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Robust dimension reduction, fusion frames, and Grassmannian packings

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Abstract

We consider estimating a random vector from its measurements in a fusion frame, in presence of noise and subspace erasures. A fusion frame is a collection of subspaces, for which the sum of the projection operators onto the subspaces is bounded below and above by constant multiples of the identity operator. We first consider the linear minimum mean-squared error (LMMSE) estimation of the random vector of interest from its fusion frame measurements in the presence of additive white noise. Each fusion frame measurement is a vector whose elements are inner products of an orthogonal basis for a fusion frame subspace and the random vector of interest. We derive bounds on the mean-squared error (MSE) and show that the MSE will achieve its lower bound if the fusion frame is *tight*. We then analyze the robustness of the constructed LMMSE estimator to erasures of the fusion frame subspaces. We limit our erasure analysis to the class of tight fusion frames and assume that all erasures are equally important. Under these assumptions, we prove that tight fusion frames consisting of equi-dimensional subspaces have maximum robustness (in the MSE sense) with respect to erasures of one subspace among all tight fusion frames, and that the optimal subspace dimension depends on signal-to-noise ratio (SNR). We also prove that tight fusion frames consisting of equi-dimensional subspaces with equal pairwise chordal distances are most robust with respect to two and more subspace erasures, among the class of equi-dimensional tight fusion frames. We call such fusion frames *equi-distance tight fusion frames*. We prove that the squared chordal distance between the subspaces in such fusion frames meets the so-called *simplex bound*, and thereby establish connections between equi-distance tight fusion frames and *optimal Grassmannian packings*. Finally, we present several examples for the construction of equi-distance tight fusion frames.

Keywords: Distributed processing; Fusion frames; Grassmannian packings; LMMSE estimation; Packet encoding; Parallel processing; Robust dimension reduction; Subspace erasures

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1. Introduction

The notion of a *fusion frame* (or *frame of subspaces*) was introduced by Casazza and Kutyniok in [7] and further developed by Casazza et al. in [8]. A fusion frame for $\mathbb{R}^M$ is a finite collection of subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{R}^M$ such that there exist constants $0 < A \leqslant B < \infty$ satisfying

$$A\|\mathbf{x}\|^2 \leqslant \sum_{i=1}^{N} \|\mathbf{P}_i\mathbf{x}\|^2 \leqslant B\|\mathbf{x}\|^2, \quad \text{for any } \mathbf{x} \in \mathbb{R}^M,$$

where $\mathbf{P}_i$ is the orthogonal projection onto $\mathcal{W}_i$. Alternatively, $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame if and only if

$$A\mathbf{I} \leqslant \sum_{i=1}^{N} \mathbf{P}_i \leqslant B\mathbf{I}. \tag{1}$$

The constants A and B are called (*fusion*) *frame bounds*. An important class of fusion frames is the class of *tight fusion frames,* for which A and B can be chosen to be equal and hence $\sum_{i=1}^N \mathbf{P}_i = A\mathbf{I}$. We note that the definition given in [7] and [8] for fusion frames applies to closed and weighted subspaces in any Hilbert space. However, since the scope of this paper is limited to non-weighted subspaces in $\mathbb{R}^M$, the definition of a fusion frame is only presented for this case.

A fusion frame can be viewed as a frame-like collection of low-dimensional subspaces. In frame theory, an input signal is represented by a collection of *scalars*, which measure the magnitudes of the projections of the signal onto frame vectors, whereas in fusion frame theory an input signal is represented by a collection of *vectors*, whose elements are the inner products of the signal and the orthogonal bases for the fusion frame subspaces. Similar to frames, fusion frames can be used to provide a redundant and non-unique representation of a signal. In fact, in many applications, where data has to be processed in a distributed manner by combining several locally processed data vectors, fusion frames can provide a more natural mathematical framework than frames. A few examples of such applications are as follows.

Distributed sensing. In distributed sensing, typically a large number of inexpensive sensors are deployed in an area to measure a physical quantity such as temperature, sound, vibration, pressure, etc., or to keep an area under surveillance for target detection and tracking. Due to practical and economical factors, such as low communication bandwidth, limited signal processing power, limited battery life, or the topography of the surveillance area, the sensors are typically deployed in clusters, where each cluster includes a unit with higher computational and transmission power for local data processing. Thus, a typical large sensor network can be viewed as a redundant collection of subnetworks forming a set of subspaces. The gathered subspace information is submitted to a central processing station for joint processing. Some references that consider fusion frames for distributed sensing are [16,17] and [9].

Parallel processing. If a frame system is simply too large to handle effectively (from the numerical standpoint), we can divide it into multiple small subsystems for more simple, and perhaps parallelizable, processing. By introducing redundancy, when splitting the large system, we can introduce robustness against errors due to failure of a subsystem. Fusion frames provide a natural framework for splitting a large frame system into smaller subsystems and then recombining the subsystems. The use of fusion frames for parallel computing has been considered in [1].

Packet encoding. In digital media transmission, information bearing source symbols are typically encoded into a number of packets and then transmitted over a communication network, e.g., the internet. The transmitted packet may be corrupted during the transmission or completely lost due to, for example, buffer overflows. By introducing redundancy in encoding the symbols, according to an error-correcting scheme, we can increase the reliability of the communication scheme. Fusion frames, as redundant collections of subspaces, can be used to produce a redundant representation of a source symbol. In the simplest form, we can think of each fusion frame measurement as a packet that carries some new information about the symbol. At the destination the packets can be decoded jointly to recover the transmitted symbol. The use of fusion frames for packet encoding is considered in [2].

The optimal reconstruction (in ℓ_2 norm sense) of a deterministic signal $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements is considered in [8]. In this paper, we consider the linear minimum mean-squared error (LMMSE) estimation (cf. [18, Chapter 8]) of a *random vector* $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements, in presence of additive white noise and subspace erasures. Each fusion frame measurement is a low-dimensional (smaller than M) vector whose elements are inner products of $\mathbf{x}$ and an orthogonal basis for a fusion frame subspace. As far as we know, optimal (in the mean-squared error sense) linear estimation of a random vector from its fusion frame measurements (or even frame measurements) in presence of erasures has not been considered before, despite the fact that random vectors provide a natural way of modeling signals in many applications.

Erasures of subspaces can occur due to many factors in practice. In the distributed sensing example, a subspace erasure can occur due to a faulty or out of battery cluster of sensors, or due to loss of data during the transmission of local subspace information to the central processor. In scenarios where one or more sensor clusters are believed to be out of range for measuring the signal, or blocked by obstacles, their corresponding subspaces can be discarded on purpose. In the parallel processing example, an erasure can occur when a local processor crashes. In the packet encoding example, an erasure can occur when buffers in the network overflow.

