Constructing Fusion Frames with Desired Parameters

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ABSTRACT

A *fusion frame* is a frame-like collection of subspaces in a Hilbert space. It generalizes the concept of a frame system for signal representation. In this paper, we study the existence and construction of fusion frames. We first introduce two general methods, namely the *spatial complement* and the *Naimark complement*, for constructing a new fusion frame from a given fusion frame. We then establish existence conditions for fusion frames with desired properties. In particular, we address the following question: Given $M, N, m \in \mathbb{N}$ and $\{\lambda_j\}_{j=1}^M$, does there exist a fusion frame in $\mathbb{R}^M$ with N subspaces of dimension m for which $\{\lambda_j\}_{j=1}^M$ are the eigenvalues of the associated fusion frame operator? We address this problem by providing an algorithm which computes such a fusion frame for almost any collection of parameters $M, N, m \in \mathbb{N}$ and $\{\lambda_j\}_{j=1}^M$. Moreover, we show how this procedure can be applied, if subspaces are to be added to a given fusion frame to force it to become tight.

Keywords: Fusion frames with desired fusion frame operators, Naimark complement, spatial complement

1. INTRODUCTION

Fusion frames (or *frames of subspaces*) are a recent development that provide a natural mathematical framework for two-stage (or, more generally, hierarchical) data processing. The notion of a fusion frame was introduced in[12] with the main ideas already contained in.[9] A fusion frame is a frame-like collection of subspaces in a Hilbert space. In frame theory,[*] a signal is represented by a collection of *scalars*, which measure the amplitudes of the projections of the signal onto the frame vectors, whereas in fusion frame theory the signal is represented by the projections of the signal onto the fusion frame subspaces. In a two-stage data processing setup, these projections serve as locally processed data, which can be combined to reconstruct a signal of interest (see, e.g.,[13,20,23]).

Given a Hilbert space $\mathcal{H}$ and a family of closed subspaces $\{\mathcal{W}_i\}_{i\in I}$ with associated positive weights v_i, $i \in I$, a *fusion frame* for $\mathcal{H}$ is a collection of weighted subspaces $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i\in I} v_i^2 \|P_i f\|^2 \leq B\|f\|^2 \qquad \text{for any } f \in \mathcal{H},$$

where P_i is the orthogonal projection onto $\mathcal{W}_i$. The constants A and B are called *fusion frame bounds*. A fusion frame is called *tight*, if A and B can be chosen to be equal, and *Parseval* if $A = B = 1$. If $v_i = 1$ for all $i \in I$, for the sake of brevity, we sometimes write $\{\mathcal{W}_i\}_{i\in I}$ instead of $\{(\mathcal{W}_i, 1)\}_{i\in I}$.

Any signal $f \in \mathcal{H}$ can be reconstructed[12] from its fusion frame measurements $\{v_i P_i f\}_{i\in I}$ by performing

$$f = \sum_{i\in I} v_i S^{-1}(v_i P_i f), \tag{1.1}$$

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[*]The reader is referred to[18,19] and the references therein for numerous applications of frame theory in signal and image processing.

where $S = \sum_{i\in I} v_i P_i f$ is the *fusion frame operator* known to be positive and self-adjoint. If $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ is Parseval then $S = I$.

The value of fusion frames for signal processing is that the interplay between local-global processing and redundant representation provides resilience to noise and erasures due to, for instance, sensor failures or buffer overflows.[2, 11, 20, 22] It also provides robustness to subspace perturbations,[12] which may arise due to imprecise knowledge of sensor network topology. In most cases, extra structure on fusion frames is required to guarantee satisfactory performance. For instance, our recent work[20, 22] shows that in order to minimize the mean-squared error in the linear minimum mean-squared error estimation of a random vector from its fusion frame measurements in white noise the fusion frame needs to be Parseval or tight. The Parseval property is also desirable for managing signal processing complexity. To provide maximal robustness against erasures of one fusion frame subspace the fusion frame subspaces must also be equi-dimensional. If maximal robustness with respect to two or more subspace erasures is desired then the fusion frame subspaces must all have the same pairwise chordal distance as well. Other examples of optimality of *structured* fusion frames for signal reconstruction can be found in.[2, 3, 11, 12, 20, 22]

A natural question is: how can one construct fusion frames with desired fusion frame operators? More specifically, how can one construct fusion frames for which the fusion frame operator has a desired set of eigenvalues? In this paper, we present a complete answer to this question.

