

20 The Laplace Transform

Recommended Problems

P20.1

Consider the signal $x(t) = 3e^{2t}u(t) + 4e^{3t}u(t)$.

- (a) Does the Fourier transform of this signal converge?
- (b) For which of the following values of σ does the Fourier transform of $x(t)e^{-\sigma t}$ converge?
 - (i) $\sigma = 1$
 - (ii) $\sigma = 2.5$
 - (iii) $\sigma = 3.5$
- (c) Determine the Laplace transform $X(s)$ of $x(t)$. Sketch the location of the poles and zeros of $X(s)$ and the ROC.

P20.2

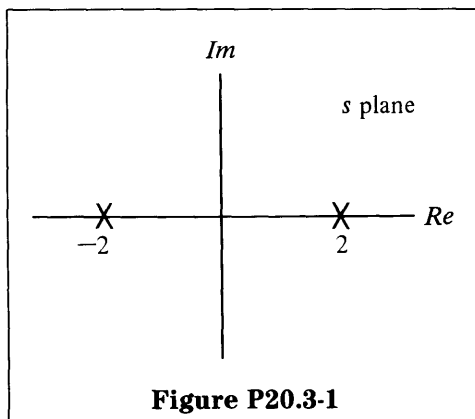
Determine the Laplace transform, pole and zero locations, and associated ROC for each of the following time functions.

- (a) $e^{-at}u(t)$, $a > 0$
- (b) $e^{-at}u(t)$, $a < 0$
- (c) $-e^{-at}u(-t)$, $a < 0$

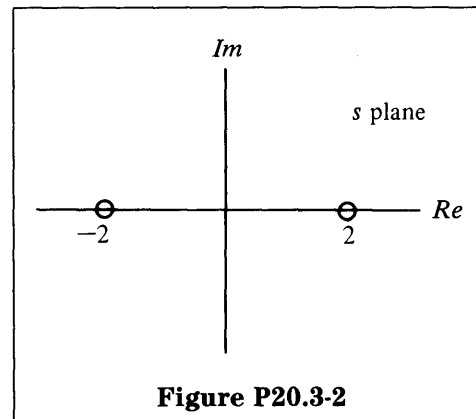
P20.3

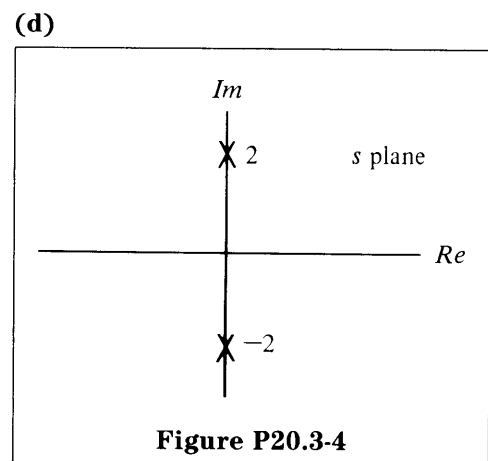
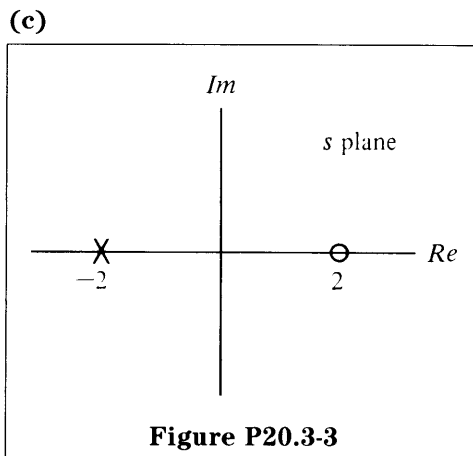
Shown in Figures P20.3-1 to P20.3-4 are four pole-zero plots. For each statement in Table P20.3 about the associated time function $x(t)$, fill in the table with the corresponding constraint on the ROC.

(a)



(b)





Constraint on ROC for Pole-Zero Pattern

$x(t)$	(a)	(b)	(c)	(d)
(i) Fourier transform of $x(t)e^{-t}$ converges				
(ii) $x(t) = 0, t > 10$				
(iii) $x(t) = 0, t < 0$				

Table P20.3

P20.4

Determine $x(t)$ for the following conditions if $X(s)$ is given by

$$X(s) = \frac{1}{(s + 1)(s + 2)}$$

- (a) $x(t)$ is right-sided
- (b) $x(t)$ is left-sided
- (c) $x(t)$ is two-sided

P20.5

An LTI system has an impulse response $h(t)$ for which the Laplace transform $H(s)$ is

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt = \frac{1}{s + 1}, \quad \text{Re}\{s\} > -1$$

Determine the system output $y(t)$ for all t if the input $x(t)$ is given by

$$x(t) = e^{-t/2} + 2e^{-t/3} \quad \text{for all } t.$$

P20.6

- (a) From the expression for the Laplace transform of $x(t)$, derive the fact that the Laplace transform of $x(t)$ is the Fourier transform of $x(t)$ weighted by an exponential.
- (b) Derive the expression for the inverse Laplace transform using the Fourier transform synthesis equation.

Optional Problems

P20.7

Determine the time function $x(t)$ for each Laplace transform $X(s)$.

- (a) $\frac{1}{s+1}$, $Re\{s\} > -1$
- (b) $\frac{1}{s+1}$, $Re\{s\} < -1$
- (c) $\frac{s}{s^2+4}$, $Re\{s\} > 0$
- (d) $\frac{s+1}{s^2+5s+6}$, $Re\{s\} > -2$
- (e) $\frac{s+1}{s^2+5s+6}$, $Re\{s\} < -3$
- (f) $\frac{s^2-s+1}{s^2(s-1)}$, $0 < Re\{s\} < 1$
- (g) $\frac{s^2-s+1}{(s+1)^2}$, $-1 < Re\{s\}$
- (h) $\frac{s+1}{(s+1)^2+4}$, $Re\{s\} > -1$

Hint: Use the result from part (c).

P20.8

The Laplace transform $X(s)$ of a signal $x(t)$ has four poles and an unknown number of zeros. $x(t)$ is known to have an impulse at $t = 0$. Determine what information, if any, this provides about the number of zeros.

P20.9

Determine the Laplace transform, pole-zero location, and associated ROC for each of the following time functions.

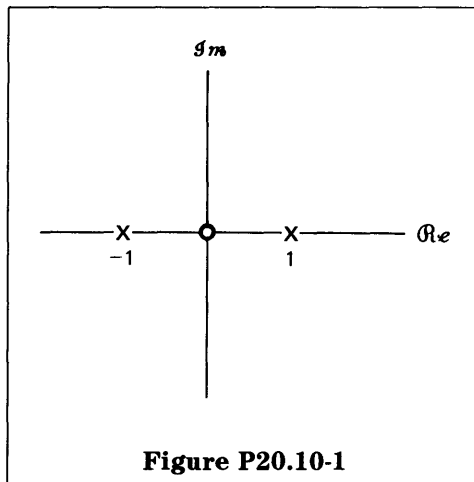
- (a) $e^{-at}u(t)$, $a < 0$
- (b) $-e^{at}u(-t)$, $a > 0$
- (c) $e^{at}u(t)$, $a > 0$
- (d) $e^{-a|t|}$, $a > 0$

- (e) $u(t)$
- (f) $\delta(t - t_0)$
- (g) $\sum_{k=0}^{\infty} a^k \delta(t - kT), \quad a > 0$
- (h) $\cos(\omega_0 t + b)u(t)$
- (i) $\sin(\omega_0 t + b)e^{-at}u(t), \quad a > 0$

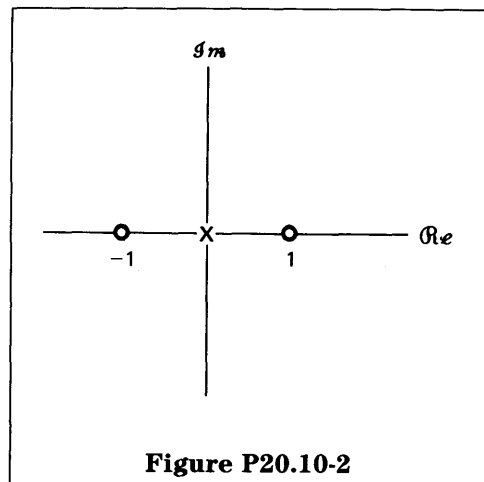
P20.10

- (a) If $x(t)$ is an even time function such that $x(t) = x(-t)$, show that this requires that $X(s) = X(-s)$.
- (b) If $x(t)$ is an odd time function such that $x(t) = -x(-t)$, show that $X(s) = -X(-s)$.
- (c) Determine which, if any, of the pole-zero plots in Figures P20.10-1 to P20.10-4 could correspond to an even time function. For those that could, indicate the required ROC.

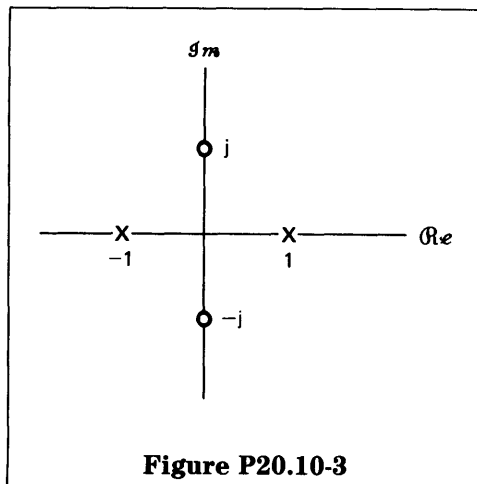
(i)



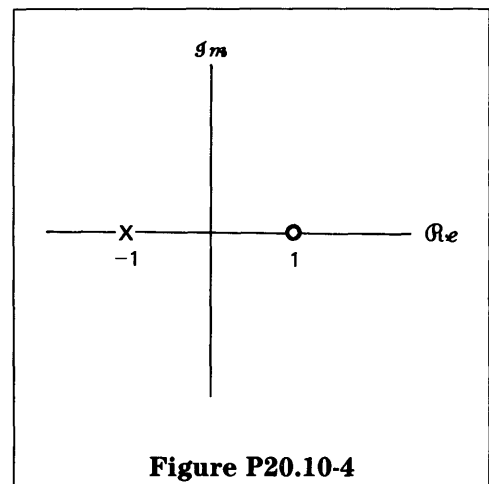
(ii)



(iii)



(iv)



- (d) Determine which, if any, of the pole-zero plots in part (c) could correspond to an odd time function. For those that could, indicate the required ROC.

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21 Continuous-Time Second-Order Systems

Recommended Problems

P21.1

Consider the following four impulse responses:

- (i) $h_1(t) = e^{-at}u(t), \quad a > 0,$
- (ii) $h_2(t) = -e^{-at}u(-t), \quad a > 0,$
- (iii) $h_3(t) = e^{-at}u(t), \quad a < 0,$
- (iv) $h_4(t) = -e^{-at}u(-t), \quad a < 0$

(a) Verify that

$$H_2(s) = \frac{1}{s + a},$$

with an ROC corresponding to $Re\{s\} < -a$.

(b) It is easily verified that in all four cases the system functions (i.e., the Laplace transforms of the impulse responses) have the same algebraic expression. However, the ROC is different. For each of the four systems, identify which of the s plane diagrams in Figures P21.1-1 to P21.1-6 correctly represents the associated Laplace transform, including the ROC. Also indicate which diagrams correspond to stable systems.