Constructing frames that allow for robust reconstruction of a *deterministic signal* in the presence of frame element erasures has been considered by a number of authors. In [13], Goyal et al. show that a normalized frame is optimally robust against noise and one erasure (erasure of one element of the frame) if the frame is tight. Some ideas concerning multiple erasures were also presented. The work of Casazza and Kovačević [5] focuses mainly on designing frames, which maintain completeness under a particular number of erasures. Holmes and Paulsen [15] and Bodmann and Paulsen [3] study the robustness of frames under multiple erasures and show that maximal robustness with respect to the worst-case (maximum) Euclidean reconstruction error is achieved when the frame elements are equi-angular. The connection between equi-angular frames and equi-angular lines has also been explored by Sustik et al. in [20] and by Strohmer and Heath in [19], where the so-called Grassmannian frames are introduced.

There are also a few papers that consider the construction of fusion frames for robust reconstruction of *deterministic signals* in the presence of subspace erasures. The main result in this context is due to Bodmann [2], who shows that a tight fusion frame is optimally robust against one subspace erasure if the dimensions of the subspaces are equal. He also proves that a tight fusion frame is optimally robust against multiple erasures if the subspaces satisfy the so-called *equi-isoclinic* condition. The performance measure considered in [2] is the worst-case (maximum) Euclidean reconstruction error. The equi-isoclinic condition requires all pairs of subspaces to have the same set of principal angles. The single erasure case discussed in [2] has also been studied by Casazza and Kutyniok in [6]. We emphasize that all the above work on robustness with respect to erasures in frames and fusion frames deal with the case where the signal of interest is deterministic.

In this paper, we consider the LMMSE estimation of a zero-mean random vector $\mathbf{x} \in \mathbb{R}^M$ from its fusion frame measurements in presence of additive white noise and subspace erasures. We limit our analysis to the case where the signal covariance matrix $\mathbf{R}_{xx} = E[\mathbf{x}\mathbf{x}^T]$ is $\mathbf{R}_{xx} = \sigma_x^2\mathbf{I}$. The case of a general $\mathbf{R}_{xx}$ is more involved and is outside the scope of this paper.

We first derive bounds on the MSE in the absence of erasures and show that the lower bound will be achieved if the fusion frame is *tight*. We then analyze the effect of subspace erasures on the performance of LMMSE estimators. We determine how the MSE of an LMMSE estimator, constructed based on the second-order statistics of the data in the absence of erasures, is affected by erasures. We restrict our analysis to the class of tight fusion frames to maintain optimality under no subspace erasures. We further restrict our analysis to the case where all subspace erasures are equally important. In other words, we wish to minimize the *maximal MSE* due to the erasure of k subspaces for $k = 1$, $k = 2$, and $k > 2$. We prove that maximum robustness against one subspace erasure is achieved when all subspaces in a tight-fusion frame have equal dimensions, and that the optimal dimension depends on SNR. We also prove that a tight fusion frame consisting of equi-dimensional subspaces with equal pairwise *chordal distances* is maximally robust with respect to two and more subspace erasures. We call such fusion frames *equi-distance tight fusion frames*. We prove that the squares of the pairwise chordal distances between the subspaces in equi-distance tight fusion frames meet the so-called *simplex bound*, and thereby establish an intriguing connection between the construction of such fusion frames and *optimal Grassmannian packings* (cf. the excellent survey by Conway et al. [10]). This connection shows that optimal Grassmannian packings are fundamental for signal processing applications where robust dimension reduction is required.

The paper is organized as follows. In Section 2, we derive the MSE in LMMSE estimation of a random vector from its noisy fusion frame measurements. In Section 3, we analyze the robustness of LMMSE estimators to erasures of fusion frame subspaces and derive conditions for the construction of maximally robust fusion frames. Section 4 establishes a connection between equi-distance tight fusion frames and optimal Grassmannian packings. In Section 5, we give several examples for the construction of equi-distance tight fusion frames. Conclusions are drawn in Section 6.

2. LMMSE estimation from fusion frame measurements

Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for $\mathbb{R}^M$ with bounds $A \leqslant B$ and m_i be the dimension of the ith subspace $\mathcal{W}_i$, $i = 1, \dots, N$. Let $\mathbf{x} \in \mathbb{R}^M$ be a zero-mean random vector with covariance matrix $E[\mathbf{x}\mathbf{x}^T] = \mathbf{R}_{xx} = \sigma_x^2\mathbf{I}$. We wish to estimate $\mathbf{x}$ from N low-dimensional (smaller than M) measurement vectors $\mathbf{z}_i \in \mathbb{R}^{m_i}$ given by

$$\mathbf{z}_i = \mathbf{U}_i^T\mathbf{x} + \mathbf{n}_i, \quad i = 1, \dots, N,$$

where $\mathbf{U}_i \in \mathbb{R}^{M\times m_i}$ is a known but otherwise arbitrary *left-orthogonal* basis for $\mathcal{W}_i$, $i = 1, \dots, N$. That is $\mathbf{U}_i^T\mathbf{U}_i = \mathbf{I}_{m_i}$, where $\mathbf{I}_{m_i}$ is the $m_i \times m_i$ identity matrix, and $\mathbf{U}_i\mathbf{U}_i^T = \mathbf{P}_i$, where $\mathbf{P}_i$ is the orthogonal projection matrix onto the m_i-dimensional subspace $\mathcal{W}_i$. The vector $\mathbf{n}_i \in \mathbb{R}^{m_i}$ is a realization of an additive white noise vector with zero mean and covariance matrix $E[\mathbf{n}_i\mathbf{n}_i^T] = \sigma_n^2\mathbf{I}$, $i = 1, \dots, N$. We assume that the noise vectors for different subspaces are mutually uncorrelated. We also assume that the signal vector $\mathbf{x}$ and the noise vectors $\mathbf{n}_i$, $i = 1, \dots, N$, are uncorrelated.

We define the composite measurement vector $\mathbf{z} \in \mathbb{R}^L$ and the composite basis matrix $\mathbf{U} \in \mathbb{R}^{M\times L}$ as $\mathbf{z} = (\mathbf{z}_1^T\ \mathbf{z}_2^T\ \cdots\ \mathbf{z}_N^T)^T$ and $\mathbf{U} = (\mathbf{U}_1\ \mathbf{U}_2\ \cdots\ \mathbf{U}_N)$, where

$$L = \sum_{i=1}^N m_i.$$

Then, the composite covariance matrix between $\mathbf{x}$ and $\mathbf{z}$ can be written as

$$E\left[\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}\begin{pmatrix}\mathbf{x}^T & \mathbf{z}^T\end{pmatrix}\right] = \begin{pmatrix}\mathbf{R}_{xx} & \mathbf{R}_{xz}\\ \mathbf{R}_{zx} & \mathbf{R}_{zz}\end{pmatrix} \in \mathbb{R}^{(M+L)\times(M+L)},$$

where

$$\mathbf{R}_{xz} = E\big[\mathbf{x}\mathbf{z}^T\big] = \mathbf{R}_{xx}\mathbf{U} = \mathbf{R}_{xx}\begin{pmatrix}\mathbf{U}_1 & \cdots & \mathbf{U}_N\end{pmatrix}$$

is the $M \times L$ cross-covariance matrix between $\mathbf{x}$ and $\mathbf{z}$, $\mathbf{R}_{zx} = \mathbf{R}_{xz}^T$, and

$$\mathbf{R}_{zz} = E\big[\mathbf{z}\mathbf{z}^T\big] = \mathbf{U}^T\mathbf{R}_{xx}\mathbf{U} + \sigma_n^2\mathbf{I}_L = \begin{pmatrix}\mathbf{U}_1^T\\ \vdots\\ \mathbf{U}_N^T\end{pmatrix}\mathbf{R}_{xx}\begin{pmatrix}\mathbf{U}_1 & \cdots & \mathbf{U}_N\end{pmatrix} + \sigma_n^2\mathbf{I}_L \tag{2}$$

is the $L \times L$ composite measurement covariance matrix.

