DOPPLER RESILIENT GOLAY COMPLEMENTARY PAIRS FOR RADAR

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ABSTRACT

We present a systematic way of constructing a Doppler resilient sequence of Golay complementary waveforms for radar, for which the composite ambiguity function maintains ideal shape at small Doppler shifts. The idea is to determine a sequence of Golay pairs that annihilates the low-order terms of the Taylor expansion of the composite ambiguity function. The Prouhet-Thue-Morse sequence plays a key role in the construction of Doppler resilient sequences of Golay pairs. We extend this construction to multiple dimensions. In particular, we consider radar polarimetry, where the dimensions are realized by two orthogonal polarizations. We determine a sequence of two-by-two Alamouti matrices, where the entries involve Golay pairs and for which the matrix-valued composite ambiguity function vanishes at small Doppler shifts.

1. INTRODUCTION

Golay complementary sequences [1] have the property that the sum of their autocorrelation functions vanishes at all nonzero integer delays. This means that the sum of the ambiguity functions (composite ambiguity function) of Golay complementary sequences is sidelobe free along the zero-Doppler axis, making them ideal for radar range imaging. However, in practice a major barrier exists in adoption of complementary sequences for radar; the perfect auto-correlation property of these sequences is sensitive to Doppler shift. Off the zero-Doppler axis, the composite ambiguity function of complementary sequences can have large sidelobes in delay, which prohibit unambiguous range imaging. Most generalizations of complementary sequences, including multiple complementary sequences and polyphase sequences suffer from the same problem to some degree. Examples of polyphase sequences that exhibit some tolerance to Doppler are Frank sequences [2], $P1$, $P2$, $P3$, and $P4$ sequences [3], and $P(n,k)$ sequences [4]. Subcomplementary codes [5],[6] are another class of near complementary codes, which exhibit some tolerance against Doppler.

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In this paper, we present a novel method of constructing a Doppler resilient sequence of Golay pairs, whose composite ambiguity function maintains impulse-like shape at small Doppler shifts. We determine a sequence of Golay pairs that forces the low-order terms of the Taylor expansion (around zero Doppler) of the composite ambiguity function to zero. We then extend this construction to multiple dimensions and integrate it with instantaneous radar polarimetry [7],[8], where the dimensions are realized by employing two orthogonal polarizations. Here, we construct a sequence of two-by-two Alamouti matrices [9], where the entries involve Golay pairs and for which the matrix-valued composite ambiguity function vanishes at small Doppler shifts. As we show, the Prouhet-Thue-Morse (PTM) sequence [10] plays a key role in the construction of Doppler resilient sequences of Golay pairs.

Finally, we note that this paper is intended to provide a summary of the results reported in [11]. We have omitted the proofs and derivations have been shortened or left out entirely.

2. GOLAY COMPLEMENTARY PAIRS FOR RADAR

Definition 1: Two length L unimodular sequences of complex numbers $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}, \quad \text{for } k = -(L-1), \cdots, (L-1), \tag{1}$$

where $C_x(k)$ is the autocorrelation of $x(\ell)$ at lag k and $\delta_{k,0}$ is the Kronecker delta function.

Henceforth we drop the discrete time index ℓ from $x(\ell)$ and $y(\ell)$ and simply use x and y. We use the notation (x, y) whenever x and y are Golay complementary and call (x, y) a Golay pair. From (1) it follows that if (x, y) is a Golay pair then $(\pm x, \pm\widetilde{y})$, $(\pm\widetilde{x}, \pm y)$, and $(\pm\widetilde{x}, \pm\widetilde{y})$ are also Golay pairs, where $\widetilde{x} = \widetilde{x}(\ell) = x^*(-\ell)$ is the time reversed complex conjugate of x.