(A)

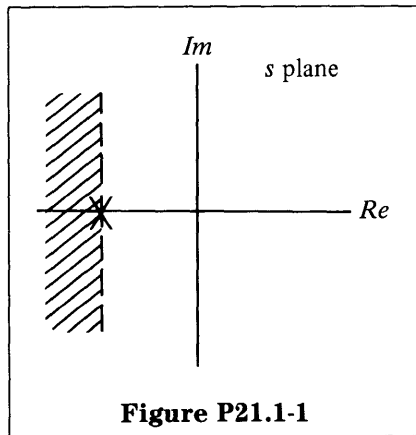


Figure P21.1-1

(B)

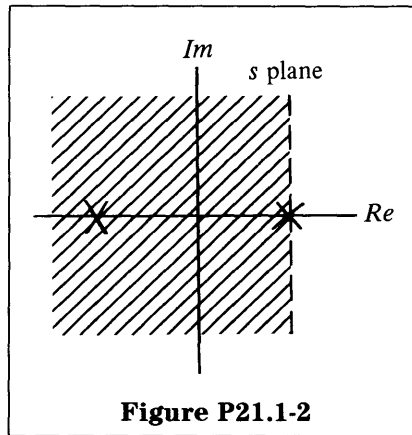


Figure P21.1-2

(C)

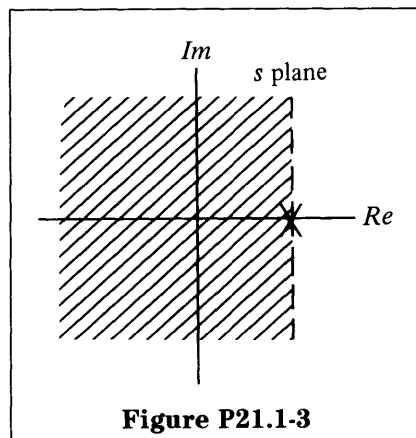


Figure P21.1-3

(D)

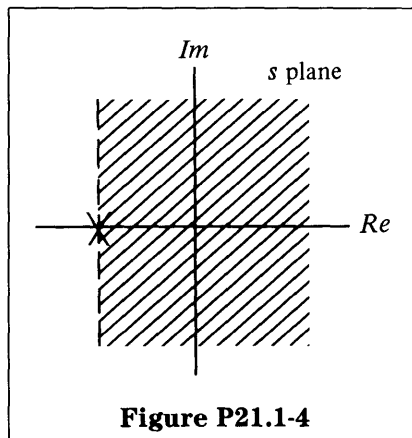
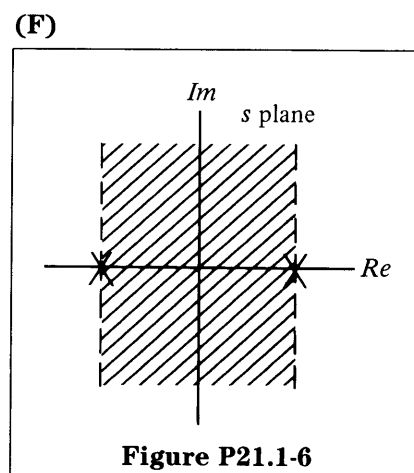
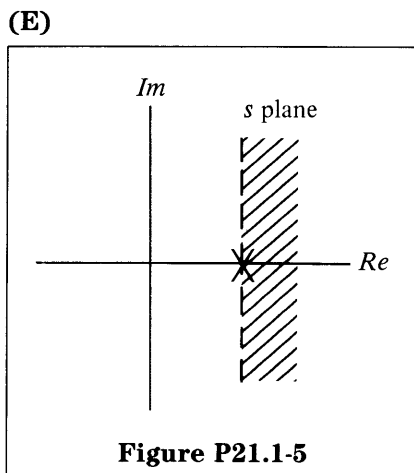


Figure P21.1-4



P21.2

Consider the LTI system with input $x(t) = e^{-t}u(t)$ and impulse response $h(t) = e^{-2t}u(t)$.

- (a) Determine $X(s)$ and $H(s)$.
- (b) Using the convolution property of the Laplace transform, determine $Y(s)$, the Laplace transform of the output, $y(t)$.
- (c) From your answer to part (b), find $y(t)$.

P21.3

As indicated in Section 9.5 of the text, many of the properties of the Laplace transform and their derivation are analogous to corresponding properties of the Fourier transform developed in Chapter 4 of the text. In this problem you are asked to outline the derivation for some of the Laplace transform properties in Section 9.5 of the text.

By paralleling the derivation for the corresponding property for the Fourier transform in Chapter 4 of the text, derive each of the following Laplace transform properties. Your derivation must include a consideration of the ROC.

- (a) Time-shifting property (Section 9.5.2)
- (b) Convolution property (Section 9.5.5)

P21.4

Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, and let $H(s)$ denote the Laplace transform of the impulse response $h(t)$ of the preceding system.

- (a) Determine $H(s)$. Sketch the pole-zero plot.

(b) Sketch the ROC for each of the following cases:

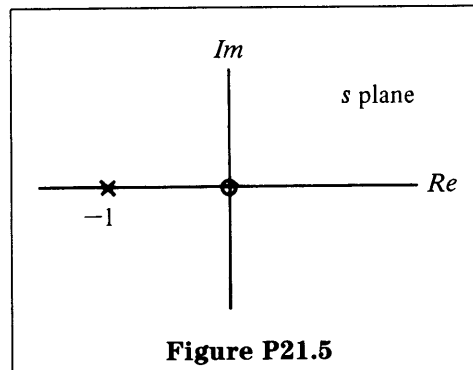
- (i) The system is stable.
- (ii) The system is causal.
- (iii) The system is neither stable nor causal.

(c) Determine $h(t)$ when the system is causal.

P21.5

Consider the following system function $H(s)$ and its corresponding pole-zero plot in Figure P21.5.

$$H(s) = \frac{s}{s + 1}$$



Using the graphical method discussed in the lecture, find $|H(0)|$, $\angle H(0)$, $|H(j1)|$, $\angle H(j1)$, $|H(j\infty)|$, and $\angle H(j\infty)$. Sketch the functions $|H(j\omega)|$ and $\angle H(j\omega)$.

P21.6

For the following system function, plot the pole-zero diagram and graphically determine the approximate locations of the peaks and valleys of $|H(j\omega)|$:

$$H(s) = \frac{(s + 5)s}{[s - (-0.1 + j2)][s - (-0.1 - j2)]}$$

Optional Problems

P21.7

(a) Draw the block diagram for the following second-order system in terms of integrators, coefficient multipliers, and adders.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(b) Sketch the pole-zero plot of $H(s)$ and plot $|H(j\omega)|$ under the following conditions.

- (i) ω_n is kept constant, but ζ is varied from close to 0 to close to 1.
- (ii) ζ is kept constant, but ω_n is varied from about 0 to infinity.

You don't have to be precise but show how the bandwidth and location of the peak changes for the two cases above.

P21.8

(a) Consider the following system function $H(s)$.

$$H(s) = \frac{s}{s^2 + s + 1} + \frac{1}{s^2 + 2s + 2}$$

Draw the block diagram for $H(s)$ implemented as

- (i) a parallel combination of second-order systems,
- (ii) a cascade combination of second-order systems.

(b) Is implementation (ii) unique?

P21.9

Complete the Laplace transform pairs given in Table P21.9. You may use the following transform of $e^{-\alpha t}u(t)$ as well as general properties of Laplace transforms to derive your answer. You may also use earlier entries in the table to help you derive later results.

$$e^{-\alpha t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + \alpha}, \quad \text{ROC: } \text{Re}\{s\} > \text{Re}\{\alpha\}$$

$x(t)$	$X(s)$	ROC
(a) $\sin(\omega_0 t)u(t)$		
(b) $e^{-2t}\sin(\omega_0 t)u(t)$		
(c) $te^{-2t}u(t)$		
(d)	$\frac{s + 1}{(s + 2)(s + 3)}$	$\text{Re}\{s\} > -2$
(e)	$\frac{1 - e^{-2s}}{s + 3}$	$\text{Re}\{s\} > -3$

Table P21.9

P.21.10

Figures P21.10-1, P21.10-2, and P21.10-3 contain impulse responses $h(t)$, frequency responses $H(j\omega)$, and pole-zero plots, respectively.

(a) For each $h(t)$, find the best matching $|H(j\omega)|$.

(b) For each $|H(j\omega)|$, find the best matching pole-zero plot of $H(s)$.

Consider entries with references to other plots, such as (3) and (5) or (b) and (h), as a pair.

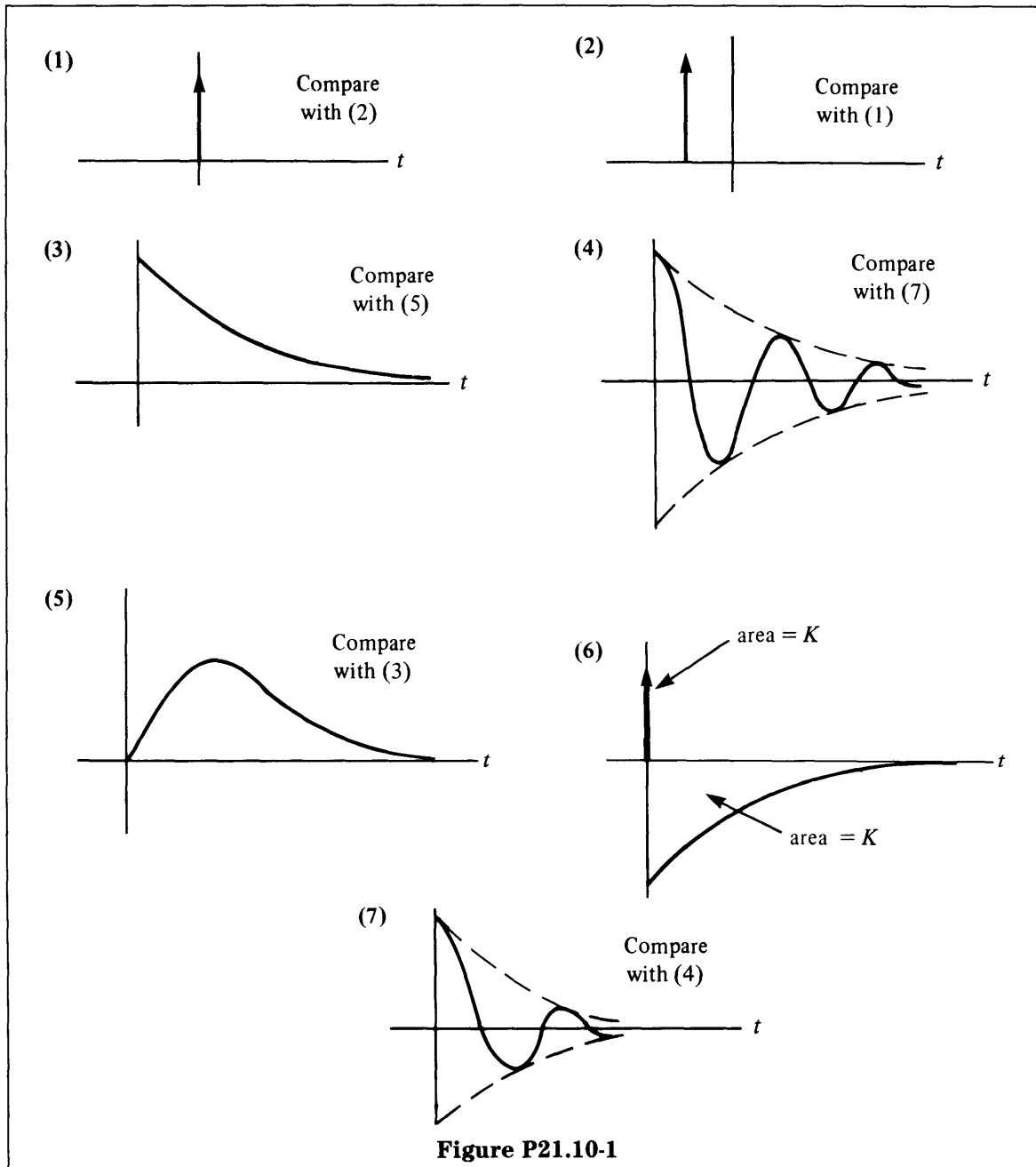
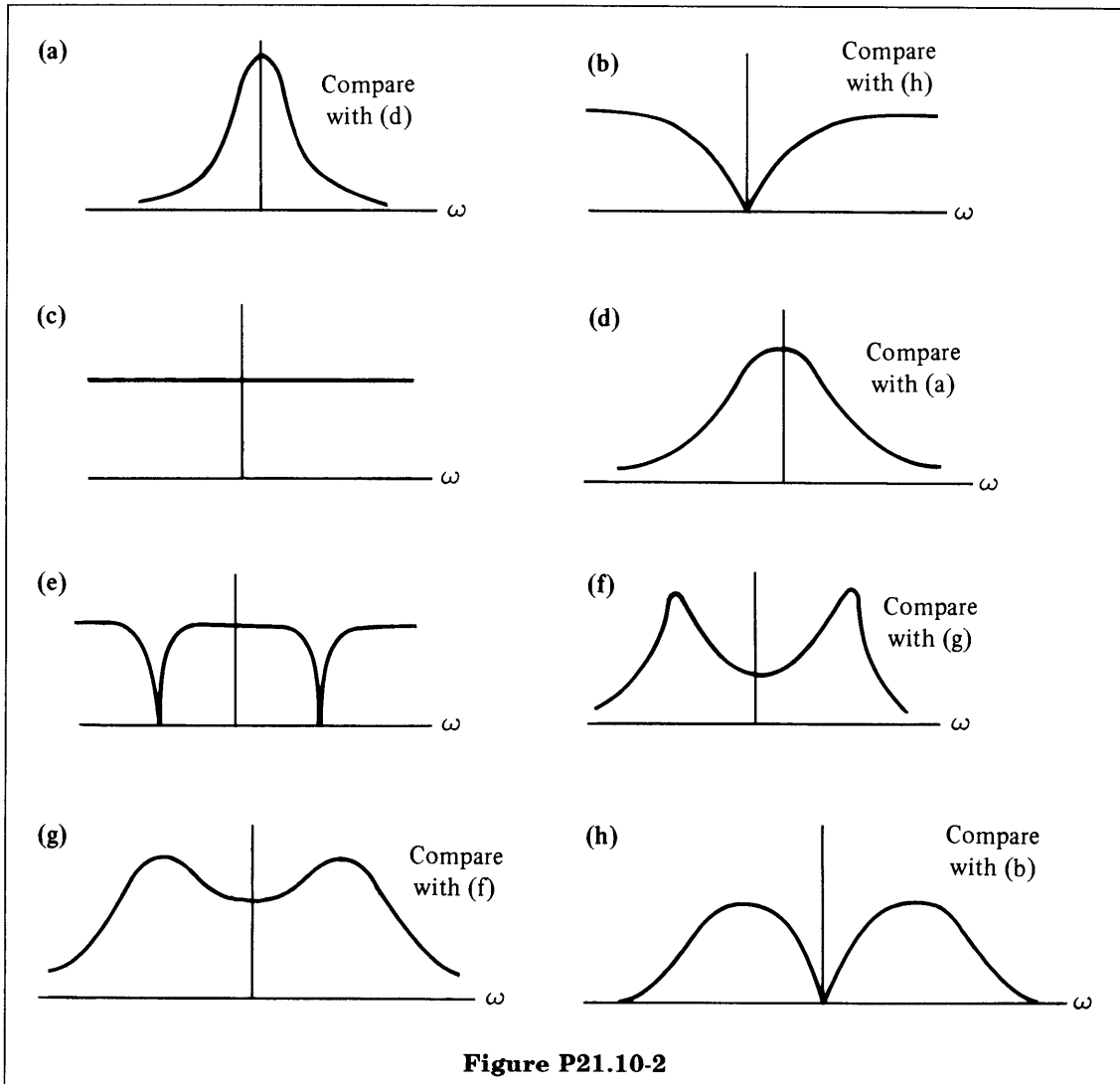