We wish to minimize the MSE in linearly estimating $\mathbf{x}$ from $\mathbf{z}$. The linear MSE minimizer is known to be the *Wiener filter* or the *LMMSE filter* $\mathbf{F} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}$, which estimates $\mathbf{x}$ by $\hat{\mathbf{x}} = \mathbf{F}\mathbf{z}$ (e.g., see [18]). The error covariance matrix $\mathbf{R}_{ee}$ in this estimation is given by

$$\begin{aligned}\mathbf{R}_{ee} &= E\big[\mathbf{e}\mathbf{e}^T\big] = E\big[(\mathbf{x}-\mathbf{F}\mathbf{z})(\mathbf{x}-\mathbf{F}\mathbf{z})^T\big] = \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}\\ &= \mathbf{R}_{xx} - \mathbf{R}_{xx}\mathbf{U}\big(\mathbf{U}^T\mathbf{R}_{xx}\mathbf{U} + \sigma_n^2\mathbf{I}_L\big)^{-1}\mathbf{U}^T\mathbf{R}_{xx} = \left(\mathbf{R}_{xx}^{-1} + \frac{1}{\sigma_n^2}\mathbf{U}\mathbf{U}^T\right)^{-1},\end{aligned}$$

where the last equality follows from the Sherman–Morrison–Woodbury formula [12]. Noting that

$$\mathbf{U}\mathbf{U}^T = \sum_{i=1}^N \mathbf{U}_i\mathbf{U}_i^T = \sum_{i=1}^N \mathbf{P}_i, \tag{3}$$

we can express $\mathbf{R}_{ee}$ as

$$\mathbf{R}_{ee} = \left(\mathbf{R}_{xx}^{-1} + \frac{1}{\sigma_n^2}\sum_{i=1}^N \mathbf{P}_i\right)^{-1}. \tag{4}$$

The MSE is obtained by taking the trace of $\mathbf{R}_{ee}$. Let ϕ_i, $i = 1, 2, \dots, M$, be the ith eigenvalue of $\mathbf{R}_{xx}^{-1} + (1/\sigma_n^2)\sum_{i=1}^N \mathbf{P}_i$. Then, the MSE is

$$\mathrm{MSE} = \mathrm{tr}[\mathbf{R}_{ee}] = \sum_{i=1}^M \frac{1}{\phi_i}.$$

Since $\mathbf{R}_{xx} = \sigma_x^2 \mathbf{I}$, it follows from (1) that

$$\frac{1}{\sigma_x^2} + \frac{A}{\sigma_n^2} \leqslant \phi_i \leqslant \frac{1}{\sigma_x^2} + \frac{B}{\sigma_n^2}.$$

Therefore, we have the following lower and upper bounds for the MSE:

$$\frac{M}{\frac{1}{\sigma_x^2} + \frac{B}{\sigma_n^2}} \leqslant \left(\mathrm{MSE} = \sum_{i=1}^{M} \frac{1}{\phi_i}\right) \leqslant \frac{M}{\frac{1}{\sigma_x^2} + \frac{A}{\sigma_n^2}}.$$

The lower bound will be achieved, if the fusion frame is tight. That is, when $A = B$ and

$$\sum_{i=1}^{N} \mathbf{P}_i = A\mathbf{I}. \tag{5}$$

Taking the trace from both sides of (5) yields the bound A as

$$A = \frac{\sum_{i=1}^{N} m_i}{M} = \frac{L}{M}. \tag{6}$$

Thus, the MSE is given by

$$\mathrm{MSE} = \frac{M\sigma_n^2\sigma_x^2}{\sigma_n^2 + \frac{\sigma_x^2 L}{M}}. \tag{7}$$

3. Robustness to subspace erasures

We now consider the case where subspace erasures occur, that is, when measurement vectors from one or more subspaces are lost or discarded. We wish to determine the MSE when the LMMSE filter $\mathbf{F}$, which is calculated based on the full composite covariance matrix in (2), is applied to the composite measurement vector with erasures. We do not wish to recalculate the LMMSE filter every time an erasure occurs. Recalculating the LMMSE filter requires calculating the inverse of the composite covariance matrix of the remaining measurement vectors, which in some cases can be intractable from a computational standpoint.

In this section, we show how the subspaces in the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ must be selected so that the MSE is minimized under subspace erasures. In our analysis we assume that the MSE with respect to no erasures is already minimized, i.e., that $\{\mathcal{W}_i\}_{i=1}^N$ is tight with bound A given by (6). We consider the case where all k-subspace erasures are equally important. In other words, we aim to minimize the maximal MSE due to k erasures.

Let $\mathbb{S} \subset \{1, 2, \ldots, N\}$ be the set of indices corresponding to the erased subspaces. Then, the composite measurement vector with erasures $\tilde{\mathbf{z}} \in \mathbb{R}^L$ may be expressed as

$$\tilde{\mathbf{z}} = (\mathbf{I} - \mathbf{E})\mathbf{z},$$

where $\mathbf{E}$ is an $L \times L$ block-diagonal erasure matrix whose ith diagonal block is an $m_i \times m_i$ zero matrix, if $i \notin \mathbb{S}$, or an $m_i \times m_i$ identity matrix, if $i \in \mathbb{S}$. In other words, in $\tilde{\mathbf{z}}$ the measurement vectors associated with the erased subspaces are set to zero.

The estimate of $\mathbf{x}$ is given by $\tilde{\mathbf{x}} = \mathbf{F}\tilde{\mathbf{z}}$, where $\mathbf{F} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}$ is the (no-erasure) LMMSE filter. The error covariance matrix $\tilde{\mathbf{R}}_{ee}$ for this estimate is given by

$$\begin{aligned}\tilde{\mathbf{R}}_{ee} &= E\big[(\mathbf{x} - \tilde{\mathbf{x}})(\mathbf{x} - \tilde{\mathbf{x}})^T\big] = E\big[\big(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z}\big)\big(\mathbf{x} - \mathbf{F}(\mathbf{I} - \mathbf{E})\mathbf{z}\big)^T\big] \\ &= \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}(\mathbf{I} - \mathbf{E})\mathbf{R}_{zx} - \mathbf{R}_{xz}(\mathbf{I} - \mathbf{E})^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx} + \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}(\mathbf{I} - \mathbf{E})\mathbf{R}_{zz}(\mathbf{I} - \mathbf{E})^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}.\end{aligned}$$

