# Multi-Rank Adaptive Beamforming with Linear and Quadratic Constraints

Henry Cox,[1] Ali Pezeshki,[2] Louis L. Scharf,[2;3] Olivier Besson,[4] and Hung Lai[1]
[1]Lockheed Martin Orincon Defense
4350 N. Fairfax DR., Ste. 470, Arlington, VA 22203
[2]Department of Electrical and Computer Engineering, [3]Department of Statistics
Colorado State University, Fort Collins, CO 80523
[4]Department of Avionics and Systems, ENSICA, Toulouse, FR

*Abstract* – **Extensions of MVDR and Capon estimation techniques are presented for the situation in which the signal is either rank-one of unknown orientation in a subspace or multi-rank. Only signal-plus-noise snapshots are available. The relationships among linearly and quadratically-constrained approaches are clarified and a unified treatment is given that includes both direct and sidelobe canceller architectures. The unifying component is the multi-rank MVDR beamformer followed by post processing. Detection statistics are presented for the situation in which there is no signal-free training data. Simulations are used to compare rank-one and multi-rank performance.**

## I. INTRODUCTION

The standard beamforming problem assumes that the signal is rank-one corresponding to ideal propagation from a point source to a receiving array. When only samples of signal-plus-noise are available, one is led to the well known MVDR beamformer or the closely related Capon [1] power estimator. These solutions involve the inverse of the signal-plus-noise covariance matrix, multiplied by a steering vector that specifies the relative amplitude and phase at each element of the array due to a signal propagating from the assumed source location. Usually the covariance matrix is based on measurements and the steering vector is assumed known based on a model of propagation from the assumed location to the array. Signal mismatch occurs when the assumed signal vector differs from the actual signal amplitude and phase pattern on the array. This leads to signal suppression [2]. A number of techniques have been used to limit signal suppression due to mismatch in rank-one beamformers. These include diagonal loading of the covariance matrix, white noise gain or norm constraint which places a quadratic inequality constraint on the norm of the weight vector resulting in angle dependent diagonal loading, use of multiple linear constraints, and combinations [3, 4, 15]. The common assumption is that the signal is rank-one or at least that the beamformer performs a vector multiply on the data.

There are many cases in which information about the signal "direction" is not well represented by a single vector [5]. An example is coherent multi-path when the signal is actually rank-one but the phases among the multi-paths are unknown. In this situation, information about the directions of the multi-paths defines a sub-space but the location of the signal in the subspace is unknown [11-13]. Sometimes the signal is truly multi-rank such as when the multi-paths are incoherent or when there is motion during the observation interval such as a source that moves, a moving receiving array, or propagation through a random medium whose characteristics vary with space and time. Under these conditions, the signal is more properly represented by a signal covariance matrix that does not have rank-one [5, 7, 8]. The eigenvectors of this signal covariance matrix span the signal subspace. Its eigenvalues determine how well the subspace may be approximated by a subspace of lower rank. In non-stationary situations there is a tradeoff between sample support and covariance matrix smearing. This leads to rank deficient sample covariance matrices, especially for large arrays.

In this paper we describe the relationships among a number of variations of multi-rank formulations including linear constraints, quadratic constraints, linear constraints plus a norm constraint. Alternative processor structures are presented including sidelobe cancellers. Finally, detection approaches are presented for both the rank-one signal or steered direction case and the multi-rank signal case. Simulations are used to compare the performance of rank-one and multi-rank processors.