Our main contributions are as follow. In Section 2, we present two general ways for constructing a new fusion frame from a given fusion frame, by developing the notions of spatial complement and Naimark complement, and establish the relationship between the parameters of the two fusion frames. In Section 3, we establish existence conditions and develop simple algorithms for constructing fusion frames with desired fusion frame operators.[†] Our construction produces frames with desired frame operators as a special case.

We note that the construction of frames with arbitrary frame operators has been studied by several authors (see, e.g.,[1, 5, 10, 15]). However, the fusion frame counterparts are far less exploited. In fact, even establishing existence conditions for fusion frames is a deep and involved problem. Frame potentials, introduced in (cf.[1]), have proven to be a valuable tool in asserting the existence of tight frames. Two recent papers[7, 21] have introduced and studied *fusion frame potentials* to address the existence of fusion frames, but with limited success. The problem here is that minimizers of the fusion frame potential are not necessarily tight fusion frames. Also, the fusion frame potential is a very complex notion and it requires some deep ideas to make it work. However, until recently, no general construction method was known for the construction of fusion frames with desired properties. A significant advance for the construction of equi-dimensional tight fusion frames was presented in.[8] The authors have provided a complete characterization of triples (M, N, m) for which tight fusion frames exist. Here M is the total dimension of the Hilbert space, N is the number of subspaces, and m is the dimension of the fusion frame subspaces. They have also developed an elegant and simple algorithm which can produce a tight fusion frame for most (M, N, m) triples.

Our paper is concerned with a more general question than that answered in,[8] that is, the construction of a fusion frame (not necessarily tight) for which the fusion frame operator can possess any desired set of eigenvalues. This includes fusion frames with desired bounds as a special case, as the fusion frame bounds are simply the smallest and largest eigenvalues of the associated fusion frame operator. More specifically, given $M, N, m \in \mathbb{N}$, and a set of real positive values $\{\lambda_j\}_{j=1}^M$, we establish existence conditions for fusion frames whose fusion frame operators have eigenvalues $\{\lambda_j\}_{j=1}^M$ and develop a simple algorithm that produces such a fusion frame. Our solution also provides an answer to the construction of frames with arbitrary frame operators as a special case. The construction of frames with arbitrary frame operators was originally considered in,[5, 14] where the authors establish necessary and sufficient conditions for deriving a frame with a desired frame operator and prescribed norms for the frame vectors. The novelty of our approach is that we can arrive at any arbitrary frame operator from sets of orthogonal vectors.

[†]Throughout this paper whenever we say a fusion frame with a desired fusion frame operator we mean a fusion frame for which the fusion frame operator has a desired set of eigenvalues. A similar language is used to refer to a frame for which the frame operator has a desired set of eigenvalues.

Finally, we note that this paper is a summary of results. We have omitted the proofs, and derivations have been shortened or left out entirely. A comprehensive treatment of the subject along with detailed proofs is presented in.[6]

2. CONSTRUCTION OF NEW FUSION FRAMES FROM EXISTING ONES

In this section, we present two general ways, namely the spatial complement and the Naimark complement, for constructing a new fusion frame from a given fusion frame and establish the relationship between the parameters of the two fusion frames. A special case of the construction methods presented here is reported in.[8] The result of[8] deals only with Parseval fusion frames in a finite dimensional Hilbert space and does not investigate the relation between the new and the original fusion frame parameters.

2.1 The Spatial Complement

Definition 2.1. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. If the family $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$, where $W_i^\perp$ is the orthogonal complement of W_i, is also a fusion frame, then we call $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ the orthogonal fusion frame to $\{(\mathcal{W}_i, v_i)\}_{i\in I}$.*

Theorem 2.2. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with optimal fusion frame bounds $0 < A \leq B < \infty$. Then the following conditions are equivalent.*

(i) $\bigcap_{i\in I} \mathcal{W}_i = \{0\}$.