Figure P21.10-1



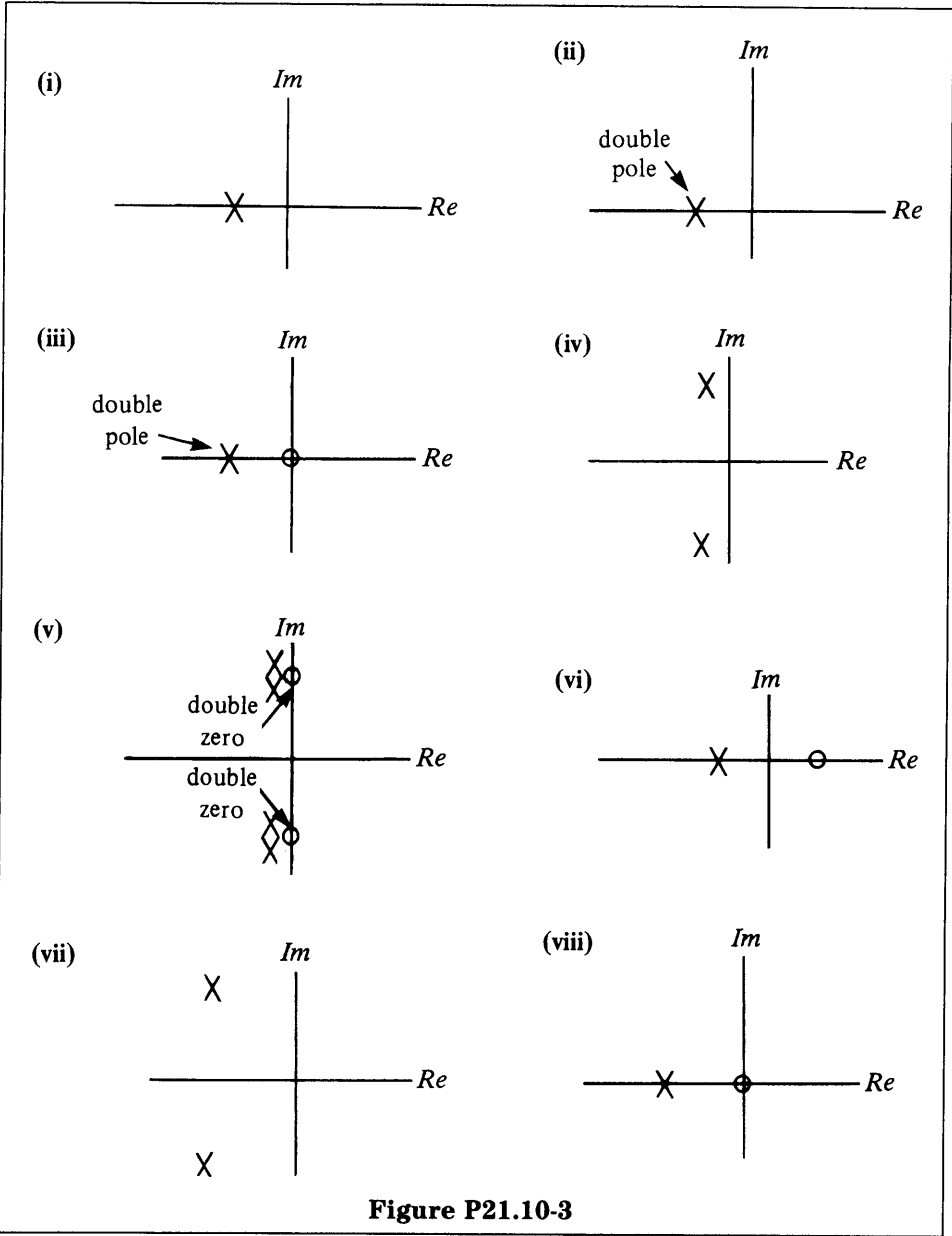


Figure P21.10-3

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20 The Laplace Transform

Solutions to Recommended Problems

S20.1

(a) The Fourier transform of the signal does not exist because of the presence of growing exponentials. In other words, $x(t)$ is not absolutely integrable.

(b) (i) For the case $\sigma = 1$, we have that

$$x(t)e^{-\sigma t} = 3e^t u(t) + 4e^{2t} u(t)$$

Although the growth rate has been slowed, the Fourier transform still does not converge.

(ii) For the case $\sigma = 2.5$, we have that

$$x(t)e^{-\sigma t} = 3e^{-0.5t} u(t) + 4e^{0.5t} u(t)$$

The first term has now been sufficiently weighted that it decays to 0 as t goes to infinity. However, since the second term is still growing exponentially, the Fourier transform does not converge.

(iii) For the case $\sigma = 3.5$, we have that

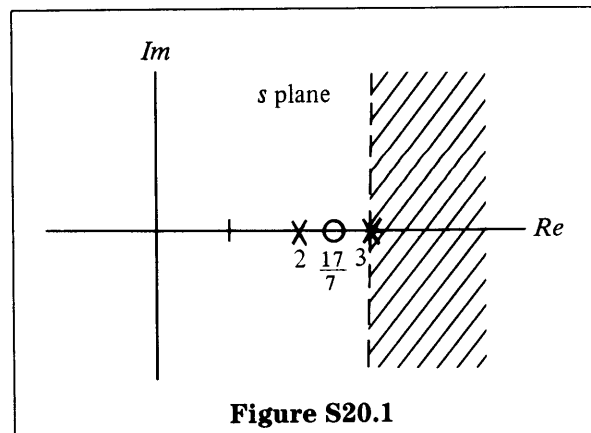
$$x(t)e^{-\sigma t} = 3e^{-1.5t} u(t) + 4e^{-0.5t} u(t)$$

Both terms do decay as t goes to infinity, and the Fourier transform converges. We note that for any value of $\sigma > 3.0$, the signal $x(t)e^{-\sigma t}$ decays exponentially, and the Fourier transform converges.

(c) The Laplace transform of $x(t)$ is

$$X(s) = \frac{3}{s-2} + \frac{4}{s-3} = \frac{7(s - \frac{17}{7})}{(s-2)(s-3)},$$

and its pole-zero plot and ROC are as shown in Figure S20.1.



Note that if $\sigma > 3.0$, $s = \sigma + j\omega$ is in the region of convergence because, as we showed in part (b)(iii), the Fourier transform converges.

S20.2

(a) $X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt = \frac{1}{s+a}$

The Laplace transform converges for $Re\{s\} + a > 0$, so

$$\sigma + a > 0, \quad \text{or} \quad \sigma > -a,$$

as shown in Figure S20.2-1.

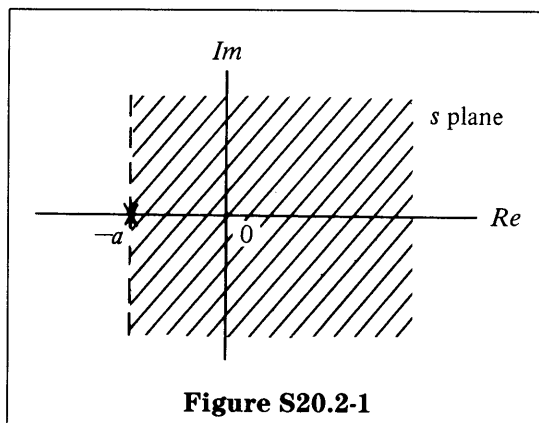


Figure S20.2-1

(b) $X(s) = \frac{1}{s+a}$

The Laplace transform converges for $Re\{s\} + a > 0$, so

$$\sigma + a > 0, \quad \text{or} \quad \sigma > -a,$$

as shown in Figure S20.2-2.

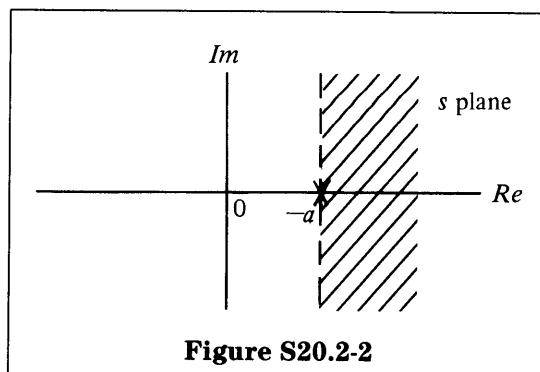
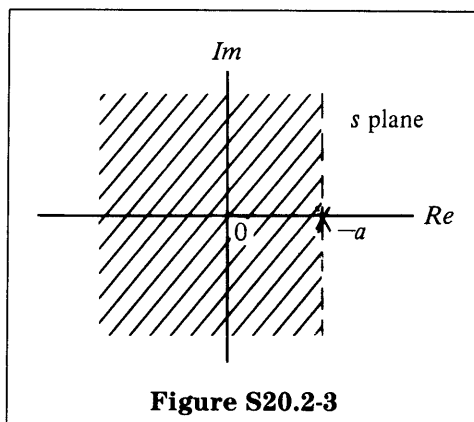


Figure S20.2-2

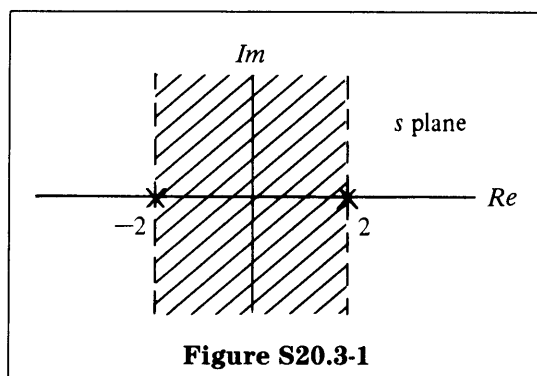
(c) $X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st} dt = \int_{-\infty}^0 -e^{-(s+a)t} dt = \frac{e^{-(s+a)t}}{s+a} \Big|_{-\infty}^0$
 $= \frac{1}{s+a}$

if $Re\{s\} + a < 0, \sigma + a < 0, \sigma < -a.$

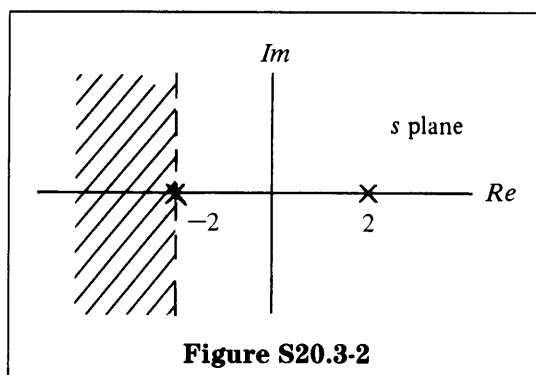


S20.3

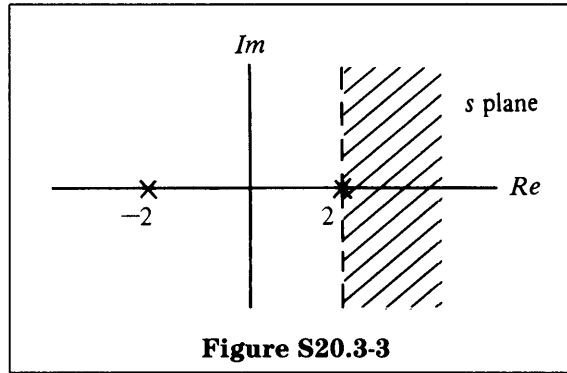
- (a) (i) Since the Fourier transform of $x(t)e^{-t}$ exists, $\sigma = 1$ must be in the ROC. Therefore only one possible ROC exists, shown in Figure S20.3-1.



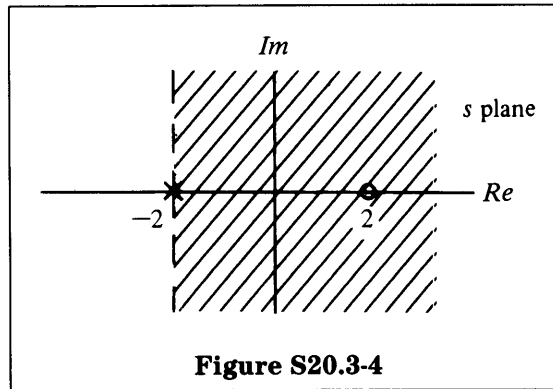
- (ii) We are specifying a left-sided signal. The corresponding ROC is as given in Figure S20.3-2.