We can rewrite $\tilde{\mathbf{R}}_{ee}$ as

$$\tilde{\mathbf{R}}_{ee} = \mathbf{R}_{ee} + \bar{\mathbf{R}}_{ee},$$

where $\mathbf{R}_{ee} = \mathbf{R}_{xx} - \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}$ is the no-erasure error covariance matrix, and

$$\bar{\mathbf{R}}_{ee} = \mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{E}\mathbf{R}_{zz}\mathbf{E}^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}$$

is the extra covariance matrix due to erasures. The MSE is given by

$$\mathrm{MSE} = \mathrm{tr}[\tilde{\mathbf{R}}_{ee}] = \mathrm{MSE}_0 + \overline{\mathrm{MSE}},$$

where $\mathrm{MSE}_0 = \mathrm{tr}[\mathbf{R}_{ee}]$ is the no-erasure MSE in (7) and

$$\begin{aligned}\overline{\mathrm{MSE}} &= \mathrm{tr}[\bar{\mathbf{R}}_{ee}] = \mathrm{tr}\big[\mathbf{R}_{xz}\mathbf{R}_{zz}^{-1}\mathbf{E}\mathbf{R}_{zz}\mathbf{E}^T\mathbf{R}_{zz}^{-1}\mathbf{R}_{zx}\big] \\ &= \mathrm{tr}\big[\sigma_x^4\mathbf{U}\big(\sigma_x^2\mathbf{U}^T\mathbf{U} + \sigma_n^2\mathbf{I}\big)^{-1}\mathbf{E}\big(\sigma_x^2\mathbf{U}^T\mathbf{U} + \sigma_n^2\mathbf{I}\big)\mathbf{E}^T\big(\sigma_x^2\mathbf{U}^T\mathbf{U} + \sigma_n^2\mathbf{I}\big)^{-1}\mathbf{U}^T\big]\end{aligned}$$

is the extra MSE due to erasures.

From the Sherman–Morrison–Woodbury formula [12], we have

$$\big(\sigma_x^2\mathbf{U}^T\mathbf{U} + \sigma_n^2\mathbf{I}\big)^{-1} = \frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^4}\mathbf{U}^T\left(\frac{1}{\sigma_n^2}\mathbf{U}\mathbf{U}^T + \frac{1}{\sigma_x^2}\mathbf{I}\right)^{-1}\mathbf{U} = \frac{1}{\sigma_n^2}\mathbf{I} - \frac{1}{\sigma_n^2}\frac{\sigma_x^2}{A\sigma_x^2 + \sigma_n^2}\mathbf{U}^T\mathbf{U}, \tag{8}$$

where the second equality follows from (3) and $\sum_{i=1}^N \mathbf{P}_i = A\mathbf{I}$.

Using (8), we can simplify the expression for $\overline{\mathrm{MSE}}$ to

$$\begin{aligned}\overline{\mathrm{MSE}} &= \alpha^2\,\mathrm{tr}\big[\mathbf{U}\mathbf{E}\big(\sigma_x^2\mathbf{U}^T\mathbf{U} + \sigma_n^2\mathbf{I}\big)\mathbf{E}\mathbf{U}^T\big] = \alpha^2\,\mathrm{tr}\left[\sigma_x^2\bigg(\sum_{i\in\mathbb{S}}\mathbf{U}_i\mathbf{U}_i^T\bigg)^2 + \sigma_n^2\bigg(\sum_{i\in\mathbb{S}}\mathbf{U}_i\mathbf{U}_i^T\bigg)\right] \\ &= \alpha^2\,\mathrm{tr}\left[\sigma_x^2\bigg(\sum_{i\in\mathbb{S}}\mathbf{P}_i\bigg)^2 + \sigma_n^2\bigg(\sum_{i\in\mathbb{S}}\mathbf{P}_i\bigg)\right],\end{aligned} \tag{9}$$

where $\alpha = \sigma_x^2/(A\sigma_x^2 + \sigma_n^2)$. The second equality in (9) follows by considering the action of the erasure matrix $\mathbf{E}$.

We now show how the subspaces in the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ must be constructed so that the *total* MSE is minimized for a given number of erasures, over the class of tight fusion frames. We consider three scenarios: one subspace erasure, two subspace erasures, and more than two subspace erasures.

3.1. One subspace erasure

If only one of the subspaces, say the ith subspace, is erased, then the MSE is given by

$$\mathrm{MSE} = \mathrm{MSE}_0 + \overline{\mathrm{MSE}} = \mathrm{MSE}_0 + \mathrm{tr}\big[\alpha^2\big(\sigma_x^2 + \sigma_n^2\big)\mathbf{P}_i\big] = \frac{M\sigma_x^2\sigma_n^2}{\sigma_n^2 + \frac{\sigma_x^2}{M}L} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)}{(\sigma_n^2 + \frac{\sigma_x^2}{M}L)^2}m_i, \tag{10}$$

where $m_i = \mathrm{tr}[\mathbf{P}_i]$ is the dimension of the ith subspace $\mathcal{W}_i$ and $L = \sum_{i=1}^N m_i$.

Since we have assumed that all one-erasures are equally important we have to choose $m_i = m$ for all $i = 1, \ldots, N$, so that any one-erasure results in the same amount of performance degradation. This strategy is equivalent to minimizing the maximal MSE due to one subspace erasure. This reduces the MSE expression (10) to

$$\mathrm{MSE} = \frac{M\sigma_x^2\sigma_n^2}{(Nm\sigma_x^2/M + \sigma_n^2)} + \frac{\sigma_x^4(\sigma_x^2 + \sigma_n^2)m}{(Nm\sigma_x^2/M + \sigma_n^2)^2}.$$

As a function of m, $\mathrm{MSE} = \mathrm{MSE}(m)$ has a maximum at $m = \tilde{m}$, where

$$\tilde{m} = \frac{M}{N}\frac{(N-1)\sigma_n^4 - \sigma_x^2\sigma_n^2}{((N+1)\sigma_n^2 + \sigma_x^2)(1 - 2\sigma_x^2)}.$$

The MSE is monotonically increasing for $m < \tilde{m}$ and monotonically decreasing for $m > \tilde{m}$. The smallest value m can take under the constraint that the set of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ remains a tight fusion frame is $m_{\min} = \lceil M/N \rceil$, where $\lceil\cdot\rceil$ denotes integer ceiling. We assume that the largest value m can take is $m_{\max} \leqslant M$. The maximum allowable dimension $m_{\max}$ is determined by practical considerations. In the distributed sensing problem it is the maximum number of sensors we can deploy in a cluster. In the parallel processing problem it is determined by the maximum computational load that the local processors can handle, and in the packet encoding problem it corresponds to the maximum amount of new information (minimum amount of redundancy) we can include in a packet, while achieving an error correction goal. Concluding, we have the following theorem.

