Homework Assignment #2: 5.50a, 7.21 a, b, g, 7.22, 7.24a, 7.26, 7.29, 7.31
Due February

5.50 a) Suppose a discrete-time LTI system has input $x[n]$ and output $y[n]$.

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$y[n] = \left(\frac{1}{3}\right)^n u[n]$$

i) Find the impulse response and frequency response of the system.

$$H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega}) = \frac{\left(\frac{1}{1 - \frac{1}{3}e^{-j\omega}}\right)}{\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) - \frac{1}{4}\left(\frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right)} = \frac{4 - 2e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})(4-e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{2(2-e^{-j\omega})}{(1-\frac{1}{3}e^{-j\omega})(4-e^{-j\omega})}$$

$$h[n] = DFT^{-1}\{H(e^{j\omega})\} = DFT^{-1}\left\{\frac{12}{4-e^{-j\omega}} - \frac{2}{1-\frac{1}{3}e^{-j\omega}}\right\} = \left[3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right] u[n]$$

$$h[n] = \left(3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right) u[n]$$

ii) Find a difference equation relating $x[n]$ and $y[n]$.

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$\left(4 - \frac{7}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}\right) Y(e^{j\omega}) = (4 - 2e^{-j\omega}) X(e^{j\omega})$$

$$\Rightarrow 4y[n] - \frac{7}{3}y[n-1] + \frac{1}{3}y[n-2] = 4x[n] - 2x[n-1]$$

$$y[n] = \frac{7}{12}y[n-1] - \frac{1}{12}y[n-2] + x[n] - \frac{1}{2}x[n-1]$$

7.21) A signal $x(t)$ with Fourier transform $X(j\omega)$ undergoes impulse-train sampling to generate $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$ where $T = 10^{-4}$. For each of the following constraints, can $x(t)$ be recovered exactly from $x_p(t)$?

a) $X(j\omega) = 0$ for $|\omega| > 5000\pi$

$$\omega_s = \frac{2\pi}{T} = 20000\pi, \quad \omega_m = 5000\pi$$

$$20000\pi > 2(5000\pi) = 10000\pi \checkmark$$

$\therefore$ can recover $x(t)$ exactly

b) $X(j\omega) = 0$ for $|\omega| > 15000\pi$

now $\omega_m = 15000\pi$ so $\omega_s < 2\omega_m$

$\therefore$ can not recover $x(t)$ exactly

g) $|X(j\omega)| = 0$ for $\omega > 5000\pi$

- due to symmetry of $X(j\omega)$, same as (a) $\Rightarrow$ recoverable

7.22) The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$ where:

$$X_1(j\omega) = 0 \text{ for } |\omega| > 1000\pi, \quad X_2(j\omega) = 0 \text{ for } |\omega| > 2000\pi.$$

Impulse train sampling is performed on $y(t)$ to obtain $y_p(t) = \sum_{n=-\infty}^{\infty} y(nT)\delta(t-nT)$. Specify the range of values for the sampling period T which ensure $y(t)$ is recoverable from $y_p(t)$.

$$Y(j\omega) = X_1(j\omega)\,X_2(j\omega)$$

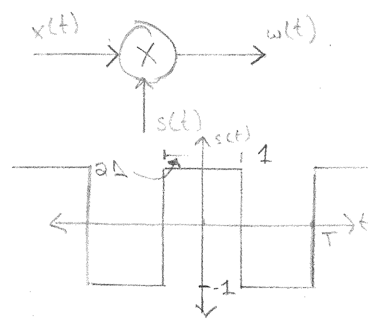
$$Y(j\omega) = 0 \text{ for } |\omega| > 1000\pi$$

$$\omega_s = \frac{2\pi}{T} > 2(1000\pi)$$

$\therefore$ $T < 10^{-3}$ ensures recoverability of $y(t)$ from $y_p(t)$

totally a word ☺!

7.24) Shown below is a system where the input signal is multiplied by a periodic square wave. The period of $s(t)$ is T. The input signal is band limited w/ $|X(j\omega)| = 0$ for $|\omega| \geq \omega_m$.



a) For $\Delta = T/3$, determine the maximum value of T for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

- shift $s(t)$ up one unit then use square wave FT per table $\Rightarrow$

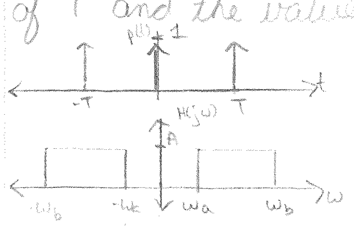
$$S(j\omega) = 2\sum_{k=-\infty}^{\infty} \frac{2\sin\left(\frac{2\pi k}{T}\left(\frac{T}{3}\right)\right)}{k}\,\delta\left(\omega - \frac{2\pi k}{T}\right) - 2\pi\delta(\omega)$$

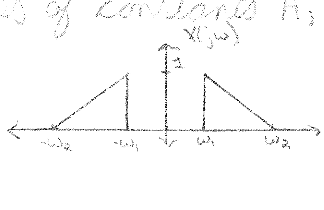
- to avoid aliasing, still need $\omega_s > 2\omega_m$

$$\omega_a = \underbrace{\frac{2\pi}{T}}_{\text{per } \delta \text{ frequency}} > 2\omega_m \Rightarrow T_{max} = \frac{\pi}{\omega_m}$$

$$T_{max} = \frac{\pi}{\omega_m}$$

7.26) Consider the system shown. Assuming $\omega_1 > \omega_2 - \omega_1$, find the maximum value of T and the values of constants A, ω_a, and ω_b such that $x_r(t) = x(t)$.