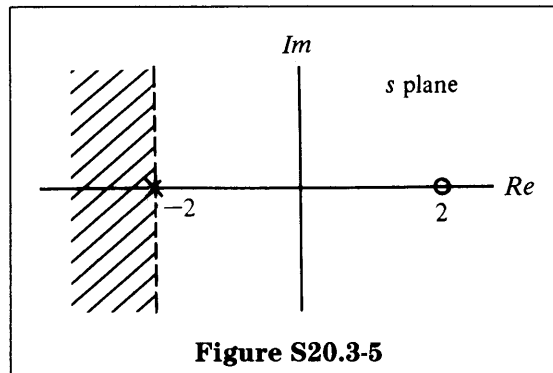
- (iii) We are specifying a right-sided signal. The corresponding ROC is as given in Figure S20.3-3.



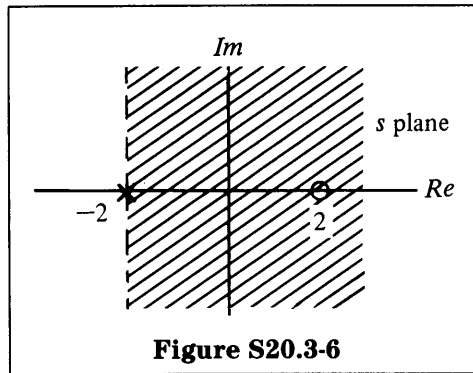
- (b) Since there are no poles present, the ROC exists everywhere in the s plane.
- (c) (i) $\sigma = 1$ must be in the ROC. Therefore, the only possible ROC is that shown in Figure S20.3-4.



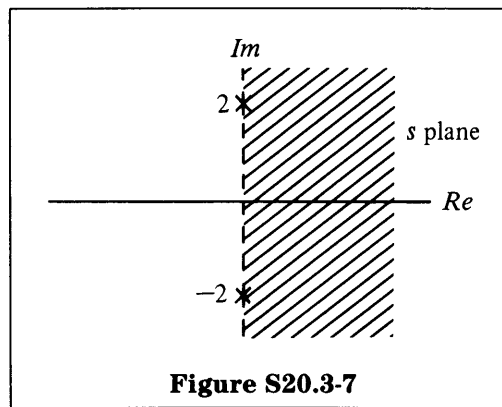
- (ii) We are specifying a left-sided signal. The corresponding ROC is as shown in Figure S20.3-5.



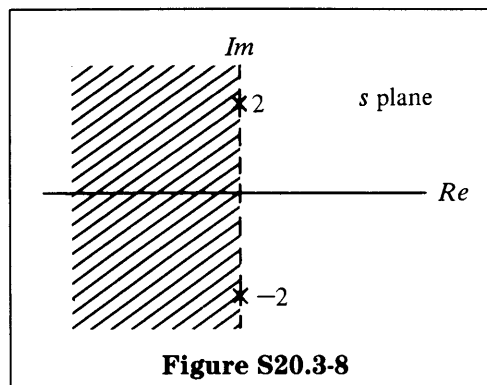
- (iii) We are specifying a right-sided signal. The corresponding ROC is as given in Figure S20.3-6.



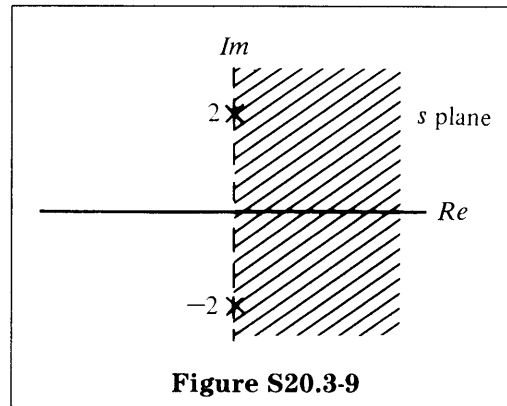
- (d) (i) $\sigma = 1$ must be in the ROC. Therefore, the only possible ROC is as shown in Figure S20.3-7.



- (ii) We are specifying a left-sided signal. The corresponding ROC is as shown in Figure S20.3-8.



(iii) We are specifying a right-sided signal. The corresponding ROC is as shown in Figure S20.3-9.



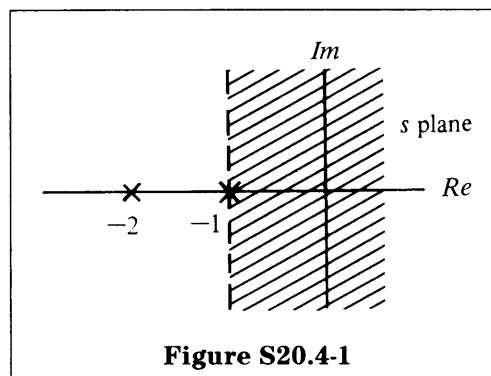
Constraint on ROC for Pole-Zero Pattern

$x(t)$	(a)	(b)	(c)	(d)
(i) Fourier transform of $x(t)e^{-t}$ converges	$-2 < \sigma < 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$
(ii) $x(t) = 0, t > 10$	$\sigma < -2$	Entire s plane	$\sigma < -2$	$\sigma < 0$
(iii) $x(t) = 0, t < 0$	$\sigma > 2$	Entire s plane	$\sigma > -2$	$\sigma > 0$

Table S20.3

S20.4

(a) For $x(t)$ right-sided, the ROC is to the right of the rightmost pole, as shown in Figure S20.4-1.



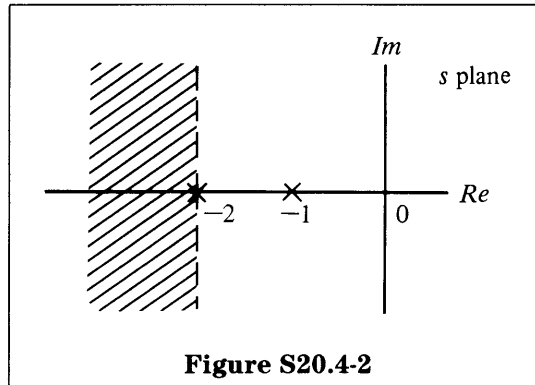
Using partial fractions,

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2},$$

so, by inspection,

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$

- (b) For $x(t)$ left-sided, the ROC is to the left of the leftmost pole, as shown in Figure S20.4-2.



Since

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2},$$

we conclude that

$$x(t) = -e^{-t}u(-t) - (-e^{-2t}u(-t))$$

- (c) For the two-sided assumption, we know that $x(t)$ will have the form

$$f_1(t)u(-t) + f_2(t)u(t)$$

We know the inverse Laplace transforms of the following:

$$\frac{1}{s+1} = \begin{cases} e^{-t}u(t), & \text{assuming right-sided,} \\ -e^{-t}u(-t), & \text{assuming left-sided,} \end{cases}$$

$$\frac{1}{s+2} = \begin{cases} e^{-2t}u(t), & \text{assuming right-sided,} \\ -e^{-2t}u(-t), & \text{assuming left-sided} \end{cases}$$

Which of the combinations should we choose for the two-sided case? Suppose we choose

$$x(t) = e^{-t}u(t) + (-e^{-2t}u(-t))$$

We ask, For what values of σ does $x(t)e^{-\sigma t}$ have a Fourier transform? And we see that there are no values. That is, suppose we choose $\sigma > -1$, so that the first term has a Fourier transform. For $\sigma > -1$, $e^{-2t}e^{-\sigma t}$ is a growing exponential as t goes to negative infinity, so the second term does not have a Fourier transform. If we increase σ , the first term decays faster as t goes to infinity, but

the second term grows faster as t goes to negative infinity. Therefore, choosing $\sigma > -1$ will not yield a Fourier transform of $x(t)e^{-\sigma t}$. If we choose $\sigma \leq -1$, we note that the first term will not have a Fourier transform. Therefore, we conclude that our choice of the two-sided sequence was wrong. It corresponds to the *invalid* region of convergence shown in Figure S20.4-3.

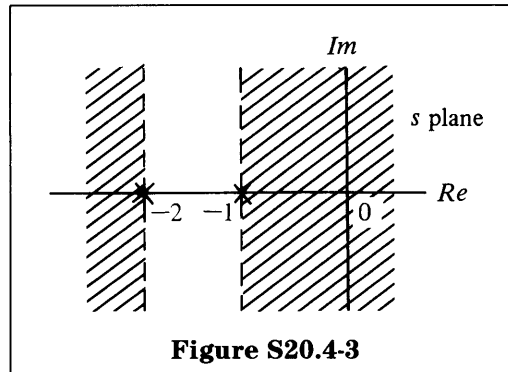


Figure S20.4-3

If we choose the other possibility,

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t),$$

we see that the valid region of convergence is as given in Figure S20.4-4.

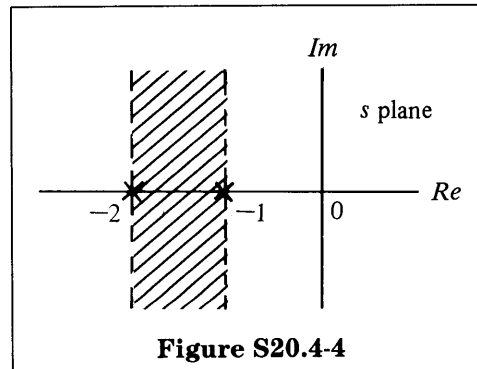


Figure S20.4-4

S20.5

There are two ways to solve this problem.

Method 1

This method is based on recognizing that the system input is a superposition of eigenfunctions. Specifically, the eigenfunction property follows from the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau) d\tau$$

Now suppose $x(t) = e^{at}$. Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{a(t-\tau)} d\tau = e^{at} \int_{-\infty}^{\infty} h(\tau)e^{-a\tau} d\tau$$

Now we recognize that

$$\int_{-\infty}^{\infty} h(\tau)e^{-a\tau} d\tau = H(s)\Big|_{s=a},$$

so that if $x(t) = e^{at}$, then

$$y(t) = \left[H(s)\Big|_{s=a} \right] e^{at},$$

i.e., e^{at} is an eigenfunction of the system.

Using linearity and superposition, we recognize that if

$$x(t) = e^{-t/2} + 2e^{-t/3},$$

then

$$y(t) = e^{-t/2}H(s)\Big|_{s=-1/2} + 2e^{-t/3}H(s)\Big|_{s=-1/3}$$

so that

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \quad \text{for all } t.$$

Method 2

We consider the solution of this problem as the superposition of the response to two signals $x_1(t)$, $x_2(t)$, where $x_1(t)$ is the noncausal part of $x(t)$ and $x_2(t)$ is the causal part of $x(t)$. That is,

$$\begin{aligned} x_1(t) &= e^{-t/2}u(-t) + 2e^{-t/3}u(-t), \\ x_2(t) &= e^{-t/2}u(t) + 2e^{-t/3}u(t) \end{aligned}$$

This allows us to use Laplace transforms, but we must be careful about the ROCs.

Now consider $\mathcal{L}\{x_1(t)\}$, where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform:

$$\mathcal{L}\{x_1(t)\} = X_1(s) = -\frac{1}{s + \frac{1}{2}} - \frac{2}{s + \frac{1}{3}}, \quad \text{Re}\{s\} < -\frac{1}{2}$$

Now since the response to $x_1(t)$ is

$$y_1(t) = \mathcal{L}^{-1}\{X_1(s)H(s)\},$$

then

$$\begin{aligned} Y_1(s) &= -\frac{1}{(s+1)(s+\frac{1}{2})} - \frac{2}{(s+\frac{1}{3})(s+1)}, \quad -1 < \text{Re}\{s\} < -\frac{1}{2}, \\ &= \frac{2}{s+1} + \frac{-2}{s+\frac{1}{2}} + \frac{-3}{s+\frac{1}{3}} + \frac{3}{s+1}, \\ &= \frac{5}{s+1} - \frac{2}{s+\frac{1}{2}} - \frac{3}{s+\frac{1}{3}}, \end{aligned}$$

so

$$y_1(t) = 5e^{-t}u(t) + 2e^{-t/2}u(-t) + 3e^{-t/3}u(-t)$$

The pole-zero plot and associated ROC for $Y_1(s)$ is shown in Figure S20.5-1.

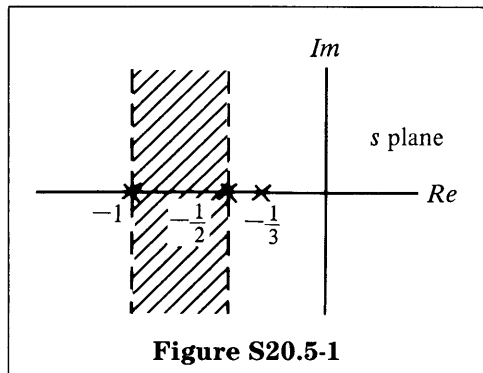


Figure S20.5-1

Next consider the response $y_2(t)$ to $x_2(t)$:

$$x_2(t) = e^{-t/2}u(t) + 2e^{-t/3}u(t),$$

$$X_2(s) = \frac{1}{s + \frac{1}{2}} + \frac{2}{s + \frac{1}{3}}, \quad \text{Re}\{s\} > -\frac{1}{3},$$

$$Y_2(s) = X_2(s)H(s) = \frac{1}{(s + \frac{1}{2})(s + 1)} + \frac{2}{(s + \frac{1}{3})(s + 1)},$$

$$Y_2(s) = \frac{2}{s + \frac{1}{2}} + \frac{-2}{s + 1} + \frac{3}{s + \frac{1}{3}} + \frac{-3}{s + 1},$$

so

$$y_2(t) = -5e^{-t}u(t) + 2e^{-t/2}u(t) + 3e^{-t/3}u(t)$$

The pole-zero plot and associated ROC for $Y_2(s)$ is shown in Figure S20.5-2.