## II. MULTI-RANK MVDR RELATIONSHIPS

### A. Linear Constraints

Consider a single frequency bin possibly containing an unknown signal **Us** that belongs to a known M-dimensional "signal" subspace. The subspace is spanned by the (not necessarily orthogonal) columns of the N by M **known** matrix **U**

$$\mathbf{x} = \mathbf{U}\mathbf{s} + \mathbf{n} \qquad (1)$$

The signal rank is unknown. It may be as small as one, as large as M, or in between. Usually, M <<N. Given a data vector **x** with N by N sample covariance matrix **R** based on L data-snapshots, we would like to build a multi-rank beamformer of the form **W*x** that will pass signals that belong to the subspace without distortion and minimize the total output power. **W** is N by M. We denote matrices by bold upper case letters and column vectors by bold lower case letters. The asterisk is used to denote complex conjugate transposition. There is no signal-free training data. Noise-plus-interference is present in the data and represented by **n**. The problem can be written as:

$$\text{Min } \mathrm{tr}\{\mathbf{W}^*\mathbf{R}\mathbf{W}\} \qquad (2)$$

subject to multiple linear constraints

$$\mathbf{W}^*\mathbf{U} = \mathbf{I} \qquad (3)$$

Here, **I** is the M by M identity matrix. This distortionless constraint controls the amplitudes and phases of the outputs. It permits subsequent coherent processing, combination across frequency bins and examination of the signal subspace to determine the characteristics of the signal after removal of interference that leaks into the signal subspace. We shall initially proceed as if **R** were full rank and later consider the rank deficiency.

This problem is a special case of the formulation of [3] where a more general constraint matrix **A*** was used in place of the distortionless **I** in (3). The matrix **A*** is a design parameter. If **W** is the solution with the distortionless constraint, **WA** is the solution with **I** replaced by the more general **A***. That is, the distortionless weight matrix **W*** is applied to the data followed by the constraint matrix **A***. We shall return to the more general case later. The solution is given in [3]:

$$\mathbf{W} = \mathbf{R}^{-1}\mathbf{U}[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1} \qquad (4)$$

Here $\mathbf{R}^{-1}$ is N by N and may be used for any signal subspace matrix **U,** while $[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1}$ which is different for each **U** is only M by M. This formulation is especially advantageous in applications in which **U** is scanned across a range of possible signal subspaces. Applying (4) to a data snapshot,

$$\hat{\mathbf{s}} = \mathbf{W}^*\mathbf{x} = [\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1}\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{x} \qquad (5)$$

The output $\hat{\mathbf{s}}$ is an estimate of the signal, the part of **x** in the signal subspace. This is called the Minimum Variance Distortionless Response (MVDR) estimate. When the weight matrix is applied to snapshots that were used to form the sample covariance matrix, the procedure is said to be inbred. Then,

$$\mathrm{tr}\{\mathbf{W}^*\mathbf{R}\mathbf{W}\} = \mathrm{tr}\{\,[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1}\,\} \qquad (6)$$

is the multi-rank Capon estimate of the total power in the signal subspace. This estimator has negative bias because of inbreeding [6]. Because interference has been suppressed by the $\mathbf{R}^{-1}$ operation in determining $\hat{\mathbf{s}}$ in (5), the output covariance matrix provides information about the rank of the signal and its location in the subspace of the signal. Thus, it is useful to examine its eigen-decomposition.

$$\mathbf{W}^*\mathbf{R}\mathbf{W} = [\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{V}^* \qquad (7)$$

Here the columns of **V** are the eigenvectors and **Σ** is a diagonal matrix of the eigenvalues. The number of dominant eigenvalues is indicative of the rank of the signal and the corresponding eigenvectors span the output signal sub-space. From (7) it follows that

$$\mathbf{V}^*\mathbf{W}^*\mathbf{R}\mathbf{W}\mathbf{V} = \mathbf{V}^*[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1}\,\mathbf{V} = \boldsymbol{\Sigma} \qquad (8)$$

The product **WV** can be viewed as the multi-rank MVDR beamformer **W*** applied to the data**,** followed by an eigenvector beamformer **V*** in the signal sub-space. **WV** is also the solution to the more general case of linear constraints with **V*** = **A*.** If **V** had been known initially it could have been used as a constraint. Fig.1 illustrates the structure of the multi-rank MVDR beamformer followed by post processing.

