + 2)^2} = \dfrac{1}{s + 1} + \dfrac{-1}{s + 2} + \dfrac{-1}{(s + 2)^2}$

Using the first-order pole result for $1/(s + 1)$ and $-1/(s + 2)$ and the second-order pole result for $-1/(s + 2)^2$, we have

$$H_d(z) = \frac{1}{1 - e^{-T}z^{-1}} - \frac{1}{1 - e^{-2T}z^{-1}} - \frac{Te^{-2T}z^{-1}}{(1 - e^{-2T}z^{-1})^2}$$

After some algebra, we obtain

$$H_d(z) = \frac{z[z(-e^{-2T} + e^{-T} - Te^{-2T}) + e^{-4T} - e^{-3T} + Te^{-3T}]}{(z - e^{-T})(z - e^{-2T})^2}$$

The corresponding pole-zero pattern is shown in Figure S23.13.

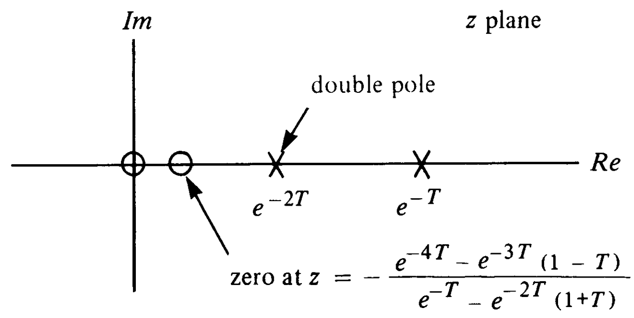

Figure S23.13