Theorem 3.1. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a tight fusion frame consisting of subspaces with dimensions $m_{\min} \leqslant m_i \leqslant m_{\max}$, $i = 1, \ldots, N$. Then the maximal* MSE *due to the erasure of one subspace is minimized when all subspaces in $\{\mathcal{W}_i\}_{i=1}^N$ have equal dimension $m = m^*$, where*

$$m^* = \begin{cases} m_{\min}, & \text{if } m_{\max} \leqslant \tilde{m} \text{ or} \\ & \text{if } m_{\min} \leqslant \tilde{m} \leqslant m_{\max} \text{ and } \mathrm{MSE}(m_{\min}) \leqslant \mathrm{MSE}(m_{\max}), \\ m_{\max}, & \text{otherwise.} \end{cases}$$

3.2. Two subspace erasures

When two subspaces, say the ith subspace and the jth subspace, are erased or discarded, the total MSE is given by

$$\mathrm{MSE} = \mathrm{MSE}_0 + \overline{\mathrm{MSE}} = \mathrm{MSE}_0 + \alpha^2 \operatorname{tr}\big[\sigma_x^2(\mathbf{P}_i + \mathbf{P}_j)^2 + \sigma_n^2(\mathbf{P}_i + \mathbf{P}_j)\big].$$

We take the fusion frame to be tight and assume all subspaces have equal dimension m^* to maintain MSE optimality with respect to no erasures and one-erasures. This reduces the minimization of MSE to minimizing the extra MSE, which is given by

$$\overline{\mathrm{MSE}} = 2\alpha^2\big(\sigma_x^2 + \sigma_n^2\big)m^* + 2\alpha^2\sigma_x^2 \operatorname{tr}[\mathbf{P}_i\mathbf{P}_j].$$

To minimize $\overline{\mathrm{MSE}}$ we have to choose $\mathcal{W}_i$ and $\mathcal{W}_j$, so that $\operatorname{tr}[\mathbf{P}_i\mathbf{P}_j]$ is minimized. Since $\mathbf{P}_i$ and $\mathbf{P}_j$ are orthogonal projection matrices onto $\mathcal{W}_i$ and $\mathcal{W}_j$, the eigenvalues of $\mathbf{P}_i\mathbf{P}_j$ are squares of the cosines of the *principal angles* $\theta_\ell(i, j)$, $\ell = 1, \ldots, M$, between $\mathcal{W}_i$ and $\mathcal{W}_j$. Therefore,

$$\operatorname{tr}[\mathbf{P}_i\mathbf{P}_j] = \sum_{\ell=1}^{M} \cos^2\theta_\ell(i, j) = M - d_c^2(i, j), \tag{11}$$

where

$$d_c(i, j) = \left(\sum_{\ell=1}^{M} \sin^2\theta_\ell(i, j)\right)^{1/2}$$

is known as the *chordal distance* [10] between $\mathcal{W}_i$ and $\mathcal{W}_j$.

Thus, we need to maximize the chordal distance $d_c(i, j)$. Since we have assumed that all two subspace erasures are equally important, we have to construct the subspaces $\{\mathcal{W}_i\}_{i=1}^N$ so that any such pair has maximum chordal distance. This strategy is equivalent to minimizing the maximal MSE due to two subspace erasures.

In Section 4, we will prove that the subspaces in a fusion frame consisting of equi-dimensional and equi-distance (equi-chordal distance) subspaces have maximal chordal distance if and only if the fusion frame is tight. We call such a fusion frame an *equi-distance tight fusion frame* and the subspaces corresponding to it *maximal equi-distance subspaces*. We note that maximal equi-distance does not mean that the principal angles between any pair of subspaces must be equal. Therefore, this is a more relaxed requirement than the equi-isoclinic condition in [2].

We have the following theorem.

Theorem 3.2. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a tight fusion frame with equal dimensional subspaces, where* $\dim[\mathcal{W}_i] = m^*$, $i = 1, \ldots, N$. *Then the maximal* MSE *due to two subspace erasures is minimized when the subspaces $\mathcal{W}_i$ are maximal equi-distance subspaces.*

We defer the construction of maximal equi-distance subspaces to Section 4, where we explain the connection between this construction and the problem of optimal packing of N equi-dimensional subspaces in a Grassmannian space [11,10].

3.3. More than two subspace erasures

We now consider the case where more than two subspaces are erased or discarded. Here we take the fusion frame to be an equi-distance tight fusion frame to maintain MSE optimality with respect no erasures, one-erasures, and two-erasures.

Let $\{\mathcal{W}_i\}_{i=1}^N$ be an equi-distance tight fusion frame, where the subspaces $\mathcal{W}_i$, $i = 1, \ldots, N$, have equal dimension m^* and equal pairwise chordal distance d_c. Then, $\overline{\text{MSE}}$ can be written as

$$\overline{\text{MSE}} = \alpha^2 \operatorname{tr}\left[\sigma_x^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i\right)^2 + \sigma_n^2 \left(\sum_{i \in \mathbb{S}} \mathbf{P}_i\right)\right] = \alpha^2\left(\sigma_x^2 + \sigma_n^2\right) \sum_{i \in \mathbb{S}} \operatorname{tr}[\mathbf{P}_i] + \alpha^2 \sigma_x^2 \sum_{i \in \mathbb{S}} \sum_{j \in \mathbb{S},\, j \neq i} \operatorname{tr}[\mathbf{P}_i \mathbf{P}_j]$$

$$= \alpha^2\left(\sigma_x^2 + \sigma_n^2\right)|\mathbb{S}| m^* + \alpha^2 \sigma_x^2 |\mathbb{S}|\left(|\mathbb{S}| - 1\right)\left(M - d_c^2\right).$$

Noting that m^*, d_c^2 and $|\mathbb{S}|$ are fixed, we have the following theorem.

Theorem 3.3. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an equi-distance tight fusion frame with* $\dim[\mathcal{W}_i] = m^*$*, $i = 1, \ldots, N$. Then the* MSE *due to k subspace erasures, $3 \leqslant k < N$ is constant.*

We would like to note that, in general, it is not known whether the fusion frame required by Theorem 3.3 exists. In such cases, this theorem gives a lower bound on the MSE which is not known to be sharp.

4. Connections between tight fusion frames and optimal packings

In this section, we show that tight fusion frames that consist of equi-dimensional and equi-distance subspaces are closely related to optimal packings of subspaces. We start by reviewing the classical packing problem for subspaces [11,10].

Classical packing problem. For given m, M, N, find a set of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{R}^M$ such that $\min_{i \neq j} d_c(i, j)$ is as large as possible. In this case we call $\{\mathcal{W}_i\}_{i=1}^N$ an *optimal packing*.

