Sidelobe Suppression in a Desired Range/Doppler Interval

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***Abstract*—We present simple methods for constructing radar waveforms whose ambiguity functions are free of sidelobes inside a desired range or Doppler interval. We exploit the time-frequency duality between pulse amplitude modulation (PAM) and orthogonal frequency division multiplexing (OFDM) to sequence Golay complementary codes across time or frequency and clear out range/Doppler sidelobes. Proper sequencing of complementary codes in time (PAM design) enables the annihilation of range sidelobes along a desired Doppler interval. The dual design, i.e., OFDM signaling of complementary codes, enables the annihilation of Doppler sidelobes along a desired range interval. The two designs can be used sequentially to bring weak targets out of the sidelobes of nearby strong reflectors inside a range-Doppler interval of interest.**

***Index Terms*—Golay complementary sequences; OFDM/PAM signalling; Sidelobe suppression, Thue-Morse sequences**

I. Introduction

In radar, we illuminate a scene with a waveform and then matched filter the radar return with the transmit waveform to resolve targets in range and Doppler. Under the point scatter assumption, much of the analysis can be performed by looking at the ambiguity function of the transmit waveform. Ideally, we wish to have an impulse ambiguity function that is completely free of sidelobes in range and Doppler, as the the presence of sidelobes means that weak target can be masked by nearby strong reflectors. However, as Moyal identity [1] shows, this is impossible and the best we can hope for is a "thumbtack" ambiguity function, where the width of the thumbtack in range and in Doppler is restricted by the time-bandwidth product of the waveform. The design of waveforms with thumbtack ambiguity functions has been one of the main focuses in radar signal processing over the past six decades. The reader is referred to [1]–[4] and the references therein for a review of some of the relevant literature. Conventional approaches to waveform design either exploit the available time-bandwidth product to achieve a desired ambiguity response over the *entire* range-Doppler plane, or they solve an optimization problem to match to a presumed signal-plus-clutter model. These approaches are generally complicated, seldom result in closed form solutions, and often lead to non-unimodular waveforms that do not satisfy the power constraints of the transmitter.

Asking for a true thumbtack ambiguity response is restrictive and often not necessary. Rather it is sufficient to have an ambiguity response that is thumbtack-like inside the range-Doppler region that is of interest for imaging. This approach is particularly important to the operation of future generations of radar systems where waveform design and target detection/tracking are expected to be performed in a closed loop fashion in real-time to enable adaptive operation. "Pushing sequences" introduced in [5] and a set of Costas sequences [5],[6] can produce waveforms for which the ambiguity functions are free of sidelobes inside a range-Doppler neighborhood around zero. These approaches rely on frequency hopping at a fast rate and in general cannot annihilate sidelobes inside an arbitrary range-Doppler interval.

Our aim in this paper is to develop a systematic way for clearing out range/Doppler sidelobes inside a Doppler or range region of interest. Recently, we showed [7]–[9] that by proper sequencing of only two complementary waveforms across time it is possible to achieve an ambiguity response that is free of range sidelobes inside a desired Doppler interval. The complementary codes used are Golay complementary codes [10]–[12] invented Marcel Golay, which have the property that the sum of their autocorrelation functions is an impulse in delay. These codes were rediscovered by Welti [13] and were proposed for use in pulsed radar but they never found wide applicability due to their sensitivity to Doppler effect [1],[14]. We showed [8]–,[15] that if the transmission of a Golay pair of phase coded waveforms in time is coordinated according to the location of zeros and ones in a binary sequence, then the magnitude of the

This work is supported in part by the NSF under grant 0701226, by the ONR under grant N00173-06-1-G006, and by the AFOSR under grant FA9550-05-1-0443.

range sidelobes of the pulse train ambiguity function will be proportional to the magnitude spectrum of the binary sequence. Range sidelobes inside a desired Doppler interval are suppressed by selecting a binary sequence whose spectrum has a high-order null at a Doppler frequency inside the desired interval. We showed that the spectra of *Prouhet-Thue-Morse (PTM) sequences* have high-order nulls at all rational (in multiple of 2π) Doppler shifts. T