Definition 2: The *composite ambiguity function* [6] of a set of waveforms $\{x_0(t), x_1(t), \ldots, x_{N-1}(t)\}$ is defined as

$$\mathcal{A}(\tau, \nu) = \sum_{n=0}^{N-1} A_{x_n}(\tau, \nu), \tag{2}$$

where $A_{x_n}(\tau, \nu)$ is the (auto) ambiguity function of $x_n(t)$ in delay variable τ and Doppler variable ν.

Remark 1: The composite ambiguity function $\mathcal{A}(\tau, \nu)$ corresponds to the main lobe of the ambiguity function of the pulse train $[x_0(t) \;\; x_1(t) \;\; \ldots \;\; x_{N-1}(t)]$, where consecutive pulses are separated by a pulse repetition interval (PRI).

Consider a single transmitter/single receiver radar system. Suppose Golay pairs $(x_0, x_1), (x_2, x_3), \ldots, (x_{N-2}, x_{N-1})$ are transmitted over N PRIs. Then, the (discretized) composite ambiguity function $\mathcal{A}(k, \theta)$ of $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$, sampled at waveform chip rate, can be expressed as

$$\mathcal{A}(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k), \tag{3}$$

where θ is the relative Doppler shift during a PRI. In writing (3), we have assumed that the Doppler shift at the chip rate is negligible compared to the Doppler shift at the pulse repetition rate.

When $\theta = 0$, $\mathcal{A}(k, \theta)$ is equal to $\mathcal{A}(k, 0) = NL\delta_{k,0}$. This means that the composite ambiguity function of Golay pairs is sidelobe free along the zero-Doppler axis at all integer delays (integer multiples of the chip rate). However, off the zero-Doppler axis ($\theta \neq 0$) this is not the case. In fact, even small Doppler shifts can result in large sidelobes. One way to solve this problem is to use a bank of Doppler filters to estimate the unknown Doppler shift θ and then compensate for it. However, since even a slight mismatch in Doppler can result in large sidelobes, we have to cover the possible Doppler range at a fine resolution, which requires the use of many Doppler filters. This motivates the question of whether it is possible to design *Doppler resilient* Golay pairs (x_0, x_1) , $\ldots$, (x_{N-2}, x_{N-1}) so that $\mathcal{A}(k, \theta) \approx NL\delta_{k,0}$ for a reasonable range of Doppler shifts θ.

3. DOPPLER RESILIENT GOLAY PAIRS

In this section, we consider the design of a Doppler resilient sequence of Golay pairs for single channel radar. We construct a sequence of Golay pairs $(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ so that in the Taylor expansion of $\mathcal{A}(k, \theta)$ around $\theta = 0$ all terms up to a certain order, say M, vanish at all nonzero delays. As we will show, the PTM sequence plays a key role in constructing the Doppler resilient sequence of Golay pairs.

Definition 3:[10] The *Prouhet-Thue-Morse (PTM) sequence* $\mathcal{S} = (s_k)_{k \geq 0}$ over $\{0, 1\}$ is defined by the following recursions:

1. $s_0 = 0$
2. $s_{2k} = s_k$
3. $s_{2k+1} = \overline{s}_k = 1 - s_k$

for all $k > 0$, where $\overline{s} = 1 - s$ denotes the binary complement of $s \in \{0, 1\}$.

For example, the PTM sequence of length 16 is

$$\mathcal{S} = (s_k)_{k=0}^{15} = 0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0. \tag{4}$$

Let (x, y) be a Golay pair. Define the pulse trains $\mathbf{x}_0$ and $\mathbf{x}_1$ as $\mathbf{x}_0 = [x \;\; y]$ and $\mathbf{x}_1 = [-\widetilde{y} \;\; \widetilde{x}]$, where pulses are separated by a PRI. Construct the *length-*$(N/2)$ *PTM sequence of Golay pairs* $\mathcal{X}_{(N/2)}^{PTM}$ using the following recursions:

1. Start with $\mathcal{X}_{(1)}^{PTM} = \mathbf{x}_0$.
2. For $n = 2, \ldots, N/2$, construct $\mathcal{X}_{(n)}^{PTM}$ as

$$\mathcal{X}_{(n)}^{PTM} = [\mathcal{X}_{(n-1)}^{PTM} \;\; \mathbf{x}_{s_{n-1}}], \tag{5}$$

where $(s_k)_{k=0}^{N/2-1}$ is the length-$(N/2)$ PTM sequence, and $[\mathbf{w} \;\; \mathbf{z}]$ is a concatenation of $\mathbf{w}$ and $\mathbf{z}$.

