Homework #5: 10.32, 10.35, 10.38, 10.39, 10.47

10.32) Consider an LTI system w/ impulse response:

$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and input:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the output $y[n]$ by explicitly evaluating the discrete convolution.

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} (u[k]-u[k-N])\,a^{n-k}u[n-k] = \sum_{k=0}^{N-1} a^{n-k}u[n-k]$$

if $0 \leq n \leq N-1$

$$y[n] = \sum_{k=0}^{N-1} a^{n-k} = a^n \sum_{k=0}^{N-1} a^{-k} = \frac{a^n(1-a^{-n-1})}{1-a^{-1}}$$

if $n > N-1$

$$y[n] = \frac{a^n(1-a^{-N})}{1-a^{-1}}$$

$$\therefore y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{a^n - a^{-1}}{1-a^{-1}}, & 0 \leq n \leq N-1 \\ \dfrac{a^n - a^{n-N}}{1-a^{-1}}, & N-1 < n \end{cases}$$

b) Determine the output $y[n]$ using z-transforms.

$$Y(z) = H(z)X(z) = \frac{1}{1-az^{-1}}\left(\frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}}\right) =$$

$$\frac{1-z^{-N}}{(1-az^{-1})(1-z^{-1})} = \frac{A}{(1-az^{-1})} + \frac{B}{(1-z^{-1})} - z^{-N}\left(\frac{A}{1-az^{-1}} + \frac{B}{1-z^{-1}}\right)$$

$$A(1-z^{-1}) + B(1-az^{-1}) = 1$$

$$\left.\begin{array}{l} A+B=1 \\ -A-aB=0 \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{-a}{1-a} \\ B = \frac{1}{1-a} \end{array}$$

$$Y(z) = \frac{\left(\frac{-a}{1-a}\right)}{1-az^{-1}} + \frac{\left(\frac{1}{1-a}\right)}{1-z^{-1}} - z^{-N}\left(\frac{\left(\frac{-a}{1-a}\right)}{1-az^{-1}} + \frac{\left(\frac{1}{1-a}\right)}{1-z^{-1}}\right)$$

$$y[n] = \frac{a^n}{1-a^{-1}}u[n] + \left(\frac{1}{1-a}\right)u[n] - \left(\frac{a^{n-N}}{1-a^{-1}}\right)u[n-N] - \left(\frac{1}{1-a}\right)u[n-N] =$$

$$\frac{1}{1-a^{-1}}\left(a^n u[n] - a^{n-N}u[n-N]\right) + \frac{1}{1-a}\left(u[n]-u[n-N]\right)$$

if $n < 0$, $y[n] = 0$

if $0 \leq n \leq N-1$

$$y[n] = \frac{1}{1-a^{-1}}\left(a^n - a^{(N-1)-N}\right) + \frac{1}{1-a}(0) = \frac{a^n - a^{-1}}{1-a^{-1}}$$

if $n > N-1$

$$y[n] = \frac{1}{1-a^{-1}}\left(a^n - a^{n-N}\right) + \frac{1}{1-a}(0) = \frac{a^n - a^{n-N}}{1-a^{-1}}$$

$$\therefore y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{a^n - a^{-1}}{1-a^{-1}}, & 0 \leq n \leq N-1 \\ \dfrac{a^n - a^{n-N}}{1-a^{-1}}, & N-1 < n \end{cases}$$

10.35) Consider an LTI system w/ input $x[n]$ and output $y[n]$ for which:

$$y[n-1] - \frac{5}{2} y[n] + y[n+1] = x[n]$$

The system may or may not be stable or causal. Determine three possible choices for the ~~input~~ unit sample response. Show each choice satisfies the difference equation.

$$\left(z^{-1} - \frac{5}{2} + z\right) Y(z) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{5}{2} + z} = \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{2/3}{1-2z^{-1}} - \frac{2/3}{1-\frac{1}{2}z^{-1}}$$

for ROC $|z| < \frac{1}{2}$:

$$h[n] = \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{2}{3}(2)^n u[-n-1]$$

for ROC $\frac{1}{2} < |z| < 2$

$$h[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{3}(2)^n u[-n-1]$$

for ROC $|z| > 2$:

$$h[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(2)^n u[n]$$

To check eqns. satisfy difference eqns use idea that $h[n]$ is the unit sample response,

if $x[n] = \delta[n]$, $y[n] = h[n]$: show $h[n-1] - \frac{5}{2}h[n] + h[n+1] = \delta[n]$