In this paper we extend this result to produce impulse-like ambiguity responses in *Doppler* over desired *range* intervals. The key idea is to view the PTM pulse trains of [7]–[9] as pulse amplitude modulation (PAM) waveforms and to exploit the time-frequency duality between PAM and orthogonal frequency division multiplexing (OFDM). The pulse trains reported in [7]–[9] are indeed PAM designs, and they are constructed by amplitude modulating a narrow pulse shape by a PTM sequence of Golay codes. Depending on the choice of the PTM sequence, the PAM waveform produces an impulse-like ambiguity response in delay (time shift) over a particular Doppler interval. If the very same PAM design is used in the frequency domain then the ambiguity response will be an impulse in *Doppler* (frequency shift) across a *range* interval. This frequency-domain PAM signal is an OFDM signal in the time domain, obtained by stacking the PTM sequence of Golay complementary codes across different frequency tones. In other words, if we have a good PAM design in time that annihilates range sidelobes inside a Doppler interval we can build an OFDM signal from it that annihilates Doppler sidelobes inside a range interval, and vice versa.

A key feature of our dual designs is their simplicity. Our waveform library consists of only a pair of complementary component waveforms and the rest are enabled by properly sequencing these components in time or in frequency. Thus, the waveform generators required at the transmitter can be considerably simpler than those required by waveform agile schemes (e.g. see [3],[4],[16]) that exercise a large waveform library. Unlike Pushing and Costas sequences, our dual designs do not exploit frequency hopping.

II. Impulse-like Ambiguity Response in Range over a Desired Doppler Interval

In this section, we review the key results of [7]–[9] to lay a foundation for our new developments, which will be discussed in Section III.

Definition 1: Two length-L unimodular sequences of complex numbers x_ℓ and y_ℓ are Golay complementary if for $k = -(L-1), \ldots, (L-1)$ the sum of their autocorrelation functions satisfies

$$C_{x,k} + C_{y,k} = 2L\delta_k, \tag{1}$$

where $C_{x,k}$ is the autocorrelation of x_ℓ at lag k and δ_k is the Kronecker delta function. Each member of the pair (x, y) is called a Golay sequence or a Golay code.

The baseband waveform $x(t)$ phase coded by x_ℓ is given by

$$x(t) = \sum_{\ell=0}^{L-1} x_\ell s(t - \ell T_c) \tag{2}$$

where $s(t)$ is a unit energy pulse shape of chip duration T_c. The ambiguity function $\chi_x(\tau, \nu)$ of $x(t)$ is given by

$$\chi_x(\tau, \nu) = \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}e^{-j\nu t} dt \tag{3}$$

where τ is delay, ν is Doppler frequency, $\overline{x(t)}$ is the complex conjugate of $x(t)$.

If the complementary waveforms $x(t)$ and $y(t)$ are transmitted separately in time, with a T sec time interval between the two transmissions, then the ambiguity function of the radar waveform $z(t) = x(t) + y(t-T)$ can be approximated by

$$\chi_z(\tau, \nu) = \sum_{k=-(L-1)}^{L-1} [C_{x,k} + e^{j\nu T} C_{y,k}]\chi_s(\tau + kT_c, \nu). \tag{4}$$

where $\chi_s(\tau, \nu)$ is the ambiguity function of the pulse shape $s(t)$ and νT is the relative Doppler shift over a PRI. Along the zero-Doppler axis ($\nu = 0$), the ambiguity function $\chi_z(\tau, \nu)$ is "free" of range sidelobes. [1] Off the zero-Doppler axis however, the ambiguity function has large sidelobes in delay (range). The range sidelobes in the ambiguity function can cause masking of a weak target that is situated near a strong reflector. The reader is referred to [7]–[9] for more details and numerical results that show the sensitivity of Golay complementary waveforms to Doppler.

Let $\mathcal{P} = \{p_n\}_{n=0}^{N-1}$ be a discrete binary sequence of length N and consider the pulse train $z_{\mathcal{P}}(t)$ given by

$$z_{\mathcal{P}}(t) = \sum_{n=0}^{N-1} p_n x(t - nT) + \overline{p}_n y(t - nT) \tag{5}$$

where $\overline{p}_n = 1 - p_n$ is the complement of p_n, and $x(t)$ and $y(t)$ are Golay complementary waveforms. The nth entry in the pulse train is $x(t)$ if $p_n = 1$ and is $y(t)$

[1]The shape of the ambiguity function depends on the autocorrelation function $\chi_\Omega(\tau, 0)$ for the pulse shape $\Omega(t)$. The Golay complementary property eliminates range sidelobes caused by replicas of $\chi_\Omega(\tau, 0)$ at nonzero integer delays.

if $p_n = 0$. Consecutive entries in the pulse train are separated in time by a PRI T.