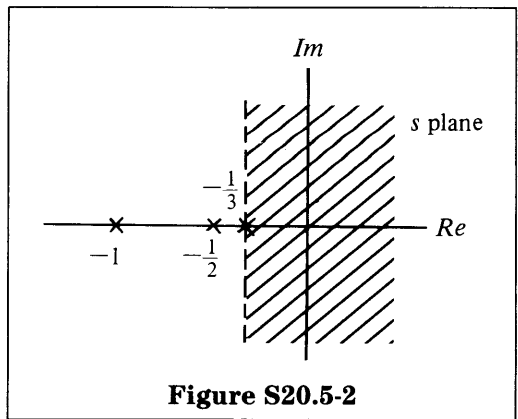


Figure S20.5-2

Since $y(t) = y_1(t) + y_2(t)$, then

$$y(t) = 2e^{-t/2} + 3e^{-t/3} \quad \text{for all } t$$

S20.6

(a) Since

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

and $s = \sigma + j\omega$, then

$$X(s) \Big|_{s=\sigma+j\omega} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\omega t} dt$$

We see that the Laplace transform is the Fourier transform of $x(t)e^{-\sigma t}$ from the definition of the Fourier analysis formula.

$$(b) x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(s) \Big|_{s=\sigma+j\omega} \right] e^{j\omega t} d\omega$$

This result is the inverse Fourier transform, or synthesis equation. So

$$\begin{aligned} x(t) &= e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(s) \Big|_{s=\sigma+j\omega} \right] e^{j\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[X(s) \Big|_{s=\sigma+j\omega} \right] e^{(\sigma+j\omega)t} d\omega, \end{aligned}$$

and letting $s = \sigma + j\omega$ yields $ds = j d\omega$:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

Solutions to Optional Problems

S20.7

$$(a) X(s) = \frac{1}{s+1}, \quad \operatorname{Re}\{s\} > -1$$

Therefore, $x(t)$ is right-sided, and specifically

$$x(t) = e^{-t}u(t)$$

$$(b) X(s) = \frac{1}{s+1}, \quad \operatorname{Re}\{s\} < -1$$

Therefore,

$$x(t) = -e^{-t}u(-t)$$

$$(c) X(s) = \frac{s}{s^2+4}, \quad \operatorname{Re}\{s\} > 0$$

Since

$$\begin{aligned} e^{j\omega_0 t} &\xleftrightarrow{\mathcal{L}} \frac{1}{s-j\omega_0} \\ e^{-j\omega_0 t} &\xleftrightarrow{\mathcal{L}} \frac{1}{s+j\omega_0} \end{aligned}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right\} = \frac{1}{2}\left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right)$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2 + \omega_0^2}$$

so

$$\text{if } X(s) = \frac{s}{s^2+4}, \quad \text{then } x(t) = \cos(2t)u(t)$$

$$(d) X(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}, \text{ so}$$

$$x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

$$(e) X(s) = \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3},$$

$$x(t) = e^{-2t}u(-t) - 2e^{-3t}u(-t)$$

$$(f) X(s) = \frac{s^2 - s + 1}{s^2(s-1)}, \quad 0 < \operatorname{Re}\{s\} < 1$$

$$= \frac{1}{s-1} - \frac{1}{s(s-1)} + \frac{1}{s^2(s-1)}$$

$$= \frac{1}{s-1} + \frac{1}{s} + \frac{-1}{s-1} + \frac{-1}{s^2} + \frac{-1}{s} + \frac{1}{s-1}$$

$$= \frac{1}{s-1} - \frac{1}{s^2},$$

$$x(t) = -e^t u(-t) - tu(t)$$

$$(g) X(s) = \frac{s^2 - s + 1}{(s+1)^2}, \quad -1 < \operatorname{Re}\{s\}$$

$$= \frac{(s+1)^2 - 3s}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2}$$

$$= 1 - \frac{3(s+1)}{(s+1)^2} + \frac{3}{(s+1)^2},$$

$$x(t) = \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t)$$

$$(h) X(s) = \frac{s+1}{(s+1)^2 + 4}$$

Consider

$$Y(s) = \frac{s}{s^2 + 4} \rightarrow y(t) = \cos(2t)u(t) \quad \text{from part (c)}$$

Now

$$f(t)e^{-at} \xleftrightarrow{\mathcal{L}} F(s+a),$$

so

$$x(t) = e^{-t} \cos(2t)u(t)$$

S20.8

The Laplace transform of an impulse $a\delta(t)$ is a . Therefore, if we expand a rational Laplace transform by dividing the denominator into the numerator, we require a *constant* term in the expansion. This will occur only if the numerator has order greater than or equal to the order of the denominator. Therefore, a necessary condition on the number of zeros is that it be greater than or equal to the number of poles.

This is only a necessary and not a sufficient condition as it is possible to construct a rational Laplace transform that has a numerator order greater than the

denominator order and that does not yield a constant term in the expansion. For example,

$$X(s) = \frac{s^2 + 1}{s} = s + \frac{1}{s},$$

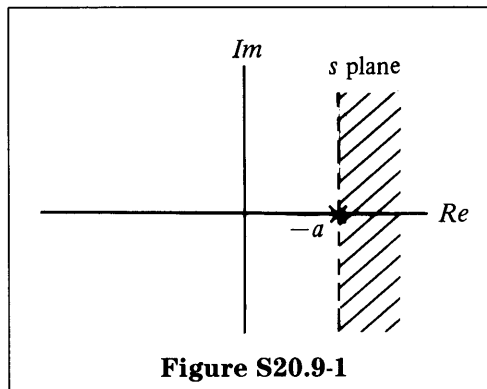
which does not have a constant term. Therefore a *necessary* condition is that the number of zeros equal or exceed the number of poles.

S20.9

(a) $x(t) = e^{-at}u(t), \quad a < 0,$

$$X(s) = \frac{1}{s + a},$$

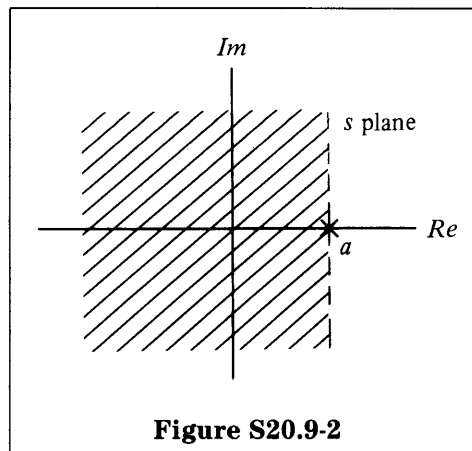
and the ROC is shown in Figure S20.9-1.



(b) $x(t) = -e^{at}u(-t), \quad a > 0,$

$$X(s) = \frac{1}{s - a},$$

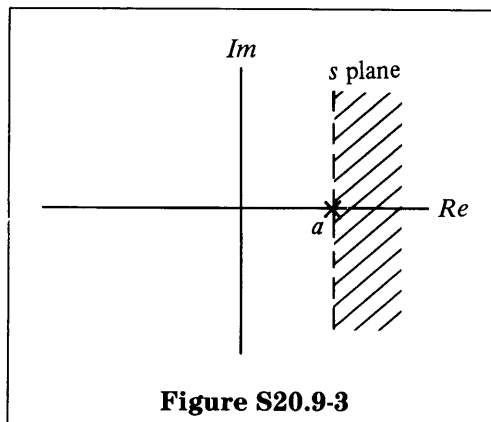
and the ROC is shown in Figure S20.9-2.



(c) $x(t) = e^{at}u(t), \quad a > 0,$

$$X(s) = \frac{1}{s - a},$$

and the ROC is shown in Figure S20.9-3.

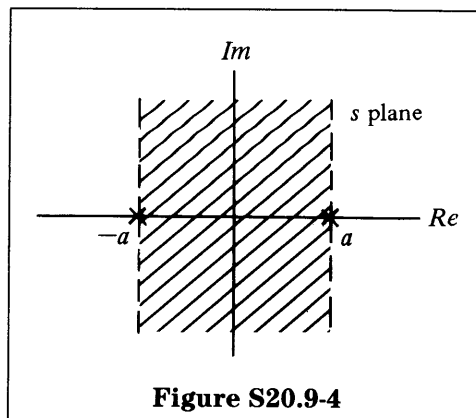


(d) $x(t) = e^{-a|t|}, \quad a > 0,$

$$= e^{-at}u(t) + e^{at}u(-t),$$

$$X(s) = \frac{1}{s + a} + \frac{-1}{s - a},$$

and the ROC is shown in Figure S20.9-4.



(e) $x(t) = u(t),$

$$X(s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s},$$

and the ROC is shown in Figure S20.9-5.

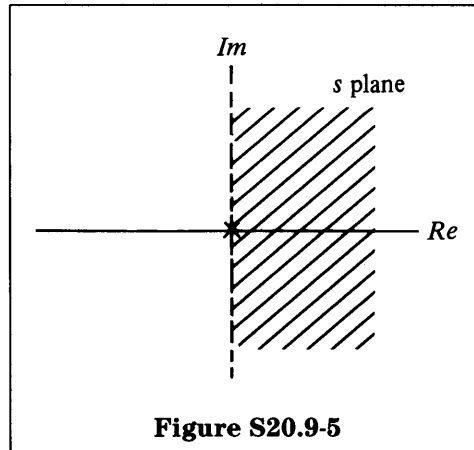


Figure S20.9-5

(f) $x(t) = \delta(t - t_0),$

$$X(s) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-st} dt = e^{-st_0},$$

and the ROC is the entire s plane.

(g) $x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT),$

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} a^k \int_{-\infty}^{\infty} \delta(t - kT)e^{-st} dt \\ &= \sum_{k=0}^{\infty} a^k e^{-skT} = \frac{1}{1 - ae^{-sT}}, \end{aligned}$$

with ROC such that $|ae^{-sT}| < 1$. Now

$$a^2 e^{-2sT} < 1 \rightarrow 2 \log a - 2sT < 0 \rightarrow s > \frac{1}{T} \log a$$

(h) $x(t) = \cos(\omega_0 t + b)u(t)$

Using the identity

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

we have that

$$x(t) = \cos b \cos(\omega_0 t)u(t) - \sin b \sin(\omega_0 t)u(t)$$

Using linearity and the transform pairs

$$\cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_0^2},$$

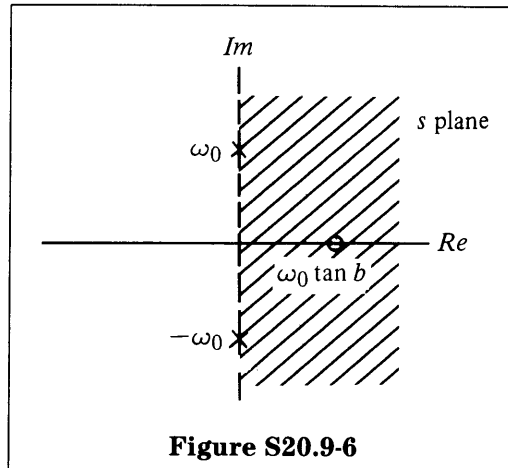
$$\sin(\omega_0 t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2 + \omega_0^2},$$

we have

$$X(s) = \cos b \frac{s}{s^2 + \omega_0^2} - \sin b \frac{\omega_0}{s^2 + \omega_0^2},$$

$$X(s) = \cos b \frac{[s - (\tan b)\omega_0]}{s^2 + \omega_0^2},$$

and the ROC is shown in Figure S20.9-6.



(i) Consider

$$\begin{aligned} x_1(t) &= \sin(\omega_0 t + b)u(t) \\ &= (\sin \omega_0 t \cos b + \cos \omega_0 t \sin b)u(t) \end{aligned}$$

Using linearity and the preceding $\sin \omega_0 t$, $\cos \omega_0 t$ pairs, we have

$$X_1(s) = \cos b \frac{\omega_0}{s^2 + \omega_0^2} + \sin b \frac{s}{s^2 + \omega_0^2},$$

$$X_1(s) = \sin b \frac{[s + (\cot b)\omega_0]}{s^2 + \omega_0^2}$$

Using the property that

$$f(t)e^{-at} \xleftrightarrow{\mathcal{L}} F(s + a),$$

we have

$$X(s) = \sin b \frac{[s + a + (\cot b)\omega_0]}{(s + a)^2 + \omega_0^2},$$

with the ROC as given in Figure S20.9-7.

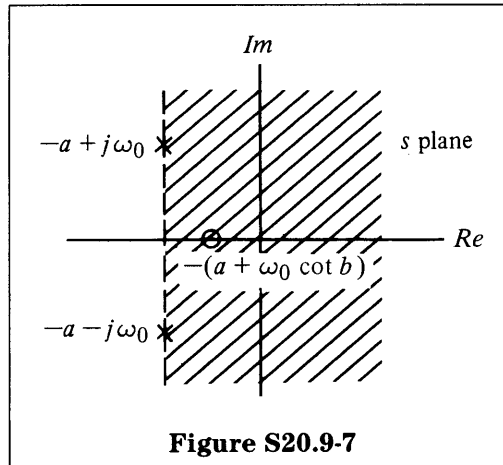


Figure S20.9-7

S20.10

(a) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$

Consider

$$X_1(s) = \int_{-\infty}^{\infty} x(-t)e^{-st} dt$$

Letting $t = -t'$, we have

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} x(t')e^{st'} dt' \\ &= X(-s), \end{aligned}$$

but $X_1(s) = X(s)$ since $x(t) = x(-t)$. Therefore, $X(s) = X(-s)$.