For example, the length-4 PTM sequence of Golay pairs $\mathcal{X}_{(4)}^{PTM}$, which is transmitted over $N = 8$ PRIs, is given by

$$\begin{aligned} \mathcal{X}_{(4)}^{PTM} &= [\mathbf{x}_0 \;\; \mathbf{x}_1 \;\; \mathbf{x}_1 \;\; \mathbf{x}_0] \\ &= [x \;\; y \;\; -\widetilde{y} \;\; \widetilde{x} \;\; -\widetilde{y} \;\; \widetilde{x} \;\; x \;\; y]. \end{aligned} \tag{6}$$

The composite ambiguity function of $\mathcal{X}_{(N/2)}^{PTM}$ is given by

$$\begin{aligned} &\mathcal{A}^{PTM}(k, \theta) \\ &= \left(\sum_{n \in \mathbb{S}_0} e^{jn\theta} \right) C_x(k) + \left(\sum_{n \in \mathbb{S}_1} e^{jn\theta} \right) C_y(k), \end{aligned} \tag{7}$$

where $\mathbb{S}_0$ is the set of all indices that correspond to zeros in the length-$(N/2)$ PTM sequence, and $\mathbb{S}_1$ is the set of all indices that correspond to ones in length-$(N/2)$ PTM sequence. In writing (7), we have used the fact that the autocorrelation functions of $\pm x, \pm \widetilde{x}$ are equal.

Theorem 1: In the Taylor expansion of $\mathcal{A}(k, \theta)$ in (3) around $\theta = 0$ all terms up to order M vanish at all nonzero integer delays $k \neq 0$, if the waveform sequence $[x_0 \;\; \ldots \;\; x_{N-1}]$ is a length-$(N/2)$ PTM sequence of Golay pairs, with $N = 2^{M+1}$.

4. EXTENSION TO MULTIPLE DIMENSIONS: INSTANTANEOUS RADAR POLARIMETRY

We now extend our Doppler resilient construction to multiple dimensions to construct a sequence of two-by-two Alamouti matrices of Golay pairs, whose *matrix-valued composite ambiguity function* vanishes at all nonzero integer delays for small Doppler shifts.

Alamouti matrices of Golay pairs were constructed in [7] and [8] for instantaneous radar polarimetry to enable target detection in range based on full polarimetric properties of the target on a pulse by pulse basis. Alamouti signal processing is used to coordinate the transmission of $(N/2)$ Golay pairs

$(x_0, x_1), \ldots, (x_{N-2}, x_{N-1})$ over vertical and horizontal polarizations during N PRIs. The waveform matrix is of the form

$$\boldsymbol{\mathcal{X}}_{2\times N} = [\mathbf{X}_{2\times 2}^{(0,1)} \ \ldots \ \mathbf{X}_{2\times 2}^{(N-2,N-1)}], \tag{8}$$

where

$$\mathbf{X}_{2\times 2}^{(2n,2n+1)} = \begin{pmatrix} x_{2n} & -\widetilde{x}_{2n+1} \\ x_{2n+1} & \widetilde{x}_{2n} \end{pmatrix}, \ n = 0, \ldots, N/2 - 1, \tag{9}$$

is a two-by-two Alamouti waveform matrix. Different rows in $\boldsymbol{\mathcal{X}}_{2\times N}$ correspond to vertical and horizontal polarizations, and different columns correspond to different time slots (PRIs).