The discretized ambiguity function of $z_{\mathcal{P}}(t)$ can be expressed as [8],[15]

$$\chi_{z_{\mathcal{P}}}(k,\theta) = \frac{1}{2}[C_{x,k} + C_{y,k}] \sum_{n=0}^{N-1} e^{jn\theta} + \frac{1}{2}[C_{x,k} - C_{y,k}] \sum_{n=0}^{N-1} (-1)^{p_n} e^{jn\theta} \quad (6)$$

where $\theta = \nu T$ is the relative Doppler shift over a PRI T. The first term on the right-hand-side of (6) is free of range sidelobes. The second term represents the range sidelobes, as $C_{x,k} - C_{y,k}$ is not an impulse. The magnitude of the range sidelobes is proportional to the magnitude of the spectrum $S_{\mathcal{P}}(\theta)$ given by

$$S_{\mathcal{P}}(\theta) = \sum_{n=0}^{N-1} (-1)^{p_n} e^{jn\theta}. \quad (7)$$

Range sidelobes inside a desired Doppler interval can be suppressed by selecting a binary sequence whose spectrum has a high-order null at a Doppler frequency inside the desired interval.

We have shown [7]–[9] that the spectrum of a binary sequence, called the Prouhet-Thue-Morse (PTM) sequence [17],[18], has a high-order null around zero and hence it suppresses range sidelobes along modest Doppler shifts.

Definition 2: [17],[18] Consider the binary representation of an integer $n = \sum b_j 2^j$. The nth element, $n \in \mathbb{N}_0$, in a Prouhet-Thue-Morse (PTM) sequence $\mathcal{P} = (p_k)_{k\geq 0}$ over $\{0,1\}$ is equal to $\sum b_j$ modulo 2.

Oversampled PTM sequences zero-force the Taylor moments of the spectrum $S_{\mathcal{P}}(\theta)$ around rational (in multiples of 2π) Doppler shifts. An oversampled PTM sequence, say by a factor m, is obtained by repeating each 0 and 1 in the PTM sequence m times.

The ambiguity function of a PTM pulse train $z_{\mathcal{P}}(t)$ of Golay complementary waveforms behaves as

$$\chi_{z_{\mathcal{P}}}(\tau,\nu) \approx \delta(\tau)\alpha(\nu), \quad \forall \nu \in \vartheta \quad (8)$$

where $\alpha(\nu)$ is a function of Doppler frequency ν and ϑ is a Doppler interval in which $\mathcal{P}$ has spectral nulls. The reader is referred to [7]–[9] for details.

Figure 1 shows the ambiguity function of a length-$(N = 2^8)$ PTM pulse train of Golay complementary waveforms, which has a seventh-order null at zero-Doppler. The horizonal axis is Doppler frequency in Hz and the vertical axis is delay in sec. The magnitude of the pulse train ambiguity function is color coded and presented in dB scale. We observe that the pulse train ambiguity function is effectively free of range sidelobes across a neighborhood of the zero-Doppler axis (approximately $[-300, 300]$ Hz). The range sidelobes in this region are approximately 80 dB below the peak of the ambiguity function. The Golay complementary waveforms used in this example are built via phase coding a raised cosine pulse with Golay complementary sequences of length 64. The chip length is $T_c = 100$ $n$sec, the carrier frequency is 17 GHz, and the PRI is $T = 50$ μsec.

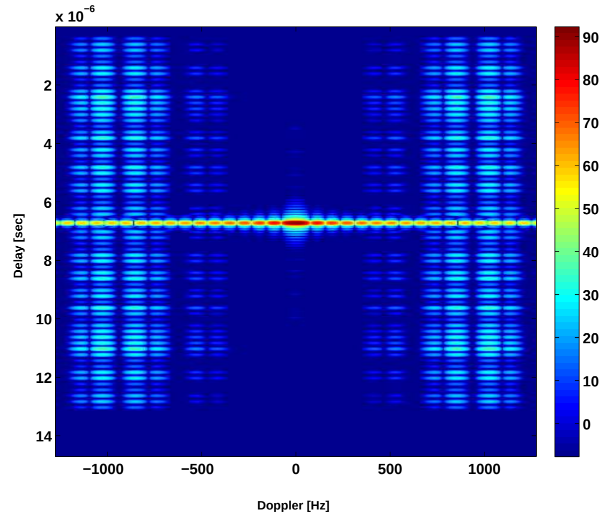

Fig. 1. Ambiguity function of a length-$(N = 2^8)$ PTM pulse train of Golay complementary waveforms is free of range sidelobes along modest Doppler frequencies.