(b) $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$

Consider

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{\infty} -x(-t)e^{-st} dt, \\ X_1(s) &= \int_{-\infty}^{\infty} -x(t')e^{st'} dt' \\ &= -X(s), \end{aligned}$$

but $X_1(s) = X(s)$ since $x(t) = -x(-t)$. Therefore, $X(s) = -X(-s)$.

(c) We note that if $X(s)$ has poles, then it must be two-sided in order for $x(t) = x(-t)$.

(i) $X(s) = \frac{Ks}{(s+1)(s-1)}$,

$$X(-s) = \frac{-Ks}{(-s+1)(-s-1)} = \frac{-Ks}{(s-1)(s+1)} \neq X(s),$$

so $x(t) \neq x(-t)$.

$$(ii) \quad X(s) = \frac{K(s+1)(s-1)}{s},$$

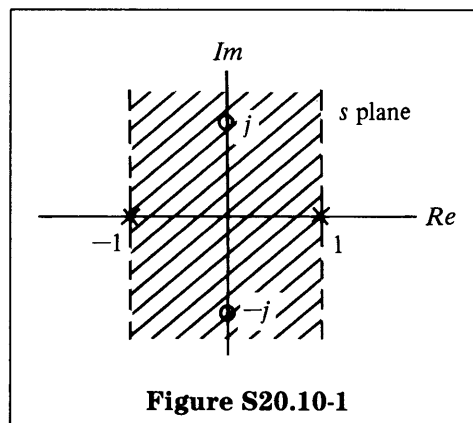
$$X(-s) = \frac{K(-s+1)(-s-1)}{-s} \neq X(s)$$

Also, this pole pattern cannot have a two-sided ROC.

$$(iii) \quad X(s) = \frac{K(s+j)(s-j)}{(s+1)(s-1)},$$

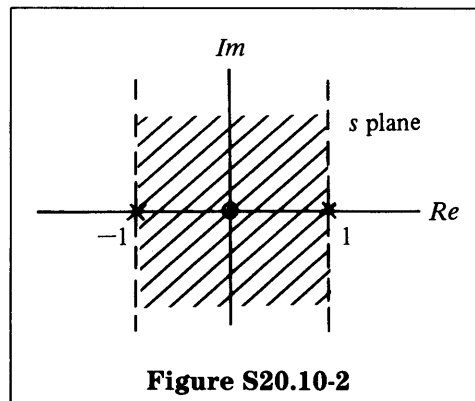
$$X(-s) = \frac{K(-s+j)(-s-j)}{(-s+1)(-s-1)} = \frac{K(s-j)(s+j)}{(s-1)(s+1)} = X(s),$$

so this can correspond to an even $x(t)$. The corresponding ROC must be two-sided, as shown in Figure S20.10-1.



(iv) This does not have any possible two-sided ROCs.

(d) We see from the results in part (c)(i) that $X(s) = -X(-s)$, so the result in part (c)(i) corresponds to an odd $x(t)$ with an ROC as given in Figure S20.10-2.



Parts (c)(ii) and (c)(iv) do not have any possible two-sided ROCs. Part (c)(iii) is even, as previously shown, and therefore cannot be odd.

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21 Continuous-Time Second-Order Systems

Solutions to Recommended Problems

S21.1

$$(a) H_2(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st} dt = - \int_{-\infty}^0 e^{-(a+s)t} dt$$

Following our previous arguments, we can integrate only if the function dies out as t goes to minus infinity. e^{-xt} will die out as t goes to minus infinity only if $Re\{x\}$ is negative. Thus we need $Re\{a + s\} < 0$ or $Re\{s\} < -a$. For s in this range,

$$H_2(s) = \frac{1}{a + s}$$

- (b) (i) $h_1(t)$ has a pole at $-a$ and no zeros. Furthermore, since $a > 0$, the pole must be in the left half-plane. Since $h_1(t)$ is causal, the ROC must be to the right of the rightmost pole, as given in D, Figure P21.1-4.
- (ii) $h_2(t)$ is left-sided; hence the ROC is to the left of the leftmost pole. Since a is positive, the pole is in the left half-plane, as shown in A, Figure P21.1-1.
- (iii) $h_3(t)$ is right-sided and has a pole in the right half-plane, as given in E, Figure P21.1-5.
- (iv) $h_4(t)$ is left-sided and has a pole in the right half-plane, as shown in C, Figure P21.1-3.

For a signal to be stable, its ROC must include the $j\omega$ axis. Thus, C, D, and F qualify. B is an ROC that includes a pole, which is impossible; hence it corresponds to no signal.

S21.2

- (a) By definition,

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st} dt \\ &= \int_0^{\infty} e^{-t}e^{-st} dt \end{aligned}$$

We limit the integral to $(0, \infty)$ because of $u(t)$, so

$$X(s) = \int_0^{\infty} e^{-(1+s)t} dt = \frac{-1}{1+s} e^{-(1+s)t} \Big|_0^{\infty}$$

If the real part of $(1 + s)$ is positive, i.e., $Re\{s\} > -1$, then

$$\lim_{t \rightarrow \infty} e^{-(1+s)t} = 0$$

Thus

$$X(s) = \frac{0(-1)}{1+s} - \frac{1(-1)}{1+s} = \frac{1}{1+s}, \quad \text{Re}\{s\} > -1$$

The condition on $\text{Re}\{s\}$ is the ROC and basically indicates the region for which $1/(1+s)$ is equal to the integral defined originally. Similarly,

$$H(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st} dt = \int_0^{\infty} e^{-(2+s)t} dt = \frac{1}{s+2}, \quad \text{Re}\{s\} > -2$$

(b) By the convolution property of the Laplace transform, $Y(s) = H(s)X(s)$ in a manner similar to the property of the Fourier transform. Thus,

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \text{Re}\{s\} > -1,$$

where the ROC is the intersection of individual ROCs.

(c) Here we can use partial fractions:

$$\begin{aligned} \frac{1}{(s+1)(s+2)} &= \frac{A}{s+1} + \frac{B}{s+2}, \\ A &= Y(s)(s+1) \Big|_{s=-1} = 1, \\ B &= Y(s)(s+2) \Big|_{s=-2} = -1 \end{aligned}$$

Thus,

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}, \quad \text{Re}\{s\} > -1$$

Recognizing the individual Laplace transforms, we have

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

S21.3

(a) The property to be derived is

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s),$$

with the same ROC as $X(s)$.

Let $y(t) = x(t - t_0)$. Then

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st} dt = \int_{-\infty}^{\infty} x(t - t_0)e^{-st} dt$$

Let $p = t - t_0$. Then $t = p + t_0$ and $dp = dt$. Substituting

$$Y(s) = \int_{-\infty}^{\infty} x(p)e^{-s(p+t_0)} dp$$

Since we are not integrating over s or t_0 , we can remove the e^{-st_0} term,

$$Y(s) = e^{-st_0} \int_{-\infty}^{\infty} x(p)e^{-sp} dp = e^{-st_0}X(s)$$

Note that wherever $X(s)$ converges, the integral defining $Y(s)$ also converges; thus the ROC of $X(s)$ is the same as the ROC of $Y(s)$.

(b) Now we study one of the most useful properties of the Laplace transform.

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s),$$

with the ROC containing $R_1 \cap R_2$. Let

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t - \tau) d\tau$$

Then

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x_1(\tau)x_2(t - \tau)e^{-st} d\tau dt \\ &= \int_{\tau=-\infty}^{\infty} x_1(\tau) \int_{t=-\infty}^{\infty} x_2(t - \tau)e^{-st} dt d\tau \end{aligned}$$

Suppose we are in a region of the s plane where $X_2(s)$ converges. Then using the property shown in part (a), we have

$$\int_{-\infty}^{\infty} x_2(t - \tau)e^{-st} dt = e^{-s\tau}X_2(s)$$

Substituting, we have

$$Y(s) = \int_{\tau=-\infty}^{\infty} x_1(\tau)e^{-s\tau}X_2(s) d\tau = X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau} d\tau$$

We can associate this last integral with $X_1(s)$ if we are also in the ROC of $x_1(t)$. Thus $Y(s) = X_2(s)X_1(s)$ for s inside at least the region $R_1 \cap R_2$. It could happen that the ROC is *larger*, but it *must* contain $R_1 \cap R_2$.

S21.4

(a) From the properties of the Laplace transform,

$$Y(s) = X(s)H(s)$$

A second relation occurs due to the differential equation. Since

$$\frac{d^k x(t)}{dt^k} \xleftrightarrow{\mathcal{L}} s^k X(s)$$

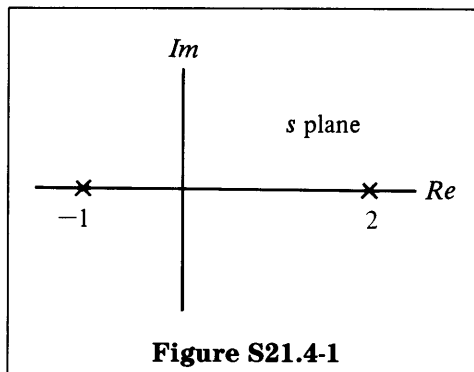
and using the linearity property of the Laplace transform, we can take the Laplace transform of both sides of the differential equation, yielding

$$s^2 Y(s) - sY(s) - 2Y(s) = X(s).$$

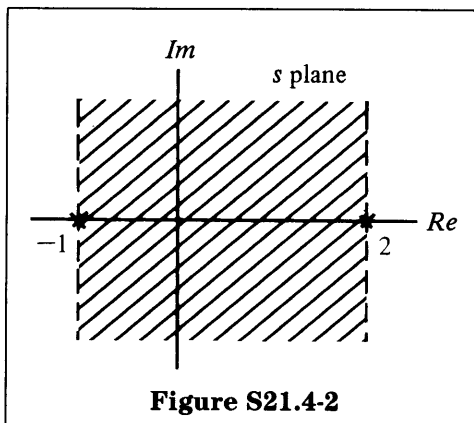
Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s - 2)(s + 1)}$$

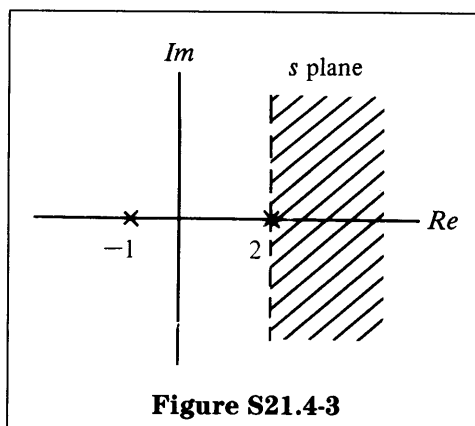
The pole-zero plot is shown in Figure S21.4-1.



- (b) (i) For a stable system, the ROC must include the $j\omega$ axis. Thus the ROC must be as drawn in Figure S21.4-2.



- (ii) For a causal system, the ROC must be to the right of the rightmost pole, as shown in Figure S21.4-3.



- (iii) For a system that is not causal or stable, we are left with an ROC that is to the left of $s = -1$, as shown in Figure S21.4-4.

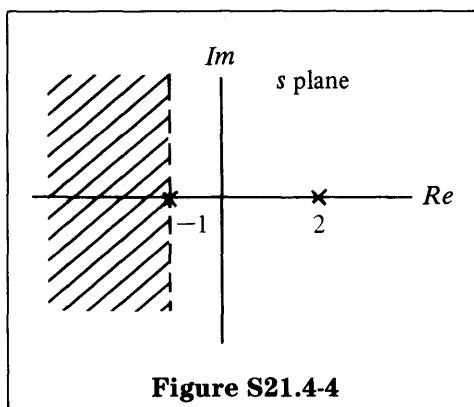


Figure S21.4-4

- (c) To take the inverse Laplace transform, we use the partial fraction expansion:

$$H(s) = \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2}$$

We now take the inverse Laplace transform of each term in the partial fraction expansion. Since the system is causal, we choose right-sided signals in both cases. Thus,

$$h(t) = -\frac{1}{3}e^{-t}u(t) + \frac{1}{3}e^{+2t}u(t)$$

S21.5

$\omega = 0$: Since there is a zero at $s = 0$, $|H(j0)| = 0$. You may think that the phase is also zero, but if we move slightly on the $j\omega$ axis, $\angle H(j\omega)$ becomes

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\omega = 1$: The distance to $s = 0$ is 1 and the distance to $s = -1$ is $\sqrt{2}$. Thus

$$|H(j1)| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The phase is

$$(\text{Angle to } s = 0) - (\text{Angle to } s = -1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} = \angle H(j1)$$

$\omega = \infty$: The distance to $s = 0$ and $s = -1$ is infinite; however, the ratio tends to 1 as ω increases. Thus, $|H(j\infty)| = 1$. The phase is given by

$$\frac{\pi}{2} - \frac{\pi}{2} = 0$$

The magnitude and phase of $H(j\omega)$ are given in Figure S21.5.