$|z| < 1/2$:

$$x[n] = \frac{2}{3}\left[\left(\frac{1}{2}\right)^{n-1} - 2^{n-1}\right]u[-n] - \frac{5}{3}\left[\left(\frac{1}{2}\right)^n - 2^n\right]u[-n-1] + \frac{2}{3}\left[\left(\frac{1}{2}\right)^{n+1} - 2^{n+1}\right]u[-n-2]$$

if $n > 0$, all unit steps are zero

$\Rightarrow x[n] = 0$

if $n = 0$

$$\frac{2}{3}\left[\left(\frac{1}{2}\right)^{-1} - 2^{-1}\right] = \frac{2}{3}\left(2 - \frac{1}{2}\right) = 1$$

$x[n] = 1$

if $n = -1$, $u[-n-2] = 0$ so only have $y[n-1]$, $y[n]$ terms

$$\frac{2}{3}\left[\left(\frac{1}{2}\right)^{-2} - 2^{-2}\right] - \frac{5}{3}\left[\left(\frac{1}{2}\right)^{-1} - 2^{-1}\right] = \frac{2}{3}\left[4 - \frac{1}{4}\right] - \frac{5}{3}\left[2 - \frac{1}{2}\right] = \frac{21}{6} - \frac{21}{6}$$

$x[n] = 0$

if $n < -1$, all terms active so shift to same time step

$$\frac{2}{3}\left[\left(\frac{1}{2}\right)^{-2}\left(\frac{1}{2}\right)^{n+1} - 2^{-2}(2)^{n+1}\right] - \frac{5}{3}\left[\left(\frac{1}{2}\right)^{-1}\left(\frac{1}{2}\right)^{n+1} - 2^{-1}2^{n+1}\right] + \frac{2}{3}\left[\left(\frac{1}{2}\right)^{n+1} - 2^{n+1}\right] =$$

$$\frac{8}{3}\left(\frac{1}{2}\right)^{n+1} - \frac{1}{6}(2)^{n+1} - \frac{10}{3}\left(\frac{1}{2}\right)^{n+1} + \frac{5}{6}(2)^{n+1} + \frac{2}{3}\left(\frac{1}{2}\right)^{n+1} - \frac{2}{3}(2)^{n+1}$$

$$\left(\frac{8}{3} - \frac{10}{3} + \frac{2}{3}\right)\left(\frac{1}{2}\right)^{n+1} + \left(-\frac{1}{6} + \frac{5}{6} - \frac{4}{6}\right)2^{n+1} = 0$$

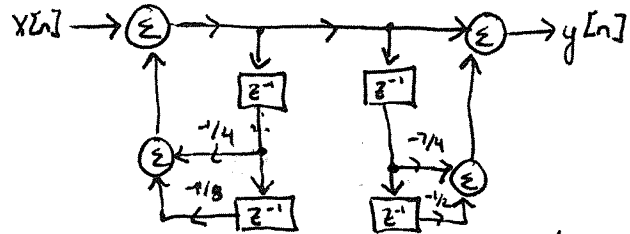
$$\therefore\ x[n] = \begin{cases} 0, & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases} = \delta[n]$$

- do similar process for other parts

10.38) Consider a causal LTI system S with input $x[n]$ and system function:

$$H(z) = \left[\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}\right]\left(1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}\right)$$

The corresponding block diagram is at right.



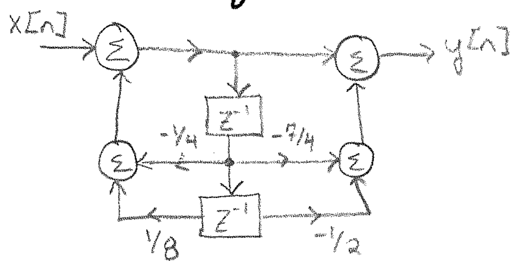
a) How is $e_1[n]$ related to $f_1[n]$?

$e_1[n] = f_1[n]$

b) How is $e_2[n]$ related to $f_2[n]$?

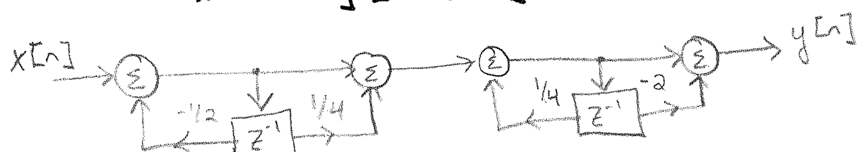
$e_2[n] = f_2[n]$

c) Construct a direct-form block diagram for S w/ 2 delay elements.