Definition 4: Let $\mathbf{X}$ be an $m \times n$ matrix, whose (i, j)-th entry is a waveform $x_{i,j}$. We define the *matrix-valued composite ambiguity function* $\boldsymbol{\mathcal{A}}(\tau, \nu) \in \mathbb{C}^{m\times m}$ of $\mathbf{X}$ as

$$\boldsymbol{\mathcal{A}}(\tau, \nu) = \begin{pmatrix} \mathcal{A}_{1,1}(\tau, \nu) & \cdots & \mathcal{A}_{1,n}(\tau, \nu) \\ \mathcal{A}_{2,1}(\tau, \nu) & \cdots & \mathcal{A}_{2,n}(\tau, \nu) \\ \vdots & \ddots & \vdots \\ \mathcal{A}_{n,1}(\tau, \nu) & \cdots & \mathcal{A}_{n,n}(\tau, \nu) \end{pmatrix}, \tag{10}$$

where $\mathcal{A}_{i,j}(\tau, \nu) = \sum_{\ell=1}^{n} A_{x_{i,\ell}x_{\ell,j}}(\tau, \nu)$ is the sum of the cross ambiguity functions $A_{x_{i,\ell}x_{\ell,j}}(\tau, \nu)$ between $x_{i,\ell}$ and $x_{\ell,j}$, $\ell = 1, \ldots, n$.

Note that similar to Section 3, we can discretize the entries of $\boldsymbol{\mathcal{A}}(\tau, \nu)$ in delay variable τ at the chip rate to obtain a discretized matrix-valued composite ambiguity function $\boldsymbol{\mathcal{A}}(k, \theta)$ in integer delay k and PRI Doppler shift θ.

The discretized matrix-valued composite ambiguity function for $\boldsymbol{\mathcal{X}}_{2\times N}$ is given by

$$\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta) = \begin{pmatrix} \mathcal{A}_{1,1}(k, \theta) & \mathcal{A}_{1,2}(k, \theta) \\ \mathcal{A}_{2,1}(k, \theta) & \mathcal{A}_{2,2}(k, \theta) \end{pmatrix}, \tag{11}$$

where

$$\mathcal{A}_{1,1}(k, \theta) = \mathcal{A}_{22}(k, \theta) = \sum_{n=0}^{N-1} e^{jn\theta} C_{x_n}(k), \tag{12}$$

$$\mathcal{A}_{1,2}(k, \theta) = \sum_{n=0}^{N/2-1} (e^{j2n\theta} - e^{j(2n+1)\theta}) C_{x_{2n}x_{2n+1}}(k), \tag{13}$$

and

$$\mathcal{A}_{2,1}(k, \theta) = \sum_{n=0}^{N/2-1} (e^{j2n\theta} - e^{j(2n+1)\theta}) C^*_{x_{2n}x_{2n+1}}(-k), \tag{14}$$

where $C_{x_{2n}x_{2n+1}}(k)$ is the cross-correlation between x_{2n} and x_{2n+1} at lag k. Note that the diagonal elements of $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ are equal to the (single channel) discretized composite ambiguity function $\mathcal{A}(k, \theta)$ in (3).

Along the zero-Doppler axis ($\theta = 0$) $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ reduces to

$$\boldsymbol{\mathcal{A}}_{2\times 2}(k, 0) = \begin{pmatrix} NL\delta_{k,0} & 0 \\ 0 & NL\delta_{k,0} \end{pmatrix}, \tag{15}$$

due to the interplay between Alamouti signal processing and complementary property of the Golay pairs. This interplay allows for target detection in range based on full polarimetric properties of the target and results in improved detection performance, without increasing the receiver signal processing complexity beyond that of single-channel matched filtering. However, off the zero-Doppler axis the perfect matrix ambiguity property in (15) does not hold and the entries of $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ may have large sidelobes in range at nonzero integer delays.