This problem was reformulated by Conway et al. in [10] by describing m-dimensional subspaces in $\mathbb{R}^M$ as points on a sphere inside of $\frac{1}{2}(M-1)(M+2)$-dimensional Euclidean space. This usually provides a lower-dimensional representation than the Plücker embedding. The reader is referred to [14] for the definition of the Plücker embedding. This idea was then used to prove the optimality of many new packings by employing results from sphere packing theory such as Rankin bounds for spherical codes. In what follows, we briefly describe the embedding of the Grassmannian manifold $G(m, M)$ of m-dimensional subspaces of $\mathbb{R}^M$, as it was described in [10]. The basic idea is to identify an m-dimensional subspace $\mathcal{W}$ with the traceless part of the projection matrix $\mathbf{Q}$ associated with $\mathcal{W}$, i.e., with $\bar{\mathbf{Q}} = \mathbf{Q} - \frac{m}{M}\mathbf{I}$. This yields an isometric embedding of $G(m, M)$ into the sphere of radius $\sqrt{m(M-m)/M}$ in $\mathbb{R}^{\frac{1}{2}(M-1)(M+2)}$, where the distance measure is the chordal distance between two projections. The chordal distance $d_c(\mathbf{Q}_1, \mathbf{Q}_2)$ between two projection matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is given by $d_c(\mathbf{Q}_1, \mathbf{Q}_2) = \frac{1}{\sqrt{2}}\|\mathbf{Q}_1 - \mathbf{Q}_2\|_2$, and is equal to $\frac{1}{\sqrt{2}}$ times the straight-line distance between the projection matrices. This is the reason that $d_c(\mathbf{Q}_1, \mathbf{Q}_2)$ is called *chordal* distance. Conway et al. [10] deduced from this particular embedding the following result.

Theorem 4.1. *(See [10].) Each packing of m-dimensional subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{R}^M$ satisfies*

$$d_c^2(i, j) \leqslant \frac{m(M-m)}{M} \frac{N}{N-1}, \quad i, j = 1, \ldots, N.$$

The upper bound is referred to as the *simplex bound*. The above theorem implies that if the squares of the pairwise chordal distances between a set of m-dimensional subspaces of $\mathbb{R}^M$ meet the simplex bound those subspaces form an optimal packing, as the minimum of chordal distances cannot grow any further.

We now establish a connection between tight fusion frames and optimal packings.

4.1. Equi-dimensional subspaces

Consider a tight fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with bound A consisting of N m-dimensional subspaces that do not necessarily have equal pairwise chordal distances. Since $\{\mathcal{W}_i\}_{i=1}^N$ is tight, we have

$$A\mathbf{I} = \sum_{i=1}^N \mathbf{P}_i. \tag{12}$$

On the one hand, we can apply the trace and employ the fact that $\mathrm{tr}[\mathbf{P}_i] = m$ for each i, to obtain

$$AM = Nm. \tag{13}$$

On the other hand, we can multiply (12) on the left by $\mathbf{P}_j$ to get

$$(A-1)\mathbf{P}_j = \sum_{i=1,\, i\neq j}^N \mathbf{P}_j\mathbf{P}_i, \quad j = 1, \ldots, N.$$

We can then take the trace, employ the fact that $\mathrm{tr}[\mathbf{P}_j] = m$ for each j, and use (11), to obtain

$$(A-1)m = \sum_{i=1,\, i\neq j}^N \mathrm{tr}[\mathbf{P}_j\mathbf{P}_i] = (N-1)m - \sum_{i=1,\, i\neq j}^N d_c^2(i,j). \tag{14}$$

Eqs. (13) and (14) together prove the following result concerning the value of the fusion frame bound.

Proposition 4.2. *A tight fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with bound A and m-dimensional subspaces satisfies*

$$A = \frac{Nm}{M} = N - \sum_{i=1,\, i\neq j}^N \frac{d_c^2(i,j)}{m}, \quad j = 1, \ldots, N.$$

4.2. Equi-dimensional and equi-distance subspaces

We now turn our attention to tight fusion frames $\{\mathcal{W}_i\}_{i=1}^N$ consisting of equi-dimensional and equi-distance subspaces, where the common dimension is m and the common chordal distance is d_c. From Proposition 4.2, it follows that

$$\frac{Nm}{M} = N - (N-1)\frac{d_c^2}{m}.$$

Thus, d_c^2 is given by

$$d_c^2 = \frac{m(M-m)}{M}\frac{N}{N-1}, \tag{15}$$

which shows that d_c^2 precisely equals the simplex bound.

Next we will study whether this condition is sufficient. That is, we wish to know whether a fusion frame consisting of equi-dimensional subspaces, for which the squares of the pairwise chordal distances are equal to the simplex bound is necessarily tight.

Consider a fusion frame $\{\mathcal{W}_i\}_{i=1}^N$, consisting of N m-dimensional subspaces with squared chordal distances d_c^2 equal to the simplex bound. Let $\pi_1, \ldots, \pi_M$ be the eigenvalues of $\sum_{i=1}^N \mathbf{P}_i$. Since $\{\mathcal{W}_i\}_{i=1}^N$ is a fusion frame for $\mathbb{R}^M$, we have $\pi_\ell > 0$, $\ell = 1, 2, \ldots, M$, and the sum of π_ℓ's is given by

$$\sum_{\ell=1}^M \pi_\ell = \sum_{i=1}^N \mathrm{tr}[\mathbf{P}_i] = Nm. \tag{16}$$

The sum of π_ℓ^2's can be written as

$$\sum_{\ell=1}^{M}\pi_\ell^2=\sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{tr}[\mathbf{P}_i\mathbf{P}_j]=\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}\operatorname{tr}[\mathbf{P}_i\mathbf{P}_j]+\sum_{i=1}^{N}\operatorname{tr}[\mathbf{P}_i]=N(N-1)\big(m-d_c^2\big)+Nm,$$

where the last equality follows from (11). Inserting the value of the simplex bound, we obtain

$$\sum_{\ell=1}^{M}\pi_\ell^2=\frac{m^2N^2}{M}. \tag{17}$$

To conclude that (16) together with (17) implies tightness of the fusion frame, we consider the problem of minimizing the function $\sum_{\ell=1}^{M}\pi_\ell^2$ under the constraint that $\pi_1,\dots,\pi_M$ is a sequence of nonnegative values which sum up to $\sum_{\ell=1}^{M}\pi_\ell=Nm$. Using the method of Lagrange multipliers, we see that the minimum is achieved when all π_ℓ's are equal to Nm/M. This implies that (16) and (17) can be simultaneously satisfied only when

$$\pi_1=\cdots=\pi_M=\frac{Nm}{M}.$$

From this relation, it follows that $\{\mathcal{W}_i\}_{i=1}^N$ is a tight fusion frame. Therefore, we have the following theorem.

Theorem 4.3. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame of equi-dimensional subspaces with equal pairwise chordal distances d_c. Then, the fusion frame is tight if and only if d_c^2 equals the simplex bound.*

An immediate consequence of Theorem 4.3 is as follows.

Corollary 4.4. *Equi-distance tight fusion frames are optimal Grassmannian packings.*

5. Construction of equi-distance tight fusion frames

In this section we present a few examples to illustrate the richness, but also the difficulty of constructing fusion frames with special properties such as tightness, equi-dimension, and equi-distance. The optimal packing of N planes in the Grassmannian space $G(m,M)$ is a difficult mathematical problem, the solution to which is known only for special values of N, m, and M. In fact, even optimal packing of lines ($m=1$) or equivalently constructing equi-angular lines is a deep mathematical problem. The reader is referred to [19] for a review of problems which are equivalent to the construction of equi-angular lines. For the construction of optimal packings with higher-dimensional subspaces we refer the reader to [10,11,4,21]. We would also like to draw the reader's attention to N.J.A. Sloane's webpage [22], which includes many examples of Grassmannian packings.