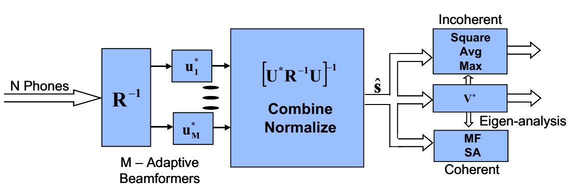


Fig. 1. Multi-rank MDVR beamformer , with post processing

*B. Quadratic Constraints*

A closely related problem with a quadratic constraint was considered in [7], [8], [14]

$$\text{Min } \mathrm{tr}\{\mathbf{H}^*\mathbf{R}\mathbf{H}\} \qquad \text{u.c.} \qquad \mathbf{H}^*\mathbf{U}\mathbf{U}^*\mathbf{H} = \mathbf{D} \qquad (9)$$

The positive definite diagonal matrix **D** is a design parameter. A solution to (9) is

$$\mathbf{H} = \mathbf{R}^{-1}\mathbf{U}[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1}\,\mathbf{V}\,\mathbf{D}^{1/2} = \mathbf{W}\,\mathbf{V}\,\mathbf{D}^{1/2} \qquad (10)$$

In this purely quadratic problem, there is a free phase parameter so that multiplying **H** by exp(iφ) will not change the result. Here **W** is the multi-rank MVDR beamformer of (4) and **V** is given by (7). Thus, the solution to the quadratic problem of (9) is the MVDR beamformer followed by the eigenbeamformer, followed by a matrix of design parameters. Then

$$\mathrm{tr}(\mathbf{H}^*\mathbf{R}\mathbf{H}) = \mathrm{tr}(\boldsymbol{\Sigma}\mathbf{D}) \qquad (11)$$

This gives the same answer as the general linearly constrained problem when the linear constraint matrix **A** is set equal to $\mathbf{V}\mathbf{D}^{1/2}$. Because **Σ** in (11) is diagonal, only the diagonal terms of **D** contribute to the trace of the product **ΣD**. If **D** is also diagonal, then an **H** that solves (9) can be multiplied by a diagonal matrix diag.$\{\exp(i\varphi_j)\}$ and still be a solution of (9).

*C. Signal and Noise Terms*

When **s** and **n** are uncorrelated, **R** can be written in terms of its unknown signal and unknown interference-plus-noise components as follows

$$\mathbf{R} = \mathbf{U}\,\boldsymbol{\Gamma}\mathbf{U}^* + \sigma^2\mathbf{Q} \qquad (12)$$

The following matrix identities are useful in relating results expressed in terms of **R**, the signal-plus-interference-plus-noise covariance matrix, to those expressed in terms of the unknown interference-plus- noise covariance matrix **Q**

$$\mathbf{W}^*\mathbf{R}\mathbf{W} = [\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1} = \boldsymbol{\Gamma} + \sigma^2\,[\mathbf{U}^*\mathbf{Q}^{-1}\mathbf{U}]^{-1} \qquad (13)$$

$$\mathbf{W} = \mathbf{R}^{-1}\mathbf{U}[\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1} = \mathbf{Q}^{-1}\mathbf{U}[\mathbf{U}^*\,\mathbf{Q}^{-1}\mathbf{U}]^{-1} \qquad (14)$$

Thus, the same signal estimate $\hat{\mathbf{s}}$ would result from (5) if the interference-plus-noise covariance matrix **Q** had been known instead of the signal-plus-interference-plus-noise covariance matrix **R**. Moreover, when the noise-plus-interference is white so that **Q** is the identity matrix, **W** reduces to a conventional beamformer**.** It is important to recognize that for this property **R** is not required to be the identity matrix, only the noise component **Q** is. In the white noise case, (**Q** = **I**), the output covariance matrix (13) reduces to

$$\mathbf{W}^*\mathbf{R}\mathbf{W} = [\mathbf{U}^*\,\mathbf{R}^{-1}\mathbf{U}]^{-1} = \boldsymbol{\Gamma} + \sigma^2\,[\mathbf{U}^*\mathbf{U}]^{-1}, \qquad (15)$$

which is an estimate of the output signal covariance **Γ**.