In what follows, we describe how a sequence of two-by-two Alamouti matrices of Golay pairs can be designed so that in the Taylor expansion of the entries of $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ around $\theta = 0$ all terms up to a certain order, say M, vanish at all nonzero delays.

Let x and y be a Golay pair. Define the two-by-two Alamouti matrices $\mathbf{X}_0$ and $\mathbf{X}_1$ as

$$\mathbf{X}_0 = \begin{pmatrix} x & -\widetilde{y} \\ y & \widetilde{x} \end{pmatrix} \quad \text{and} \quad \mathbf{X}_1 = \begin{pmatrix} -\widetilde{y} & -x \\ \widetilde{x} & -y \end{pmatrix}. \tag{16}$$

Construct the *length-*$(N/2)$ *PTM sequence of* 2 *by* 2 *Alamouti matrices of Golay pairs* $\boldsymbol{\mathcal{X}}^{PTM}_{(N/2)}$ using the following recursions:

1. Start with $\boldsymbol{\mathcal{X}}^{PTM}_{(1)} = \mathbf{X}_0$.

2. For $n = 2, \ldots, N/2$, construct $\boldsymbol{\mathcal{X}}^{PTM}_{(n)}$ as

$$\boldsymbol{\mathcal{X}}^{PTM}_{(n)} = [\boldsymbol{\mathcal{X}}^{PTM}_{(n-1)} \ \mathbf{X}_{s_{n-1}}], \tag{17}$$

   where $(s_k)_{k=0}^{N/2-1}$ is the length-$(N/2)$ PTM sequence.

For example, the length-4 PTM sequence of two-by-two Alamouti matrices of Golay pairs, which is transmitted over vertical and horizontal polarization during $N = 8$ PRIs, is given by

$$\boldsymbol{\mathcal{X}}^{PTM}_{(4)} = [\mathbf{X}_0 \ \mathbf{X}_1 \ \mathbf{X}_1 \ \mathbf{X}_0]. \tag{18}$$

Theorem 2: In the Taylor expansion of the entries of $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ in (11) all terms up to order M vanish at all nonzero integer delays $k \neq 0$ (for the off diagonal elements they also vanish at $k = 0$), if $\boldsymbol{\mathcal{X}}_{2\times N}$ is a length-$(N/2)$ PTM sequence of two-by-two Alamouti matrices of Golay pairs, with $N = 2^{M+1}$.

5. NUMERICAL EXAMPLES

In this section, we present numerical examples to verify the results of Sections 3 and 4 and compare our Doppler resilient design to a conventional scheme, where the same Golay pair is repeated. We consider the two-by-two Alamouti case described in Section 4. Considering this case also covers the single channel case, as the diagonal elements of $\boldsymbol{\mathcal{A}}_{2\times 2}(k, \theta)$ are equal to the single channel composite ambiguity function $\mathcal{A}(k, \theta)$ in (3).

Following Theorem 2, we coordinate the transmission of a length-8 PTM sequence of two-by-two Alamouti matrices of Golay pairs $\boldsymbol{\mathcal{X}}_{(8)}^{PTM}$ to annihilate the first, second, and third order terms ($M = 3$) of the Taylor expansion of $\mathcal{A}_{1,1}(k,\theta) = \mathcal{A}_{2,2}(k,\theta)$ at all nonzero integer delays $k \neq 0$, and to annihilate the first, second, and third order terms of the Taylor expansion of $\mathcal{A}_{1,2}(k,\theta)$ (or alternatively $\mathcal{A}_{2,1}(k,\theta)$) at all integer delays. The Golay pair (x, y) used in constructing $\boldsymbol{\mathcal{X}}_{(8)}^{PTM}$ is the following length-8 ($L = 8$) Golay pair:

$$\begin{aligned} x &= \{1, 1, -1, 1, 1, 1, 1, -1\} \\ y &= \{-1, -1, 1, -1, 1, 1, 1, -1\}. \end{aligned} \quad (19)$$