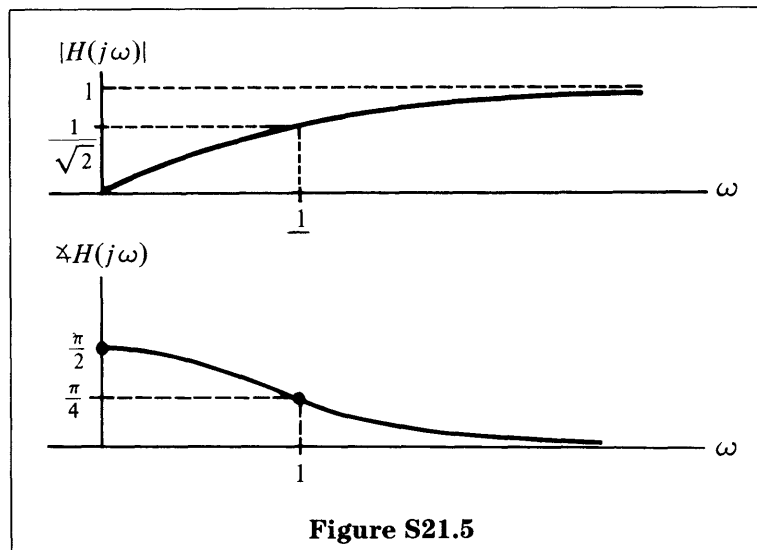


Figure S21.5

S21.6

The pole-zero plot is shown in Figure S21.6.

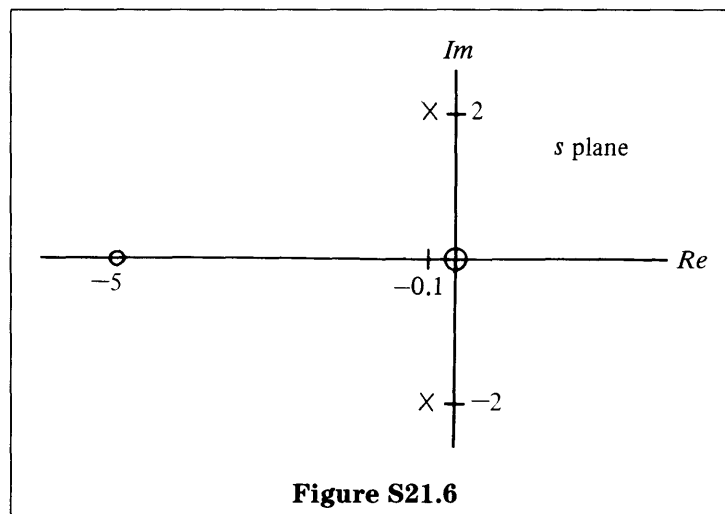


Figure S21.6

Because the zero at $s = -5$ is so far away from the $j\omega$ axis, it will have virtually no effect on $|H(j\omega)|$. Since there is a zero at $\omega = 0$ and poles near $\omega = 2$, we estimate a valley (actually a null) at $\omega = 0$ and a peak at $\omega \simeq \pm 2$.

Solutions to Optional Problems

S21.7

- (a) Let $y(t)$ be the system response to the excitation $x(t)$. Then the differential equation relating $y(t)$ to $x(t)$ is

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

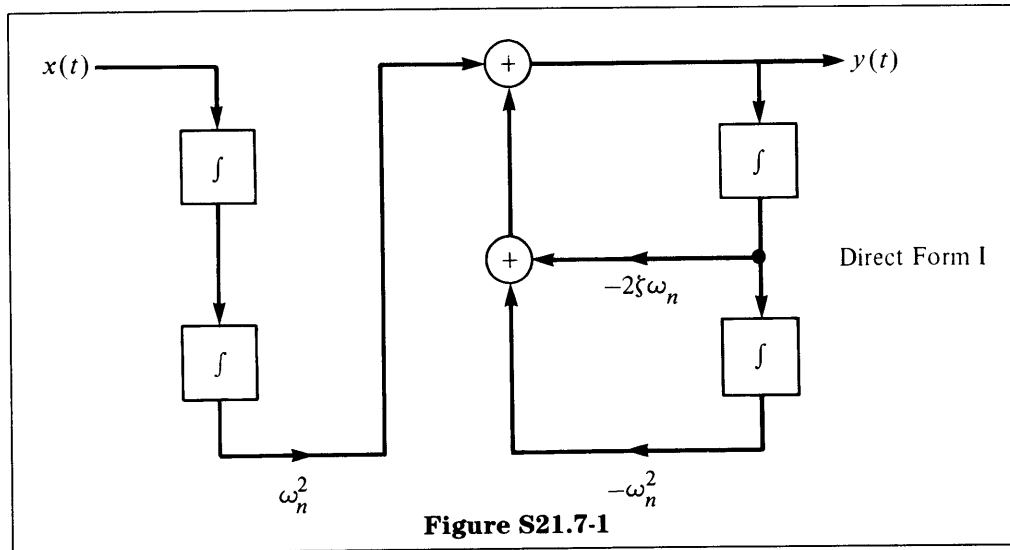
Integrating twice, we have

$$y(t) + 2\zeta\omega_n \int_{-\infty}^t y(\tau) d\tau + \omega_n^2 \int_{-\infty}^t \int_{-\infty}^{\tau'} y(\tau) d\tau d\tau' = \omega_n^2 \int_{-\infty}^t \int_{-\infty}^{\tau'} x(\tau) d\tau d\tau',$$

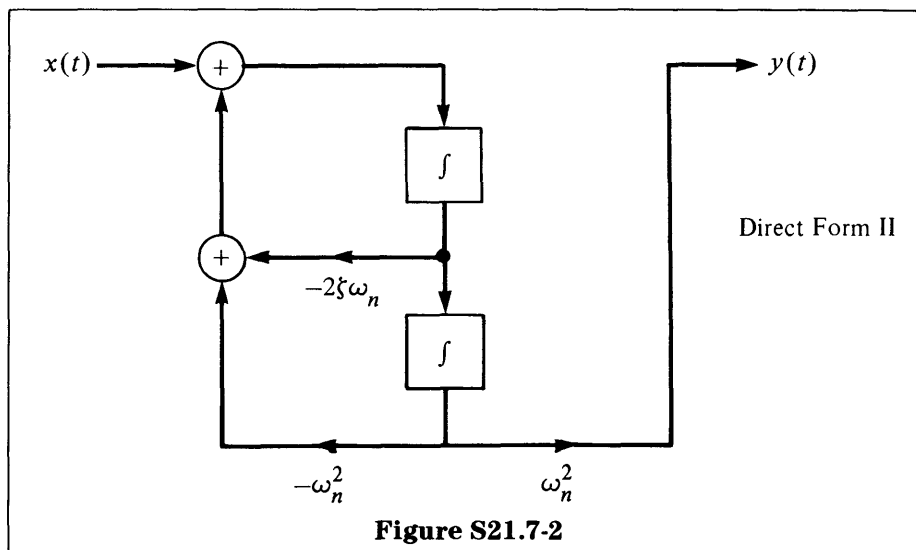
or

$$y(t) = -2\zeta\omega_n \int_{-\infty}^t y(\tau) d\tau - \omega_n^2 \int_{-\infty}^t \int_{-\infty}^{\tau'} y(\tau) d\tau d\tau' + \omega_n^2 \int_{-\infty}^t \int_{-\infty}^{\tau'} x(\tau) d\tau d\tau',$$

shown in Figure S21.7-1.



Recall that Figure S21.7-1 can be simplified as given in Figure S21.7-2.



- (b) (i) For a constant ω_n and $0 \leq \zeta < 1$, $H(s)$ has a conjugate pole pair on a circle centered at the origin of radius ω_n . As ζ changes from 0 to 1, the poles move from close to the $j\omega$ axis to $-\omega_n$, as shown in Figures S21.7-3, S21.7-4, and S21.7-5.

Figure S21.7-3 shows that for $\zeta \approx 0$ the pole is close to the $j\omega$ axis, so $|H(j\omega)|$ has a peak very near ω_n .

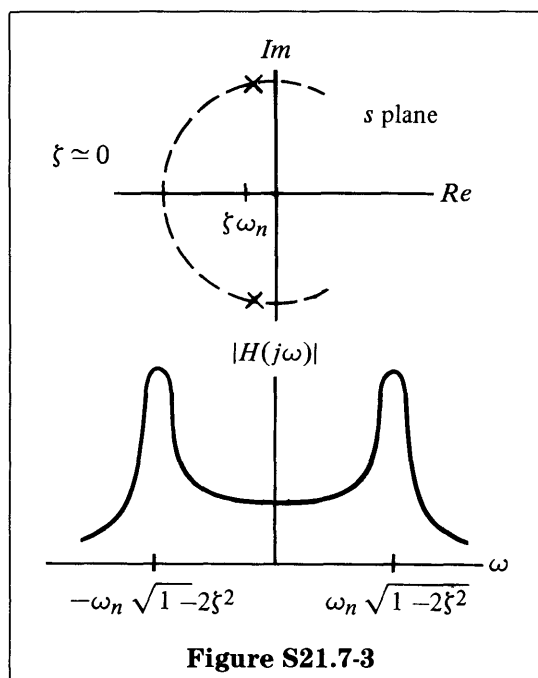


Figure S21.7-4 shows that the peaks are closer together and more spread out at $\zeta = 0.5$.

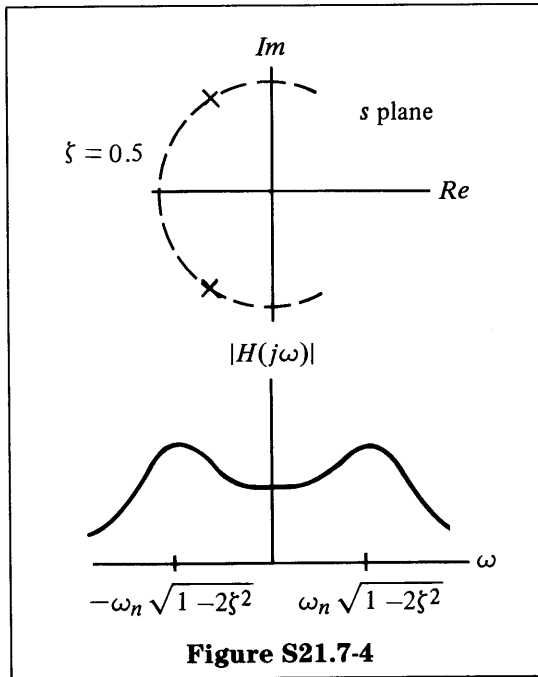
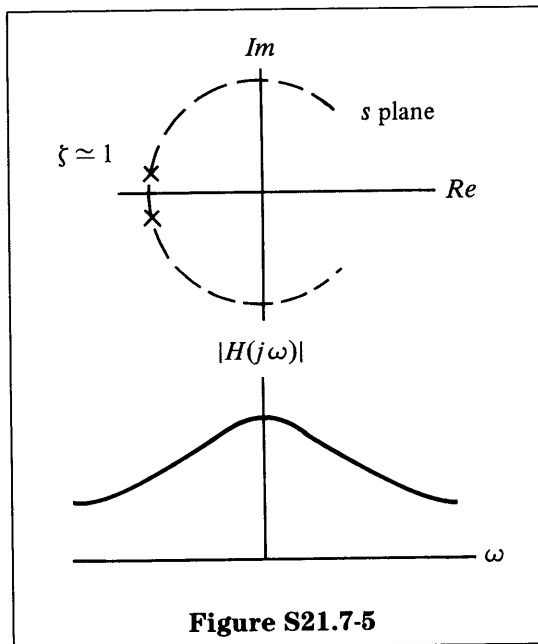
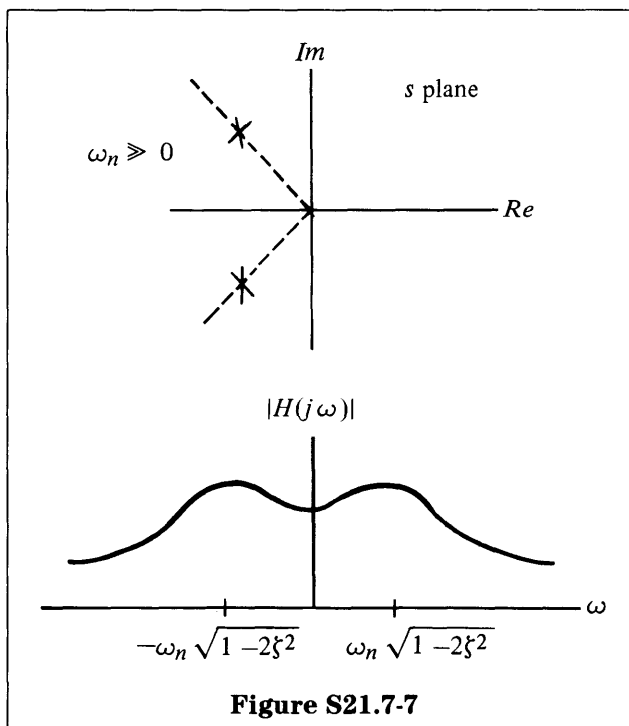
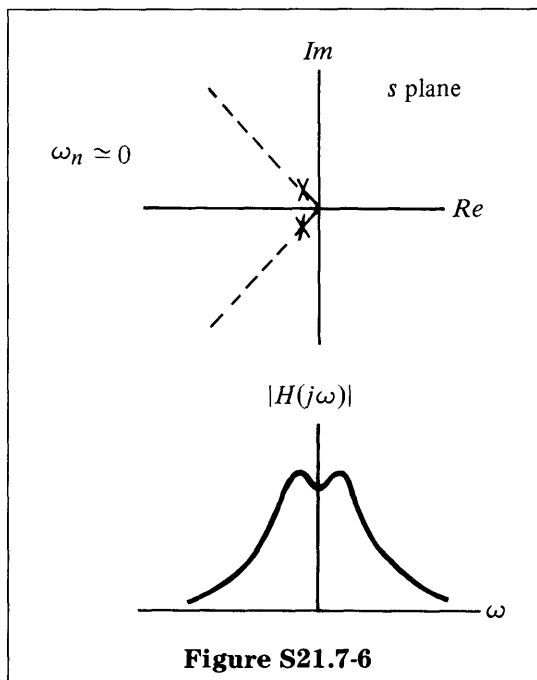


Figure S21.7-5 shows that at $\zeta \simeq 1$ the poles are so close together and far from the $j\omega$ axis that $|H(j\omega)|$ has a single peak.