*D. Orthogonal Decomposition*

The distortionless optimum weight matrix **W** can be decomposed into orthogonal components using the orthogonal projection operators

$$\mathbf{P}_\mathbf{u} = \mathbf{U}[\mathbf{U}^*\mathbf{U}]^{-1}\mathbf{U}^* \qquad \mathbf{P}_\perp = \mathbf{I} - \mathbf{P}_\mathbf{u} \qquad (16)$$

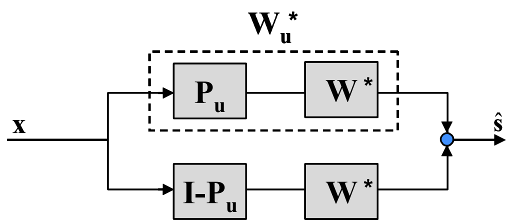


Fig. 2. MVDR structure with orthogonal decomposition.

$$\mathbf{W} = \mathbf{W_u} + \mathbf{W}_\perp \qquad (17)$$

Here $\mathbf{W_u}$ is the projection of **W** (4) onto the subspace spanned by the columns of **U.**

$$\mathbf{W_u} = \mathbf{P_u}\,\mathbf{W} = \mathbf{U}[\mathbf{U^*U}]^{-1}\mathbf{U^*}\,\mathbf{W} = \mathbf{U}[\mathbf{U^*U}]^{-1} \qquad (18)$$

This is the multi-rank conventional beamformer that is optimum in the white noise case. It is noteworthy that $\mathbf{W_u}$ does not depend on **R** and satisfies the constraint (3). $\mathbf{W_u}$ can be viewed as a set of parallel rank-one conventional beamformers each matched to a column of **U,** followed by a combination and normalization step. As illustrated in Fig.2, $\mathbf{W_U}$ and $\mathbf{W}_\perp$ can be thought of as two orthogonal parallel rank-M beamformers, a conventional beamformer and an adaptive beamformer that depends on **R**, is orthogonal to the columns of **U** and reduces sidelobe leakage into the signal subspace. Because the orthogonal projections are symmetric, an alternative interpretation is that the projections are applied to the data followed by **W***.

### *E. Norm-constraint and Multi-rank DMR*

From the orthogonal decomposition of (17), it follows that

$$\mathrm{tr}\{\mathbf{W^*W}\} = \mathrm{tr}\{\mathbf{W^*_u W_u}\} + \mathrm{tr}\{\mathbf{W^*_\perp W_\perp}\} \geq \mathrm{tr}\{\mathbf{W^*_u W_u}\} = \mathrm{tr}\,\{[\mathbf{U^*U}]^{-1}\} \qquad (19)$$

The ratio

$$\rho = \mathrm{tr}\{\mathbf{W^*W}\} \,/\, \mathrm{tr}\,\{[\mathbf{U^*U}]^{-1}\} \qquad (20)$$

represents the growth in the weight matrix over that for the white noise case, and is a measure of sensitivity to mismatch. A robust approach to controlling sensitivity to mismatch in adaptive processing is to add an additional quadratic inequality constraint that places an upper limit on ρ[3, 4]. This is called a "white noise gain" constraint or norm constraint. The norm-constrained multi-rank MVDR problem is

$$\mathrm{Min}\ \mathrm{tr}\{\mathbf{W^*RW}\}\ \ \mathrm{u.c.}\ \ \mathbf{W^*U} = \mathbf{I}\,,\ \ \mathrm{tr}(\mathbf{W^*W}) \leq \delta^2 \quad (21)$$