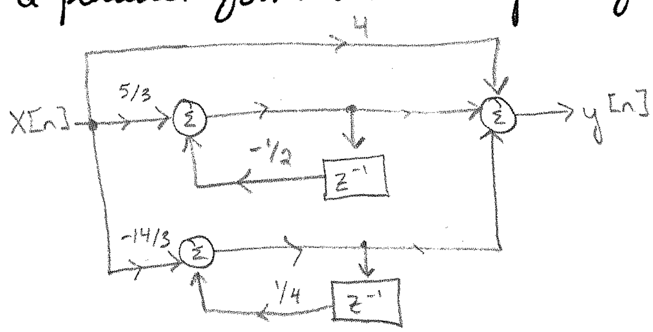


d) Draw a cascade-form block diagram for S noting

$$H(z) = \left[\frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right]\left[\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}\right].$$



e) Draw a parallel-form block diagram for $H(z) = 4 + \dfrac{5/3}{1+\frac{1}{2}z^{-1}} - \dfrac{14/3}{1-\frac{1}{4}z^{-1}}$.



10.39) Consider the following three system functions for causal LTI systems:

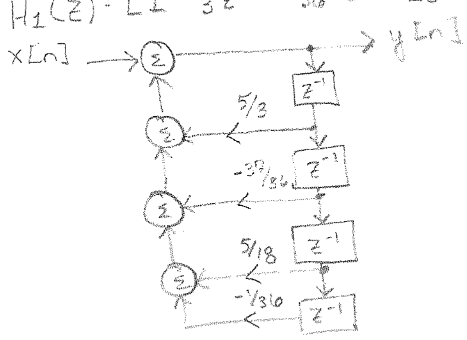
$$H_1(z) = \left[\left(1-z^{-1}+\tfrac{1}{4}z^{-2}\right)\left(1-\tfrac{2}{3}z^{-1}+\tfrac{1}{9}z^{-2}\right)\right]^{-1}$$

$$H_2(z) = \left[\left(1-z^{-1}+\tfrac{1}{2}z^{-2}\right)\left(1-\tfrac{1}{2}z^{-1}+z^{-2}\right)\right]^{-1}$$

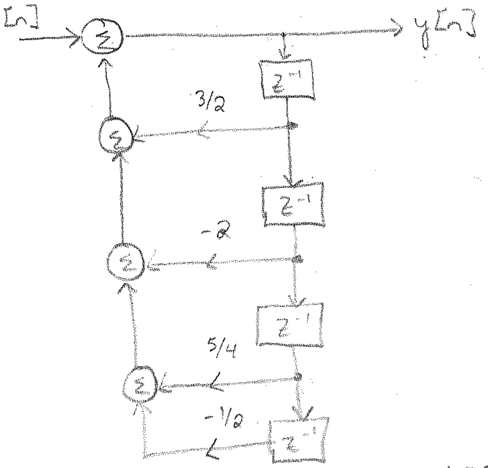
$$H_3(z) = \left[\left(1-z^{-1}+\tfrac{1}{2}z^{-2}\right)\left(1-z^{-1}+\tfrac{1}{4}z^{-2}\right)\right]^{-1}$$

a) For each system function, draw a direct-form block diagram.

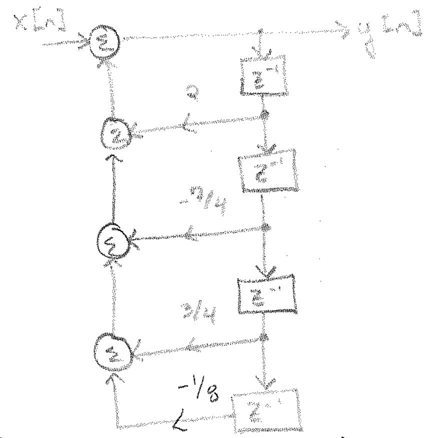
$$H_1(z) = \left[1-\tfrac{5}{3}z^{-1}+\tfrac{37}{36}z^{-2}-\tfrac{5}{18}z^{-3}+\tfrac{1}{36}z^{-4}\right]^{-1}$$



$$H_2(z) = \left[1 - \frac{3}{2}z^{-1} + 2z^{-2} - \frac{5}{4}z^{-3} + \frac{1}{2}z^{-4}\right]^{-1}$$