(ii) $B < \sum_{i\in I} v_i^2$.

(iii) The family $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ with optimal fusion frame bounds $\sum_{i\in I} v_i^2 - B$ and $\sum_{i\in I} v_i^2 - A$.

Proof. See.[6]

The following theorem shows that all the parameters of the new fusion frame can be determined from those of the generating fusion frame prior to the construction.

Theorem 2.3. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$, and let $\{(\mathcal{W}_i^\perp, v_i)\}_{i\in I}$ be its associated orthogonal fusion frame. Then the following conditions hold.*

(i) Let S denote the frame operator for $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ with eigenvectors $\{e_j\}_{j\in J}$ and respective eigenvalues $\{\lambda_j\}_{j\in J}$. Then the fusion frame operator for $\{(\mathcal{W}_i^\perp, v_i)\}_{i=1}^N$ possesses the same eigenvectors $\{e_j\}_{j\in J}$ and respective eigenvalues $\{\sum_{i\in I} v_i^2 - \lambda_j\}_{j\in J}$.

(ii) Assume that $\dim \mathcal{H} < \infty$ and $m := \dim \mathcal{W}_i$ for all $i \in I$. Then,

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = d_c^2(\mathcal{W}_i, \mathcal{W}_j) + 2m - \dim \mathcal{H} \quad \text{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

where $d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ denotes the squared chordal distance between $\mathcal{W}_i$ and $\mathcal{W}_j$ and is given by

$$d_c^2(\mathcal{W}_i, \mathcal{W}_j) = \dim \mathcal{H} - \mathrm{tr}[P_i P_j].$$

Proof. See.[6]

Corollary 2.1. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A-tight fusion frame for $\mathbb{R}^M$ such that $\mathcal{W}_k \neq \mathcal{H}$ for some $k \in \{1, \ldots, N\}$. Then $\{W_i^\perp\}_{i=1}^N$ is an $(N - A)$-tight fusion frame for $\mathbb{R}^M$. If $m := \dim \mathcal{W}_i$ for all $i \in \{1, \ldots, N\}$ and $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$, then*

$$d_c^2(\mathcal{W}_i^\perp, \mathcal{W}_j^\perp) = d^2 + 2m - M \quad \text{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

Proof. Assume that $W_k \neq \mathbb{R}^M$. Then by choosing some $0 \neq f \in W_k^\perp$, we obtain

$$A\|f\|^2 = \sum_{i=1}^N v_i^2 \|P_i f\|^2 = \sum_{i\neq k} v_i^2 \|P_i f\|^2 < \Big(\sum_{i=1}^N v_i^2\Big)\|f\|^2.$$

Thus we have $A < \sum_{i=1}^N v_i^2$, and the application of Theorem 2.2 proves the first part of the claim. The second part follows immediately from Theorem 2.3 (ii). $\square$

A straightforward application of Corollary 2.1 provides a way of constructing tight fusion frames with equi-dimensional subspaces.

Corollary 2.2. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a family of m-dimensional subspaces of $\mathbb{R}^M$. Then there exist $N(M-1)$ m-dimensional subspaces $\{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ of $\mathbb{R}^M$ so that $\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N(M-1)}$ is a tight fusion frame. Moreover, if $N = 1$ and $\dim \mathcal{W}_1 = M-1$ then the construction is minimal in the sense that it identifies the smallest number of m-dimensional subspaces which need to be added to obtain a tight fusion frame.*

Proof. For each $i = 1, \ldots, N$, we choose an orthonormal basis $\{e_j^i\}_{j=1}^M$ for $\mathbb{R}^M$ in such a way that $\{e_j^i\}_{j=1}^m$ is an orthonormal basis for $\mathcal{W}_i$. Let T_i, $i = 1, \ldots, N$, denote the circular shift operator on the orthonormal basis $\{e_j^i\}_{j=1}^M$. Then

$$\{T_i^k \mathcal{W}_i\}_{i=1,k=0}^{N\ ,M-1},$$

is a tight fusion frame for $\mathbb{R}^M$ of m-dimensional subspaces which contains $\{\mathcal{W}_i\}_{i=1}^N$.