The solution to this problem can be expressed in terms of a diagonally loaded sample covariance matrix, $\mathbf{R}(\varepsilon) = (\mathbf{R} + \varepsilon\mathbf{I})$:

$$\mathbf{W}(\varepsilon) = (\mathbf{R} + \varepsilon\mathbf{I})^{-1}\mathbf{U}[\mathbf{U^*}(\mathbf{R} + \varepsilon\mathbf{I})^{-1}\,\mathbf{U}]^{-1} \qquad (22)$$

The value of the scalar diagonal loading parameter ε must be adjusted to satisfy the norm constraint. When **U** is steered with azimuth, this results in azimuthally dependent diagonal loading. This is common practice in applications of rank-one MVDR. An important variation on this theme, called Dominant Mode Rejection (DMR) [9], is to retain only the dominant subspace, the largest K eigenvectors, of **R** when computing weights in (22).

$$\mathbf{R_K}(\varepsilon_1,\varepsilon_2) = \Sigma_{i=1}^{K}\,\lambda_i\,\mathbf{e}_i\,\mathbf{e}_i^* + \varepsilon_1\mathbf{I} + \varepsilon_2\mathbf{I} \qquad (23)$$

Here $\varepsilon_1$ is a fixed loading parameter based on a noise estimate and $\varepsilon_2$ is varied to control the norm of the weights. DMR is especially useful when **R** is rank-deficient. The diagonal loading is only applied to **R** in estimating the weights and not to the data. When the orthogonal decomposition of **W** is used, the norm constraint can be applied to the adaptive component since the conventional beamformer component is fixed. This property was used in the development of the scaled projection algorithm [4]. Using $\mathbf{R_K}(\varepsilon_1,\varepsilon_2)$ for **R** in (4) to obtain $\mathbf{W_K}(\varepsilon_1,\varepsilon_2)$, the power estimator becomes

$$\mathrm{P} = \mathrm{tr}\{\mathbf{W^*_K}(\varepsilon_1,\varepsilon_2)\ \mathbf{R}\,\mathbf{W_K}(\varepsilon_1,\varepsilon_2)\} \qquad (24)$$

### *F. Multi-Rank Generalized Sidelobe Canceller*

We saw earlier that the optimum beamformer could be decomposed into two parallel beamformers: one of which was a conventional beamformer and an orthogonal beamformer that depends on **R.** The structure is illustrated by the diagram of Fig. 3. Rather than solving the constrained optimization problem given by (1) and (2), one can consider an equivalent unconstrained optimization problem for a beamformer that depends on **R** and operates on the subspace that is orthogonal to the column vectors of **U** [10]. Let **G** be an N-M by N blocking matrix whose columns span the subspace that is orthogonal to the columns of **U**. ($\mathbf{G^*U} = \mathbf{0}$). Then

$$\mathbf{W} = \mathbf{W_u} - \mathbf{GF} \qquad (25)$$

Substituting from (4) for **W**, pre-multiplying both sides of (25) **G*R** and noting the orthogonality of **G** and **U** yields:

$$\mathbf{F} = (\mathbf{G^*RG})^{-1}\,\mathbf{G^*R}\,\mathbf{W_u} \qquad (26)$$

A disadvantage of this formulation, in applications in which **U** is scanned, is that a new **G** and **F,** that involves the N-M by N-M inverse $(\mathbf{G^*RG})^{-1}$, must be obtained for each signal subspace matrix **U. F** is also the solution to the following unconstrained optimization problem. Choose **F** to minimize:

$$\mathrm{Min}\ \mathrm{tr}\{(\mathbf{W_u} - \mathbf{GF})^*\,\mathbf{R}\,(\mathbf{W_u} - \mathbf{GF})\} \qquad (27)$$

Let $\mathbf{y} = \mathbf{G^*x}$ be the output of the blocking matrix. The same solution for **F** can also be obtained by requiring **y** to be uncorrelated with the final output **ŝ.** That is

$$\mathrm{E}[\mathbf{y}\,\hat{\mathbf{s}}^*] = \mathbf{G^*R}[\mathbf{W_u} - \mathbf{GF}] = \mathbf{0} \qquad (28)$$