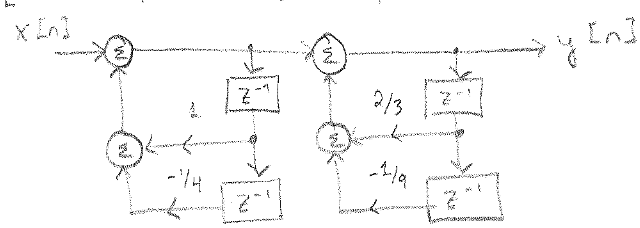


$$H_3(z) = \left[1 - 2z^{-1} + \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} + \frac{1}{8}z^{-4}\right]^{-1}$$

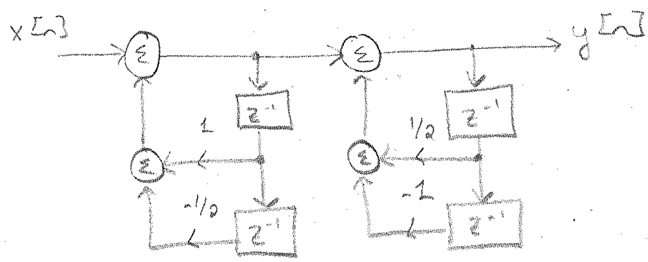


b) Draw each system function as the cascade of two direct form diagrams.

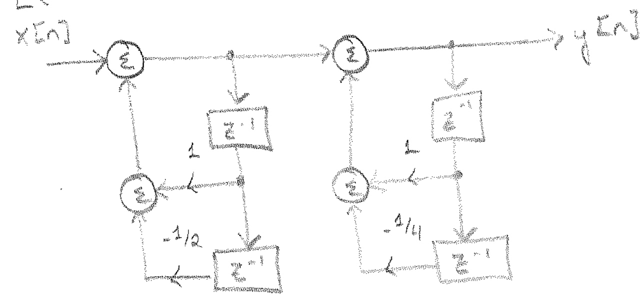
$$H_1(z) = \left[\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}\right)\right]^{-1}$$



$$H_2(z) = \left[\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 - \frac{1}{2}z^{-1} + z^{-2}\right)\right]^{-1}$$



$$H_3(z) = \left[\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\right]^{-1}$$



c) For each system function, determine if the system can be represented as the cascade of four first-order systems. all coefficient multipliers must be real.

$H_1(z) = \left[(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{3}z^{-1})^2\right]^{-1}$

w/ only real poles, the system can be represented thusly

$H_2(z)$ has complex conjugate roots so cannot be represented as a cascade of first-order systems w/ real coefficients

$H_3(z)$ also has complex conjugate roots so cannot represent as a cascade of first-order systems w/ real coefficients

10.47) The following is known about a discrete time LTI system:

1) If $x[n] = (-2)^n, \forall n$ then $y[n] = 0$ for all n.

2) If $x[n] = (\frac{1}{2})^n u[n]$ then $y[n] = \delta[n] + a(\frac{1}{4})^n u[n]$ for all n where a is a constant.

a) Determine the value of constant a.

$x[n] = (\frac{1}{2})^n u[n]$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$y[n] = \delta[n] + a(\frac{1}{4})^n u[n]$

$$Y(z) = 1 + \frac{a}{1-\frac{1}{4}z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-\frac{1}{4}z^{-1}+a)(1-\frac{1}{2}z^{-1})}{1-\frac{1}{4}z^{-1}}$$

$$H(z)\Big|_{z^{-1}=-\frac{1}{2}} = \frac{(1-\frac{1}{4}(-\frac{1}{2})+a)(1-\frac{1}{2}(-\frac{1}{2}z^{-1}))}{1-\frac{1}{4}(-\frac{1}{2})} =$$

$$\frac{(\frac{9}{8}+a)(\frac{5}{4})}{9/8} = 0$$

$$\Rightarrow a = -\frac{9}{8}$$

$a = \frac{-9}{8}$

b) Determine the response if $x[n] = 1$ for all n.

$x[n] = 1$

$y[n] = H(1)\,x[n] =$

$$\frac{(-\frac{1}{8}-\frac{1}{4})(1-\frac{1}{2})}{1-\frac{1}{4}} = \frac{(-\frac{3}{8})(\frac{1}{2})}{\frac{3}{4}} = -\frac{1}{4}$$

$y[n] = \frac{-1}{4}$