III. Impulse-like Ambiguity Response in Doppler Over a Desired Range Interval

PTM sequencing of Golay complementary waveforms in time produces impulse-like ambiguity responses in range across desired Doppler intervals. We now show that a dual design, which involves PTM sequencing of Golay pairs across frequency tones, can be used to clear out Doppler sidelobes across a range interval of interest.

Consider the PTM pulse train $z_{\mathcal{P}}(t)$ given in (5). From here on we drop the subscript $\mathcal{P}$ and simply use $z(t)$. Define the sequence $\hat{x}_\ell$ as

$$\hat{x}_\ell = \begin{cases} x_\ell, & \ell = 0,1,\ldots,L-1 \\ 0, & \ell = L, L+1, \ldots, L+K-1 \end{cases} \quad (9)$$

where $K = [T/T_c]$. Similarly define $\hat{y}_\ell$ by zero-padding the Golay sequence y_ℓ to fill up a PRI. Then we can write $z(t)$ as

$$z(t) = \sum_{m=0}^{M-1} a_m s(t - mT_c) \quad (10)$$

where the sequence $\{a_m\}_{m=0}^{M-1}$ is obtained by concatenating the sequences $\hat{x}_\ell$ and $\hat{y}_\ell$ according to a PTM sequence $\{p_n\}$ (original or oversampled), with 0 corresponding to $\hat{x}_\ell$ and 1 corresponding to $\hat{y}_\ell$.

The PTM pulse train $z(t)$ is a time-domain PAM waveform, for which the ambiguity function behaves almost as an impulse in range across the Doppler interval ϑ:

$$\begin{aligned}\chi_z(\tau,\nu) &= \int_{-\infty}^{\infty} z(t)\overline{z(t-\tau)}e^{-j\nu t} \\ &\approx \delta(\tau)\alpha(\nu), \quad \forall \nu \in \vartheta \end{aligned} \tag{11}$$

The Doppler interval ϑ, in which we wish to clear the range sidelobes, determines which PTM sequence (orginal or an oversampled version) needs to be used.

Consider now the corresponding frequency-domain PAM waveform $z(\omega)$ constructed as

$$z(\omega) = \sum_{m=0}^{M-1} a_m s(\omega - m\omega_c) \tag{12}$$

where ω_c denotes the duration of the frequency-domain pulse shape $s(\omega)$. The ambiguity function of $z(\omega)$ is given by

$$\begin{aligned}\chi_z(\nu,\tau) &= \int_{-\infty}^{\infty} z(\omega)\overline{z(\omega-\nu)}e^{-j\tau\omega}d\omega \\ &\approx \delta(\nu)\alpha(\tau), \quad \forall \tau \in \Delta \end{aligned} \tag{13}$$

Here, Doppler frequency (frequency shift) ν plays the same role, with respect to global frequency ω, that the time delay (time shift) τ plays with respect to global time t. The delay interval Δ plays the role of the Doppler interval ϑ in (11), and determines which PTM sequence and hence PAM design must be used in the frequency domain.

Comparing (11) and (13), we observe that the time-domain PAM design that clears out range sidelobes will clear out Doppler sidelobes if it is performed in the frequency domain. However, the frequency-domain PAM signal in (12) is a time-domain OFDM signal given by

$$Z(t) = \left(\sum_{m=0}^{M-1} a_m e^{-jm\omega_c t}\right) S(t) \tag{14}$$

where $Z(t) = \mathcal{F}^{-1}\{2\pi z(-\omega)\}$ and $S(t) = \mathcal{F}^{-1}\{2\pi s(-\omega)\}$. It is easy to verify that the ambiguity function $\chi_z(\nu,\tau)$ in (13) can be written as

$$\begin{aligned}\chi_z(\nu,\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z(t)\overline{z(\omega-\nu)}e^{j\omega(t-\tau)}d\omega dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(t)\left[\overline{\int_{-\infty}^{\infty} z(-\omega)e^{j\omega(t-\tau)}d\omega}\right] dt \\ &= \frac{e^{-j\nu\tau}}{2\pi}\int_{-\infty}^{\infty} Z(t)\overline{Z(t-\tau)}e^{j\nu t}dt \\ &\approx \alpha(\tau)\delta(\nu), \quad \forall \tau \in \Delta. \end{aligned} \tag{15}$$