Example 5.1. As our first example for construction of equi-distance tight fusion frames, we use a result obtained by Calderbank et al. [4] for construction of optimal packings. The procedure is as follows. Choose p to be a prime which is either 3 or congruent to -1 modulo 8. Then there exists an explicit construction which produces a tight fusion frame $\{\mathcal{W}_i\}_{i=1}^{p(p+1)/2}$ in $\mathbb{R}^p$ with

$$m_i=\frac{p-1}{2}\quad\text{and}\quad d_c^2(i,j)=\frac{(p+1)^2}{4(p+2)}\quad\text{for all } i,j=1,\dots,\frac{p(p+1)}{2},$$

where m_i denotes the dimension of the ith subspace and $d_c^2(i,j)$ is the squared chordal distance between $\mathcal{W}_i$ and $\mathcal{W}_j$. From Proposition 4.2 it follows that the bound of this fusion frame equals

$$A=\frac{p^2-1}{4}.$$

As a particular example of this construction we briefly outline the equi-distance tight fusion frame we obtain for $p=7$. For this, let $Q=\{q_i\}_{i=1}^3=\{1,2,4\}$ denote the nonzero quadratic residues modulo 7, and $R=\{3,5,6\}$ the nonresidues. Further, let $\mathbf{H}$ be a 4×4 Hadamard matrix, e.g.,

$$\mathbf{H}=\begin{pmatrix}1&1&1&1\\-1&1&-1&1\\-1&-1&1&1\\1&-1&-1&1\end{pmatrix}.$$

Finally, we denote the coordinate vectors in $\mathbb{R}^7$ by $\mathbf{e}_i$, $0 \leqslant i \leqslant 6$, and set $C = \sqrt{2}$ and $k = 3$. Then we define 4 three-dimensional planes $\mathbf{L}_j$, $1 \leqslant j \leqslant 4$, to be spanned by the vectors

$$\mathbf{e}_{q_i} + C\mathbf{H}_{ij}\mathbf{e}_{kq_i}, \quad 1 \leqslant i \leqslant 3.$$

For each $\mathbf{L}_j$, we obtain 6 further planes by applying the cyclic permutation of coordinates $\mathbf{e}_i \mapsto \mathbf{e}_{(i+1) \bmod 7}$. This yields 28 three-dimensional planes in $\mathbb{R}^7$, which form a tight fusion frame with bound 12. Moreover, the squared chordal distance between each pair of them equals $d_c^2 = \frac{16}{9}$.

This construction is based on employing properties of special groups, in this case the Clifford group. We remark that this is closely related with the construction of error-correcting codes.

Example 5.2. This example considers the construction of an equi-distance tight fusion frame for a dimension not covered by Example 5.1 by employing the theory of Eisenstein integers. More precisely, the subspaces will be generated by the minimal elements of a special lattice. For this, we let $\mathcal{E} = \{a + \omega b\colon\ a, b \in \mathbb{Z}\}$ denote the Eisenstein integers, where $\omega = \frac{-1+\mathrm{i}\sqrt{3}}{2}$ is a complex root of unity. The three-dimensional complex lattice E_6^* over $\mathcal{E}$ is then defined by its generator matrix

$$\begin{pmatrix} \sqrt{-3} & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

It can be shown that the minimal norm of a non-zero element in E_6^* is $\frac{4}{3}$. Out of the set of minimal elements, we now select the following nine:

$$(1,-1,0),\ (1,0,-1),\ (0,1,-1),\ (\omega,-1,0),\ (0,\omega,-1),\ (-1,0,\omega),\ (\omega,0,-1),\ (-1,\omega,0),\ (0,-1,\omega).$$

Multiplied by the 6th roots of unity, this yields 9 planes in $\mathbb{C}^3$. Using the canonical mapping of $\mathbb{C}^3$ onto $\mathbb{R}^6$, e.g., $(\omega,-1,0) \mapsto (-\frac{1}{2}, \frac{\sqrt{3}}{2}, -1, 0, 0, 0)$, we obtain 9 two-dimensional planes in $\mathbb{R}^6$.

In this example all principle angles between each pair of planes are in fact equal to $\frac{\pi}{3}$. In particular, the squared chordal distance is $d_c^2 = \frac{3}{2}$, which can easily be seen to satisfy the simplex bound (cf. (15)). By Theorem 4.3 it now follows that the fusion frame consisting of these planes is tight, and Proposition 4.2 shows that the frame bound equals 3.

Example 5.3. The third example explores the construction of fusion frames in $\mathbb{R}^8$ by employing a similar strategy as in Example 5.2. However, with this example we wish to illustrate the need to be particularly meticulous when generating a fusion frame from minimal vectors of a particular lattice. In fact by using a similar approach, we will generate a tight fusion frame with equi-dimensional subspaces, but not equi-distance subspaces, although with a very distinct set of chordal distances. In fact, the chordal distances do attain only two different values.

For our analysis, we choose the lattice

$$E_8 = \left\{(x_1,\ldots,x_8)\colon \left(x_i \in \mathbb{Z}\ \forall 1 \leqslant i \leqslant 8 \text{ or } x_i \in \mathbb{Z} + \frac{1}{2}\ \forall 1 \leqslant i \leqslant 8\right) \text{ and } \sum_{i=1}^{8} x_i \in 2\mathbb{Z}\right\},$$

which is again a lattice over the Eisenstein integers $\mathcal{E} = \{a + \omega b\colon\ a, b \in \mathbb{Z}\}$, $\omega = \frac{-1+\mathrm{i}\sqrt{3}}{2}$. Before studying the minimal vectors in this lattice, we consider the complex root of unity $\omega = \frac{-1+\mathrm{i}\sqrt{3}}{2}$ which was employed in the construction of $\mathcal{E}$. We first express ω in quaternions, which gives $\omega = \frac{1}{2}(-1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$. Next we define a matrix $\mathbf{H}$ by choosing as row vectors the coefficients of ω, $\mathrm{i}\omega$, $\mathrm{j}\omega$, and $\mathrm{k}\omega$, i.e.,

$$\mathbf{H} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}.$$

From this, we build an 8×8-matrix by setting

$$\boldsymbol{\Omega} = \begin{pmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{pmatrix}.$$

Realizing that this matrix satisfies $\boldsymbol{\Omega}^2 + \boldsymbol{\Omega} + \mathbf{I} = 0$, we can conclude that scaling a vector $\mathbf{v} \in E_8$ by an Eisenstein integer $a + \omega b$ can be rewritten as

$$(a + \omega b)\mathbf{v} = a\mathbf{v} + b\boldsymbol{\Omega}\mathbf{v}.$$

Now we are equipped to generate subspaces by minimal vectors, whose norm can be computed to equal 2. The lattice E_8 has 240 minimal vectors, which we assign to planes in the following way. We first consider the four minimal vectors

$$(1, -1, 0, 0, 0, 0, 0, 0),\ (1, 0, -1, 0, 0, 0, 0, 0),\ (1, 0, 0, -1, 0, 0, 0, 0),\ (0, 1, -1, 0, 0, 0, 0, 0)$$

and multiply each of them with

$$\mathbf{I},\ -\mathbf{I},\ \boldsymbol{\Omega},\ -\boldsymbol{\Omega},\ \mathbf{I} + \boldsymbol{\Omega},\ \text{and}\ -\mathbf{I} - \boldsymbol{\Omega}. \tag{18}$$