We compare the Doppler resilient transmission scheme $\boldsymbol{\mathcal{X}}_{(8)}^{PTM}$ with a conventional transmission scheme, where the Alamouti waveform matrix built from a single Golay pair ($x_0 = x, x_1 = y$) is repeated and the waveform matrix $\boldsymbol{\mathcal{X}}^{Conv}$ is of the form

$$\boldsymbol{\mathcal{X}}^{Conv} = [\mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0 \;\; \mathbf{X}_0], \quad (20)$$

with discretized matrix-valued composite ambiguity function

$$\boldsymbol{\mathcal{A}}^{Conv}(k,\theta) = \begin{pmatrix} \mathcal{A}_{1,1}^{Conv}(k,\theta) & \mathcal{A}_{1,2}^{Conv}(k,\theta) \\ \mathcal{A}_{2,1}^{Conv}(k,\theta) & \mathcal{A}_{1,1}^{Conv}(k,\theta) \end{pmatrix} \quad (21)$$

Figures 1(a),(b) show the plots of the magnitudes of the diagonal entries $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,1}^{Conv}(k,\theta)$ versus delay index k and Doppler shift θ. Comparison of $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,1}^{Conv}(k,\theta)$ at Doppler shifts $\theta = 0.05$ rad and $\theta = 0.075$ rad is provided in Figs. 2(a),(b), where the solid lines correspond to $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ and the dashed lines correspond to $\mathcal{A}_{1,1}^{Conv}(k,\theta)$. The peaks of the range sidelobes of $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ are at least 28 dB (for $\theta = 0.05$ rad) and 29 dB (for $\theta = 0.075$ rad) smaller than those of $\mathcal{A}_{1,1}^{PTM}(k,\theta)$.

Figures 3(a),(b) show the plots of the magnitudes of the off-diagonal entries $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,2}^{Conv}(k,\theta)$ versus delay index k and Doppler shift θ. Comparison of $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,2}^{Conv}(k,\theta)$ at Doppler shifts $\theta = 0.05$ rad and $\theta = 0.075$ rad is provided in Figs. 4(a),(b), where the solid lines correspond to $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ and the dashed lines correspond to $\mathcal{A}_{1,2}^{Conv}(k,\theta)$. The peaks of the range sidelobes of $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ are at least 12 dB (for $\theta = 0.05$ rad) and 5 dB (for $\theta = 0.075$ rad) smaller than those of $\mathcal{A}_{1,2}^{Conv}(k,\theta)$.

Remark 2: For a radar with carrier frequency $f_0 = 2.5$ GHz and PRI= 100 μsec, the Doppler shift range of 0 to 0.05 rad (0.075 rad) corresponds to a maximum target speed of $V \approx 35$ kmph (50 kmph). To cover a larger speed range we can use our design with a bank of Doppler filters to provide Doppler resilience within an interval around the Doppler frequency associated with each filter.

6. CONCLUSIONS

We presented a systematic way of constructing a Doppler resilient sequence of Golay complementary pairs for single channel radar. We then extended this construction to instantaneous radar polarimetry, where Doppler resilient sequences of two-by-two Alamouti matrices of Golay pairs were designed. The idea is to determine a sequence of Golay pairs that forces the low-order terms of the Taylor expansion of a composite ambiguity function to zero. The Prouhet-Thue-Morse sequence is the key to selecting the Doppler resilient sequences of Golay pairs.