This in turn implies that the output covariance matrix can be expressed as the sum

$$\mathbf{W^*RW} = \mathbf{W_u}^*\mathbf{R}\,\mathbf{W_u} - \mathbf{F^*G^*RGF} \qquad (29)$$

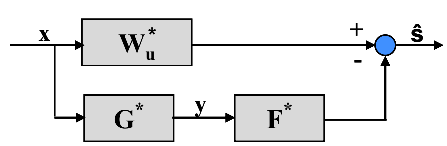


Fig. 3. Multi-rank generalized sidelobe canceller with null at CBF output.

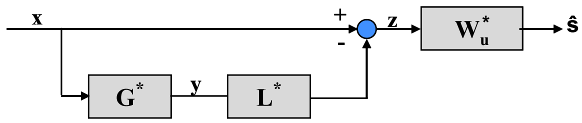


Fig. 4. Multi-rank generalized sidelobe canceller with null at CBF input

Noting that $\mathbf{W_u}$ appears as the last part of **F** makes it clear that rather than performing the cancellation at the output of the conventional processor $\mathbf{W_u}$, it could have been done at the input as shown in Fig. 4. The two figures are equivalent if

$$\mathbf{L^* = RG\,(G^*RG)^{-1}} \tag{30}$$

Let $\mathbf{K = L^*G^*}$ so that $\mathbf{W^* = W_u^*\,(I\text{-}K)}$. Earlier, we applied the orthogonal projections $\mathbf{P_u}$ and $\mathbf{I - P_u}$ to the weight matrix **W.** Here, **(I –K)** and **K** are oblique (non-symmetric) projections that are applied to the data **x.** The range and null space of (**I-K**) are **U** and **RG** respectively.

## III. Comparison with Rank-One MVDR

It is interesting to compare the multi-rank MVDR with a closely related rank-one MVDR with appropriately chosen multiple linear constraints. Consider the problem:

$$\text{Min } \mathbf{w^*Rw} \quad \text{u.c.} \quad \mathbf{w^*U = 1^*} \tag{31}$$

Here **1** a vector of ones that specifies unit gain with zero phase for signals that are perfectly matched to any linear combination of the columns of **U**. The solution is

$$\mathbf{w = R^{-1}U[U^*\,R^{-1}U]^{-1}1 = W\,1} \tag{32}$$

Thus, its output is the coherent sum of the components of the multi-rank MVDR shown in Fig. 1. The output power is

$$\mathbf{w^*Rw = 1^T[U^*\,R^{-1}U]^{-1}\,1} \tag{33}$$

This sum over all the elements of $\mathbf{[U^*\,R^{-1}U]^{-1}}$ compares to the trace or sum over the diagonal elements of the same matrix in the multi-rank case. For white noise, as seen from (15), this will give the same noise response as the multi-rank MVDR when the columns of **U** are orthogonal. The signal responses will be the same when **Γ** is diagonal.

## IV. Detection

Besson et al. [11-13] addressed a number of matched direction and adaptive matched direction problems for detecting rank-one signals of unknown orientation, using only primary data. In [13], primary and secondary data are used. Here, we consider situations in which signal-free training data is unavailable so that only **R,** an estimate of the signal-plus-noise covariance based on L snapshots, is available. Consider the problem.

$$\begin{aligned} H_1&:\ \mathbf{x = Us + n} \\ H_0&:\ \mathbf{x = n} \\ E(\mathbf{nn^*}) &= \mathbf{J} + \sigma^2\mathbf{I} \end{aligned} \tag{34}$$

Here **J** is an unknown covariance matrix of jammers and the white noise level $\sigma^2$ is also unknown. The subspace **U** is assumed known. The signal is random with unknown covariance matrix, E(**ss***) = **Γ.** Two situations are of interest: (1) the signal covariance matrix **Γ** is rank-one (or less than rank-M) and (2) the signal covariance matrix **Γ** is rank-M. In both cases we can use MVDR to reject interference and provide undistorted information about the eigen-structure of the signal subspace as given by (7). If **R** is rank-deficient, a norm constrained MVDR or DMR can be used.