This procedure generates four sets of six minimal vectors, where each set generates a two-dimensional plane in $\mathbb{R}^8$. Noticing that this construction only takes all minimal vectors which are of the form $(x_1, x_2, x_3, x_4, 0, 0, 0, 0)$ into account, we can clearly use the same idea to group all minimal vectors of the form $(0, 0, 0, 0, x_5, x_6, x_7, x_8)$. Summarizing, this construction provides us with 8 two-dimensional planes in $\mathbb{R}^8$ which we denote by $\mathcal{W}_1, \dots, \mathcal{W}_8$. Next we consider minimal vectors $(x_1, \dots, x_8)$, which have one coordinate out of x_1, x_2, x_3, x_4 and one coordinate out of x_5, x_6, x_7, x_8 equal to -1 or 1, the others being equal to zero. Again we multiply these vectors by the factors given in (18). We can easily see that this procedure generates another 32 two-dimensional planes in $\mathbb{R}^8$, denoted by $\mathcal{W}_9, \dots, \mathcal{W}_{40}$.

Although this construction seems similar to the one in Example 5.2, we found it surprising to see that in fact $\{\mathcal{W}_i\}_{i=1}^{40}$ does constitute an equi-dimension tight fusion frame, however the subspaces are not equi-distance. The fusion frame bound can be derived from Proposition 4.2 and equals 10. Most interestingly, the squared chordal distance d_c^2 takes only two values, either $d_c^2 = 2$ for mutually orthogonal subspaces, or $d_c^2 = \frac{4}{3}$.

6. Conclusions

We considered the LMMSE estimation of a zero mean random vector, with covariance matrix $\sigma_x^2\mathbf{I}$, from its measurements in low-dimensional subspaces constituting a fusion frame. We proved that, in the presence of additive white noise, the MSE in such an estimation will achieve its lower bound if the fusion frame is tight. We analyzed the effect of subspace erasures on the performance of LMMSE estimators. We restricted our erasure analysis to the class of tight fusion frames and considered minimizing the maximal MSE due to k subspace erasures for $k = 1$, $k = 2$, and $k > 2$. We proved that maximum robustness against one subspace erasures is achieved when all subspaces of the tight fusion frame have equal dimensions, where the optimal dimension depends on the SNR. We also proved that equi-distance tight fusion frames are maximally robust against two and more than two subspace erasures. In addition, we proved that equi-distance tight fusion frames are in fact optimal Grassmannian packings, and thereby showed that optimal Grassmannian packings are fundamental for signal processing applications where robust dimension reduction is required. We presented a few examples for the construction of equi-distance tight fusion frames and illustrated the interesting and sometimes challenging nature of such constructions.

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References

[1] P.J. Bjørstad, J. Mandel, On the spectra of sums of orthogonal projections with applications to parallel computing, BIT 1 (1) (1991) 76–88.
[2] B.G. Bodmann, Optimal linear transmission by loss-insensitive packet encoding, Appl. Comput. Harmon. Anal. 22 (3) (2007) 274–285.
[3] B.G. Bodmann, V.I. Paulsen, Frames, graphs and erasures, Linear Algebra Appl. 404 (2005) 118–146.
[4] A.R. Calderbank, R.H. Hardin, E.M. Rains, P.W. Shor, N.J.A. Sloane, A group-theoretic framework for the construction of packings in Grassmannian spaces, J. Algebraic Combin. 9 (2) (1999) 129–140.
[5] P.G. Casazza, J. Kovačević, Equal-norm tight frames with erasures, Adv. Comput. Math. 18 (2–4) (2003) 387–430.
[6] P.G. Casazza, G. Kutyniok, Robustness of fusion frames under erasures of subspaces and of local frame vectors, in: Radon Transforms, Geometry, and Wavelets, New Orleans, LA, 2006, in: Contemp. Math., Amer. Math. Soc., Providence, RI, in press.
[7] P.G. Casazza, G. Kutyniok, Frames of subspaces, in: Wavelets, Frames and Operator Theory, in: Contemp. Math., vol. 345, Amer. Math. Soc., Providence, RI, 2004, pp. 87–113.
[8] P.G. Casazza, G. Kutyniok, S. Li, Fusion frames and distributed processing, Appl. Comput. Harmon. Anal., doi:10.1016/j.acha.2007.10.001, in press.
[9] P.G. Casazza, G. Kutyniok, S. Li, C.J. Rozell, Modeling sensor networks with fusion frames, in: Wavelets XII, San Diego, CA, 2007, in: SPIE Proc., vol. 6701, SPIE, Bellingham, WA, 2007, 67011M-1–67011M-11.
[10] J.H. Conway, R.H. Hardin, N.J.A. Sloane, Packing lines, planes, etc.: Packings in Grassmannian spaces, Experiment. Math. 5 (2) (1996) 139–159.
[11] J.H. Conway, N.J.A. Sloane, Sphere Packings, Lattices and Groups, second ed., Springer, New York, 1993.
[12] G.H. Golub, C.F. Van Loan, Matrix Computations, third ed., Johns Hopkins Univ. Press, Baltimore, MD, 1996.
[13] V.K. Goyal, J. Kovačević, J.A. Kelner, Quantized frame expansions with erasures, Appl. Comput. Harmon. Anal. 10 (3) (2001) 203–233.
[14] J. Harris, Algebraic Geometry, Grad. Texts in Math., vol. 133, Springer-Verlag, New York, 1992, a first course.
[15] R.B. Holmes, V.I. Paulsen, Optimal frames for erasures, Linear Algebra Appl. 377 (2004) 31–51.
[16] C.J. Rozell, I.N. Goodman, D.H. Johnson, Feature-based information processing with selective attention, in: Proc. Int. Conf. Acoust., Speech, Signal Process. (ICASSP), vol. 4, 2006, pp. 709–712.
[17] C.J. Rozell, D.H. Johnson, Analyzing the robustness of redundant population codes in sensory and feature extraction systems, Neurocomputing 69 (2006) 1215–1218.
[18] L.L. Scharf, Statistical Signal Processing, Addison–Wesley, Reading, MA, 1991.
[19] T. Strohmer, R.W. Heath Jr., Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (3) (2003) 257–275.
[20] M.A. Sustik, J.A. Tropp, I. Dhillon, R.W. Heath Jr., On the existence of equiangular tight frames, Linear Algebra Appl. 426 (2–3) (2007) 619–635.
[21] A. Tropp, I. Dhillon, R.W. Heath Jr., T. Strohmer, Constructing packings in Grassmannian manifolds via alternating projections, Experiment. Math., in press.
[22] http://www.research.att.com/~njas/grass/index.html.