The last integral on the right-hand-side of (15) is the ambiguity function $\chi_z(\tau,-\nu)$ of the OFDM waveform $Z(t)$. We observe that the time-domain OFDM waveform $Z(t)$, obtained by stacking Golay complementary codes over OFDM frequencies according to the PTM sequence, can produce an impulse-like ambiguity response in Doppler across a range interval of interest.

The above analysis shows that if we have a good PAM design in time (or OFDM design in frequency) that annihilates range sidelobes inside a Doppler interval we can build a time-domain OFDM signal (or a frequency-domain PAM signal) from it that annihilates Doppler sidelobes inside a range interval, and vice versa. An example for these dual designs is shown in Fig. 2.

IV. Conclusions

In this paper, we extended the results of [7]–[9] to dual domain to design waveforms whose ambiguity functions are free of Doppler sidelobes across a range interval of interest. This was accomplished by exploiting the time-frequency duality between PAM and OFDM waveforms. The pulse trains designed in [7]–[9] for range sidelobe suppression employ PTM sequences of Golay complementary codes to amplitude modulate a pulse shape in time. Depending on the choice of the PTM sequence, the resulting PAM waveform annihilates range sidlobes inside some Doppler interval. The dual waveforms constructed in this paper stack a PTM sequence of Golay complementary codes across OFDM tones to annihilate Doppler sidelobes in a range interval of interest. Sequential application of the two designs can be used to suppress range sidelobes and Doppler sidelobes in successions. The proposed designs are simple and only require a phase code library with two components that form a complementary pair.

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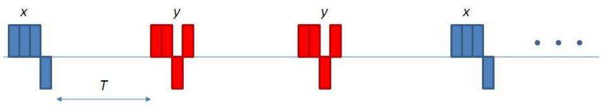
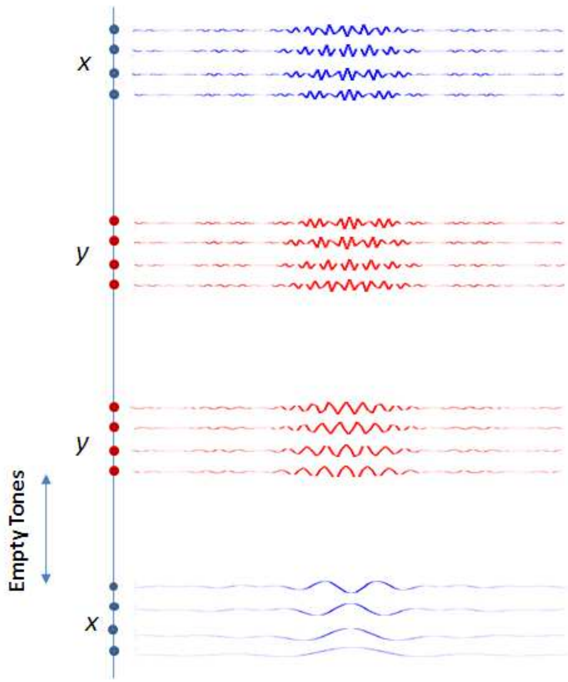


(a) Time-domain PAM



(b) Time-domain OFDM

Fig. 2. Dual (PAM and OFDM) designs for sidelobe suppression. (a) Time-domain PAM (or frequency-domain OFDM) design for range sidelobe suppression. The pulse $s(t)$ used in PAM is a square pulse. (b) Time-domain OFDM (or frequency-domain PAM) design for Doppler sidelobe suppression. Each OFDM subcarrier transmits a modulated $sinc$ function $a_m e^{-jm\omega_c t} S(t)$. The empty tones correspond to the locations where a_m is zero. The Golay complementary sequences used in this example are $\{x_\ell\} = \{1, 1, 1, -1\}$ and $\{y_\ell\} = \{1, 1, -1, 1\}$. Sequencing of x and y across time or frequency is performed according to the PTM sequence $01101001\ldots$, with 0 corresponding to x and 1 corresponding to y.

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