### *A. Rank-one Signal ( Matched Direction)*

In the rank-one case, the largest eigen-value $\sigma_1$ is an estimate of signal-plus-noise power. An estimate of the noise power can be obtained from the other eigenvalues. The test statistic may be written as

$$Z = \sigma_1 / f(\mathbf{\Sigma}) \tag{35}$$

Here $f(\mathbf{\Sigma})$ is a suitable function of the eigenvalues such as the trace [12], the average or minimum of the remaining eigenvalues or their median. For finite sample support, the ratio Z will have a bias or non-zero response to noise due to the bias in the eigenvalue estimates. One eigenvalue will be the largest even if the input is white noise. This bias can be taken into account in setting a threshold. This approach can readily be applied to signals of rank-P less than M by comparing the first P eigenvalues with the rest.

### *B. Rank-M Signal*

In the rank-M case a biased estimate of the signal-plus-noise power is tr(**W*RW).** The problem is to obtain an estimate of the noise power. To get such an estimate we examine the eigenvalues, $\{\lambda_1,\ldots\}$ of the sample covariance matrix **R.** The rank of **R** will be the lesser of the number of snapshots L or the dimension of the matrix N. An examination of the eigen-spectrum will typically show a number of large eigenvalues associated with the jammers followed by a set of smaller eigenvalues due to noise. These eigenvalues are biased, especially so in the rank deficient case, L<N. When the number of jammers is less than one-half the rank of **R,** the median eigenvalue can be used to obtain an estimate of the noise power $\sigma^2$. In the rank deficient case, a useful estimate of $\sigma^2$ is L/N times the median eigen-value $\lambda_{L/2:}$

$$\text{Est}(\sigma^2) = (L/N)\,\lambda_{L/2} \tag{36}$$

The factor L/N is an approximate correction for the bias in estimating the eigenvalues of **R**. It accounts for the fact that the power on the N elements must be shared by the L eigenvalues. An estimate of the noise power in the signal subspace is tr(**W*W)** (L/N) $\lambda_{L/2}$. This leads to the following test statistic

$$Z = \text{tr}(\mathbf{W^*RW})\,/\text{tr}(\mathbf{W^*W})(L/N)\,\lambda_{L/2} \tag{37}$$

This test statistic is biased due to the bias of the inbred Capon estimator and a small error in the approximate correction for noise bias. Again this can be accounted for in setting the detection threshold. This same detector can also be used when the signal subspace is rank-one and the weight matrix becomes the rank-one MVDR weight vector.

## V. Simulations

Simulations were conducted for a 128 element line array uniformly spaced at one-half wavelength in 0dB white noise with seven sources, including far-field point sources, coherent and incoherent multi-paths, near-field and moving as listed in

TABLE I
SOURCE CHARACTERISTICS

| Type | Strength | Location |
|---|---|---|
| Point | 10 dB | u = - 0.8 |
| Point | -10 dB | u = - 0.5 |
| Point | 20 dB | u = + 0.8 |
| Moving | 10 dB | u = - 0.25(4beams) |
| Near-field | 20 dB | u = + 0.1 |
| Multi-path Coherent | 20 dB | u = 0.42 (3 beams) |
| Multi-path Incoherent | 20 dB | u = 0.62 (4 beams) |

Table I. The results for five processors using only 128 snapshots are shown in Fig. (5). The multi-rank processors assumed a five beam spread of the signal when the multi-rank signals were spread at most over only four beams, so that there was some mismatch. The rank-one detector used the average of the remaining eigenvalues as the noise estimate. The subspace **U** was scanned across angle.