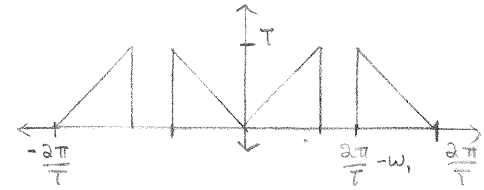
$2\omega_m < \omega_s$ to avoid aliasing

$$(2\omega_1 - \omega_2) < \left(\frac{2\pi}{T} - \omega_2\right) < \omega_2$$

$$0 = \frac{2\pi}{T_{max}} - \omega_2 \Rightarrow T_{max} = \frac{2\pi}{\omega_2}$$

- to preserve amplitude, need:

$$A\left(\frac{1}{T}\right) = 1 \Rightarrow A = T$$

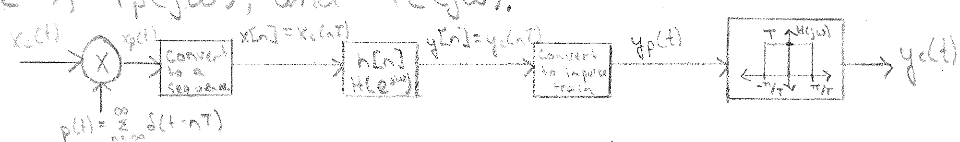


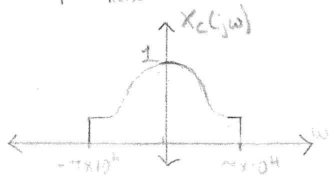
- for bounds:

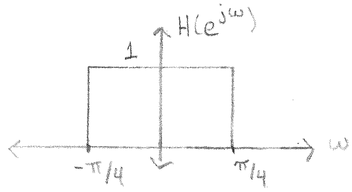
$$\omega_b = \frac{2\pi}{T}, \quad \omega_a = \frac{2\pi}{T} - \omega_1$$

$\therefore$ to ensure $x_r(t) = x(t)$, need $A = T$, $\omega_b = \frac{2\pi}{T}$, $\omega_a = \frac{2\pi}{T} - \omega_1$, and $T_{max} = \frac{2\pi}{\omega_2}$

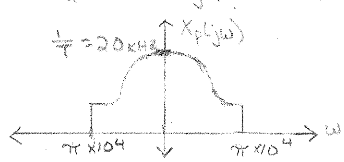
7.29) A system for filtering a continuous-time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are also shown w/ $\frac{1}{T} = 20\text{kHz}$, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.



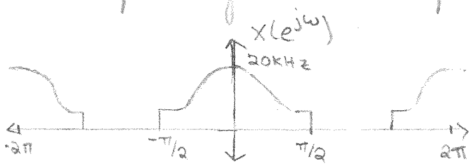




$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - \frac{2\pi k}{T}\right)\right) \quad \text{where } \frac{1}{T} = 20\text{kHz}$$

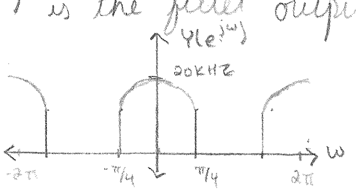


$X(e^{j\omega})$ maps signal to a periodic signal

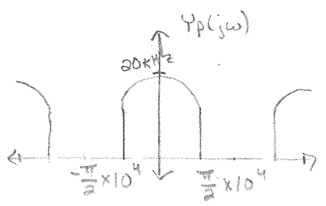


$$\omega_m T = \pi \times 10^4 \left(\frac{1}{20\text{kHz}}\right) = \pi/2$$

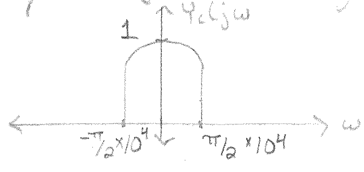
$Y(e^{j\omega})$ is the filter output



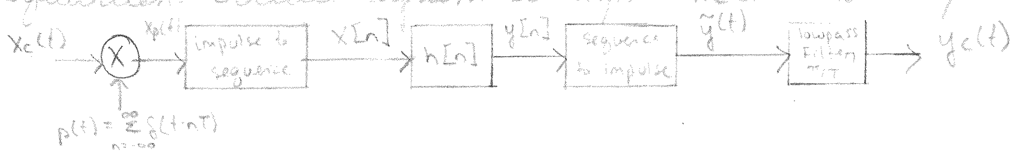
$Y_p(j\omega)$ converts signal back into impulse train



Finally, $Y_c(j\omega)$ is the filter output:



7.31) Shown is a system that processes continuous-time signals using a digital filter $h[n]$ thats linear and causal w/ difference eqn: $y[n] = \frac{1}{2}y[n-1] + x[n]$. For band limited signals w/ $X_c(j\omega) = 0$ for $|\omega| > \pi/T$, the system is equivalent to a continuous LTI system. Determine the frequency response $H_c(j\omega)$ of the equivalent overall system w/ input $x_c(t)$ and output $y_c(t)$.



- first focus on $h[n]$, per difference eqn. $\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$
- next a sketch of the logic (can support w/ arbitrary figure for $x_c(t)$)
  - $x_p(t) \rightarrow x[n]$: signal becomes 2π periodic so bounds $\pm\pi$
  - $x_c(t) \rightarrow x_p(t)$: magnitude scaled by $1/T$, bounds become $\pm\frac{\pi}{T}$
  - conversely, $y[n] \rightarrow \tilde{y}(t)$: scales magnitude by T, bounds back to $\pm\pi/T$
  - $\tilde{y}(t) \rightarrow y_c(t)$: lose data outside π/T (which preserves 'middle' signal)

$\therefore\ H_c(j\omega) = H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

If input signal band limited w/ $|\omega| > \pi/T$, discrete time system is equivalent to continuous time :)