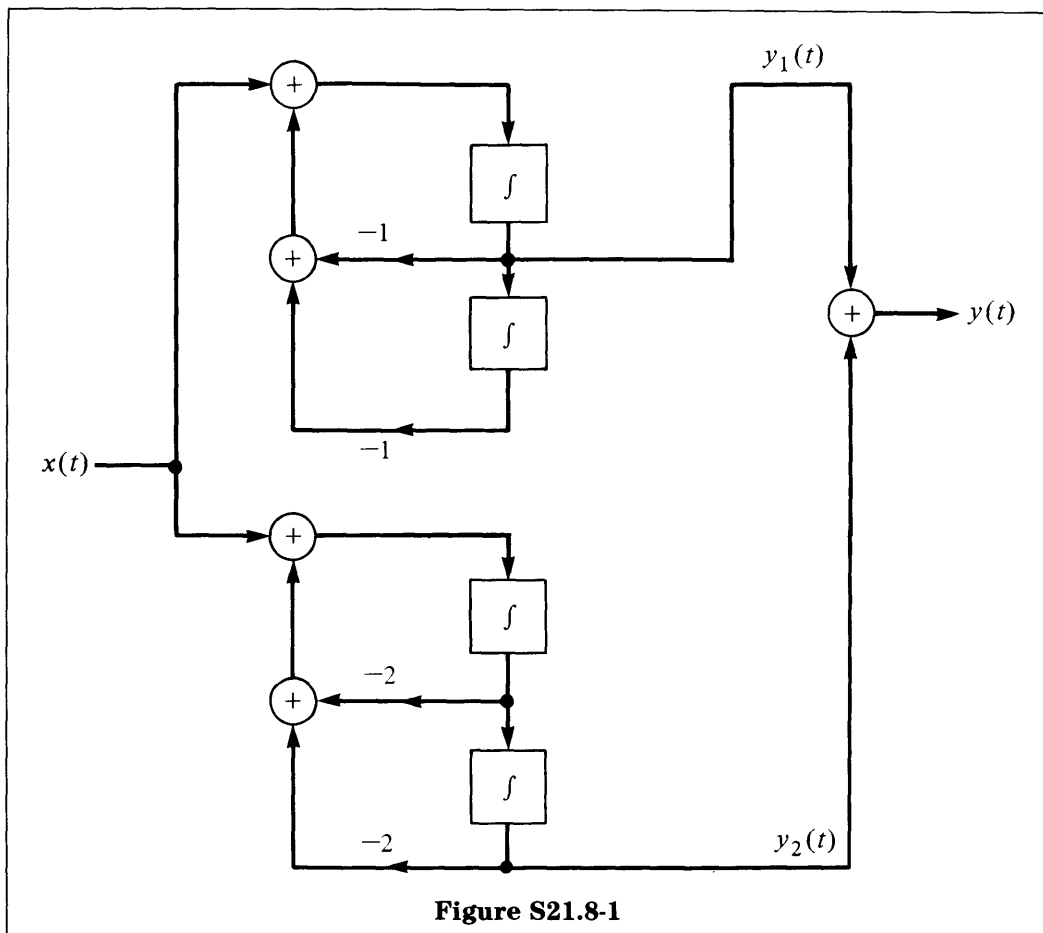
- (ii) For constant ζ between 0 and 1, the poles are located on two straight lines. As ω_n increases, the peak frequency increases as well as the bandwidth, as indicated in Figures S21.7-6 and S21.7-7.



S21.8

- (a) (i) The parallel implementation of $H(s)$, shown in Figure S21.8-1, can be drawn directly from the form for $H(s)$ given in the problem statement. The corresponding differential equations for each section are as follows:

$$\begin{aligned} \frac{d^2y_1(t)}{dt^2} + \frac{dy_1(t)}{dt} + y_1(t) &= \frac{dx(t)}{dt}, \\ \frac{d^2y_2(t)}{dt^2} + \frac{2dy_2(t)}{dt} + 2y_2(t) &= x(t), \\ y(t) &= y_1(t) + y_2(t) \end{aligned}$$



- (ii) To generate the cascade implementation, shown in Figure S21.8-2, we first express $H(s)$ as a product of second-order sections. Thus,

$$H(s) = \frac{s(s^2 + 2s + 2) + (s^2 + s + 1)}{(s^2 + s + 1)(s^2 + 2s + 2)} = \frac{s^3 + 3s^2 + 3s + 1}{(s^2 + s + 1)(s^2 + 2s + 2)}$$

Now we need to separate the numerator into two sections. In this case, the numerator equals $(s + 1)^3$, so an obvious choice is

$$(s + 1)(s^2 + 2s + 1)$$

Thus,

$$H(s) = \left(\frac{s + 1}{s^2 + s + 1} \right) \left(\frac{s^2 + 2s + 1}{s^2 + 2s + 2} \right)$$

The corresponding differential equations are as follows:

$$\frac{d^2r(t)}{dt^2} + \frac{dr(t)}{dt} + r(t) = x(t) + \frac{dx(t)}{dt},$$

$$\frac{d^2y(t)}{dt^2} + \frac{2dy(t)}{dt} + 2y(t) = \frac{d^2r(t)}{dt^2} + \frac{2dr(t)}{dt} + r(t)$$

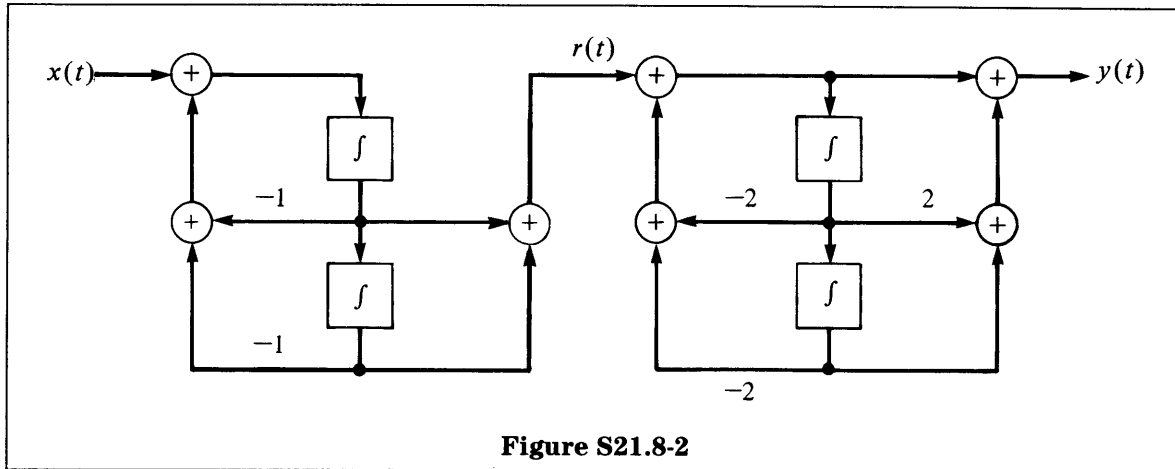


Figure S21.8-2

(b) We see that we could have decomposed $H(s)$ as

$$H(s) = \left(\frac{s^2 + 2s + 1}{s^2 + s + 1} \right) \left(\frac{s + 1}{s^2 + 2s + 2} \right)$$

Thus, the cascade implementation is not unique.

S21.9

(a) Decompose $\sin \omega_0 t$ as

$$\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

Then

$$x_1(t) = \sin(\omega_0 t)u(t) = \frac{e^{j\omega_0 t}}{2j} u(t) - \frac{e^{-j\omega_0 t}}{2j} u(t)$$

Using the transform pair given in the problem statement and the linearity property of the Laplace transform, we have

$$X_1(s) = \frac{1}{2j} \left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0} \right]$$

$$= \frac{1}{2j} \frac{2j\omega_0}{s^2 + \omega_0^2} = \frac{\omega_0}{s^2 + \omega_0^2},$$

with an ROC corresponding to $Re\{s\} > 0$.

(b) $x_2(t) = e^{-2t} \sin(\omega_0 t)u(t)$. Since

$$e^{-2t} \sin(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} X_1(s + 2),$$

the ROC is shifted by 2. Therefore,

$$e^{-2t} \sin(\omega_0 t) u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+2)^2 + \omega_0^2},$$

and the ROC is $\text{Re}\{s\} > -2$. Here we have used our answer to part (a).

(c) Since

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{dX(s)}{ds},$$

with the same ROC as $X(s)$, then

$$te^{-2t}u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds} \left[\frac{1}{(s+2)} \right],$$

Thus

$$te^{-2t}u(t) \xleftrightarrow{\mathcal{L}} -\left[\frac{-1}{(s+2)^2} \right] = \frac{1}{(s+2)^2}$$

with the ROC given by $\text{Re}\{s\} > -2$.

(d) Here we use partial fractions:

$$\begin{aligned} \frac{s+1}{(s+2)(s+3)} &= \frac{A}{s+2} + \frac{B}{s+3}, \\ A &= \left[\frac{s+1}{s+3} \right] \Big|_{s=-2} = -1, \quad B = \left[\frac{s+1}{s+2} \right] \Big|_{s=-3} = \frac{-2}{-1} = 2, \\ \frac{s+1}{(s+2)(s+3)} &= \frac{-1}{s+2} + \frac{2}{s+3} \end{aligned} \quad (\text{S21.9-1})$$

The ROC associated with the first term of eq. (S21.9-1) is $\text{Re}\{s\} > -2$ and the ROC associated with the second term is $\text{Re}\{s\} > -3$ to be consistent with the given total ROC. Thus,

$$x(t) = -e^{-2t}u(t) + 2e^{-3t}u(t)$$

(e) From properties of the Laplace transform we know that

$$x(t-T) \xleftrightarrow{\mathcal{L}} e^{-sT}X(s),$$

with the same ROC as $X(s)$. Since

$$e^{-3t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+3},$$

with an ROC given by $\text{Re}\{s\} > -3$, $(1 - e^{-2s})/(s+3)$ must correspond to

$$x(t) = e^{-3t}u(t) - e^{-3(t-2)}u(t-2)$$

S21.10

(a) (1), (2): An impulse has a constant Fourier transform whose magnitude is unaffected by a time shift. Hence, the Fourier transform magnitudes of (1) and (2) are shown in (c).

(3), (5): A decaying exponential corresponds to a lowpass filter; hence, (3) could be (a) or (d). By comparing it with (5), we see that (5) corresponds to $kte^{-at}u(t)$, which has a double pole at $-a$. Thus, (5) is a steeper lowpass filter than (3). Hence, (3) corresponds to (d) and (5) corresponds to (a).

(4), (7): These signals are of the form $e^{-at} \cos(\omega_0 t)u(t)$. For larger a , the poles are farther to the left. Hence $|H(j\omega)|$ for larger a is less peaky. Thus, (4) corresponds to (f) and (7) corresponds to (g).

(6): If we convolve $x(t) = 1$ with $h(t)$ given in (6), we find that the output is zero. Thus (6) corresponds to a null at $\omega = 0$, either (b) or (h). Note that (6) can be thought of as an $h(t)$ given by (1) minus an $h(t)$ given by (3). Thus, the Fourier transform is the difference between a constant and a lowpass filter. Therefore, (6) is a highpass filter, or (b).

(b) (a), (d): These are simple lowpass filters that correspond to (i) or (ii). Since (a) is a steeper lowpass filter, we associate (a) with (ii) and (d) with (i).

(b), (h): These require a null at zero, and thus could correspond to (iii) or (viii). In the case of (iii), as ω increases, one pole-zero pair is canceled so that for large ω , $H(s)$ looks like a lowpass filter. Hence, (b) corresponds to (viii) and (h) corresponds to (iii).

(c): Here we need a pole-zero plot that is an all-pass system. The only possible pole-zero plot is (vi).

(e): Here we need a null on the $j\omega$ axis, but not at $\omega = 0$. The only possibility is (v).

(f), (g): These are resonant second-order systems that could correspond to (iv) or (vii). Since poles closer to the $j\omega$ axis lead to peakier Fourier transforms, (f) must correspond to (iv) and (g) to (vii).

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