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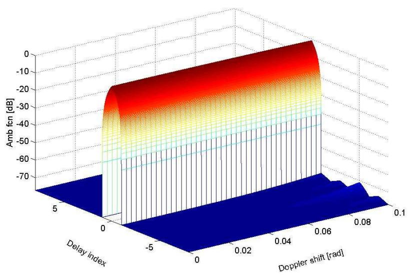
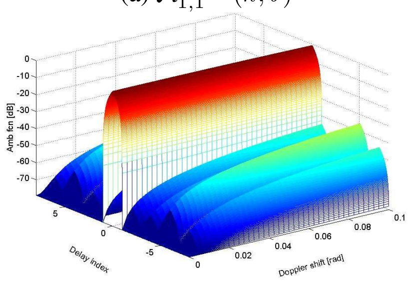
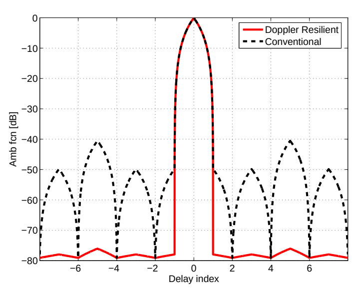


(a) $\mathcal{A}_{1,1}^{PTM}(k,\theta)$

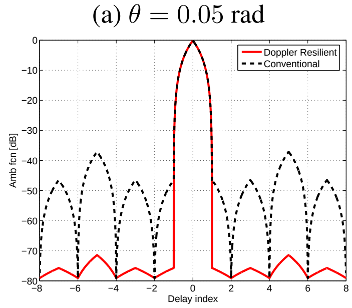


(b) $\mathcal{A}_{1,1}^{Conv}(k,\theta)$

Fig. 1. The plot of (a) $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ and (b) $\mathcal{A}_{1,1}^{Conv}(k,\theta)$ versus delay index k and Doppler shift θ.

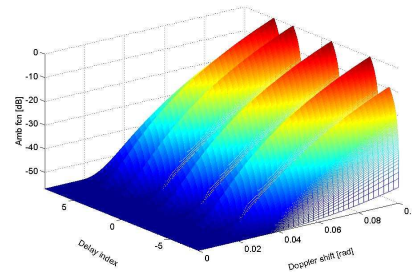


(a) $\theta = 0.05$ rad

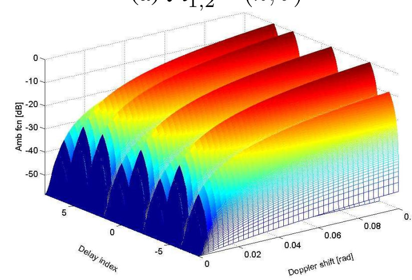


(b) $\theta = 0.075$ rad

Fig. 2. Comparison of $\mathcal{A}_{1,1}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,1}^{Conv}(k,\theta)$ at Doppler shifts (a) $\theta = 0.05$ rad and (b) $\theta = 0.075$ rad.

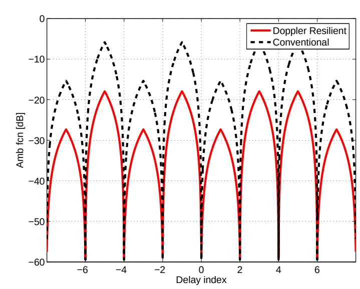


(a) $\mathcal{A}_{1,2}^{PTM}(k,\theta)$

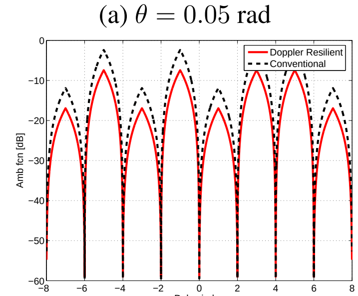


(b) $\mathcal{A}_{1,2}^{Conv}(k,\theta)$

Fig. 3. The plot of (a) $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ and (b) $\mathcal{A}_{1,2}^{Conv}(k,\theta)$ versus delay index k and Doppler shift θ.



(a) $\theta = 0.05$ rad



(b) $\theta = 0.075$ rad

Fig. 4. Comparison of $\mathcal{A}_{1,2}^{PTM}(k,\theta)$ and $\mathcal{A}_{1,2}^{Conv}(k,\theta)$ at Doppler shifts (a) $\theta = 0.05$ rad and (b) $\theta = 0.075$ rad.