The three lowest curves correspond to rank-one conventional (green), multi-rank (rank-five) DMR using 15 eigenvectors (light blue) and a 3dB norm constraint, and a corresponding rank one DMR (black). The conventional beamformer is sidelobe limited. Its output only reaches a minimum level of -10dB rather than the noise-only response of -21dB (10 log 128). The point source at u = -0.5 is nearly lost in sidelobes. The peak splits for the coherent multi-path signal. The rank-one DMR fails completely on the coherent multi-path and near-field sources. It achieves the desired noise response of -21 dB. The rank-five DMR has a noise level of -14dB that results from a 7 dB increase due to the integration over the rank-five sub-space. It responds well to all sources with a broadened peak for point sources. The two top curves are the detection statistics for the rank-one (blue) and rank-five (red) signal assumptions. The rank-one detector has a small bias of about 2db between its peaks. It responds well to all rank- one sources. It yields split peaks for multi-rank signals as the subspace is scanned to the point where only one signal eigenvector lies in the assumed signal subspace. The rank-five detector has a lower response to rank-one signals due to noise integration in the assumed signal subspace. It responds well to the multi-rank signals without peak splitting.

Fig. (6) presents similar results for the severely rank-deficient case of only 32 snapshots for a 128 element array. The multi-rank detectors have increased noise level bias due to the smaller snapshot support. Otherwise the results are very similar to the 128 snapshot case showing the effectiveness of norm constrained techniques in snapshot starved situations. Fig. (7) shows ample sample support with 2000 snapshots. Bias and variability between sources are further reduced.

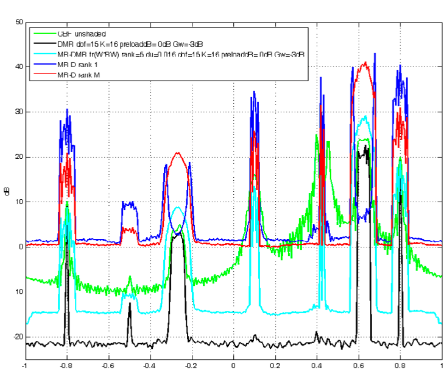

Fig. 5. Response for 128 elements line array using 128 snapshots

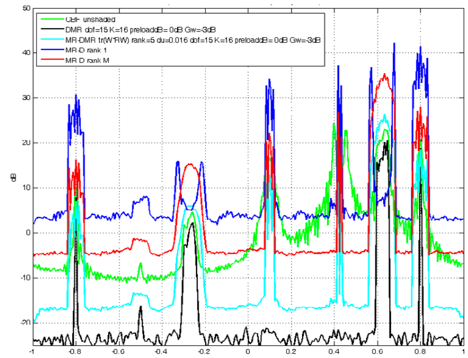

Fig. 6. Response for 128 element line array using 32 snapshots

## VI. CONCLUSIONS

Multi-rank signal models are appropriate to many real-world situations. A unifying treatment of multi-rank adaptive beamforming has been presented that includes extension of most rank-one techniques. The multi-rank MVDR beamformer is the appropriate first step in a variety of processors. Detection approaches have been presented for situations in which there are no signal-free training data to develop noise-only statistics. Results of realistic simulations show the utility of multi-rank techniques, even in severely sample starved situations.

## ACKNOWLEDGMENT

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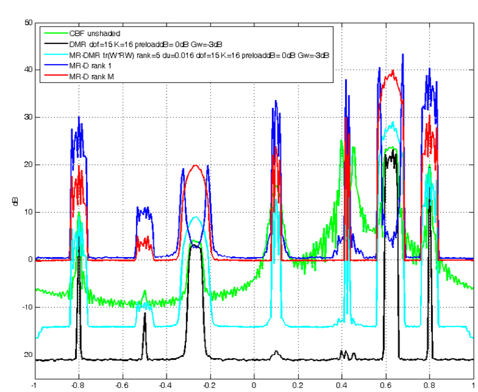

Fig. 7. Response for 128 elements line array using 2000 snapshots

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