Now consider the case where $N = 1$ and $\dim \mathcal{W}_1 = M-1$. Let $\{\mathcal{V}_i\}_{i=1}^{N_1}$ be any collection of $(M-1)$-dimensional subspaces so that $\{\mathcal{W}_1\} \cup \{\mathcal{V}_i\}_{i=1}^{N_1}$ is a tight fusion frame. By Theorem 2.2, we have $1 + N_1 = M$, hence $N_1 = M-1$, which equals $N(M-1)$. $\square$

2.2 The Naimark Complement

Another approach to constructing a new fusion frame from an existing one is to use the notion of *Naimark complement*. This approach however applies to Parseval fusion frames only, as stated in the following theorem.

Theorem 2.4. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Parseval fusion frame $\{(\mathcal{W}_i', \sqrt{1-v_i^2})\}_{i\in I}$ for $\mathcal{K} \ominus \mathcal{H}$ with the following properties.*

(i) $\dim \mathcal{W}_i' = \dim \mathcal{W}_i$ *for all* $i \in I$.

(ii) If $\dim \mathcal{H} < \infty$ *and* $\dim \mathcal{W}_i = \dim \mathcal{W}_j$ *for all* $i, j \in I$, $i \neq j$, *then*

$$d_c^2(\mathcal{W}_i', \mathcal{W}_j') = d_c^2(\mathcal{W}_i, \mathcal{W}_j) \quad \textit{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

Proof. See.[6]

Definition 2.5. *Let $\{(\mathcal{W}_i, v_i)\}_{i\in I}$ be a tight fusion frame for $\mathcal{H}$. We call the tight fusion frame $\{(\mathcal{W}_i', \sqrt{1-v_i^2})\}_{i\in I}$ for $\mathcal{K} \ominus \mathcal{H}$ from Theorem 2.4 as the Naimark fusion frame associated with $\{(\mathcal{W}_i, v_i)\}_{i\in I}$. The rationale for this terminology is that this is the fusion frame version of the Naimark theorem.*[4, 16, 17]

Corollary 2.3. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be an A-tight fusion frame for $\mathbb{R}^M$. Then there exists some $L \geq M$ and a $\sqrt{1-1/A^2}$-tight fusion frame for $\mathbb{R}^{L-M}$ which satisfies $\dim \mathcal{W}_i' = \dim \mathcal{W}_i$ for all $i \in \{1, \ldots, N\}$. If, in addition, $d^2 := d_c^2(\mathcal{W}_i, \mathcal{W}_j)$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$, then*

$$d_c^2(\mathcal{W}_i', \mathcal{W}_j') = d^2 \quad \textit{for all } i, j \in \{1, \ldots, N\},\ i \neq j.$$

Proof. This follows immediately from Theorem 2.4. $\square$

3. FUSION FRAMES WITH DESIRED FUSION FRAME OPERATORS

We now focus on the existence and construction of fusion frames whose fusion frame operators possess a desired set of eigenvalues.

Let $\lambda_1 \geq \ldots \geq \lambda_M > 0$, $M \in \mathbb{N}$, be real positive values satisfying a factorization as

$$\text{(Fac)} \qquad \sum_{j=1}^M \lambda_j = Nm \in \mathbb{N}.$$

We wish to construct a fusion frame $\{\mathcal{W}_i\}_{i=1}^N$, $\mathcal{W}_i \subseteq \mathbb{R}^M$, such that

(FF1) $\dim \mathcal{W}_i = m$ for all $i = 1 \ldots, N$, and

(FF2) the associated fusion frame operator has $\{\lambda_j\}_{j=1}^M$ as its eigenvalues.

3.1 The Integer Case

We first consider the simple case where $\lambda_j \in \mathbb{N}$ for all $i = 1, \ldots, M$. This case is central to developing intuition about the construction algorithms to be developed.

Proposition 3.1. *If the positive integers $N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M > 0$, $N \in \mathbb{N}$, and $m \in \mathbb{N}$ satisfy* (Fac)*, then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ constructed via the* (FFCIE) *algorithm outlined in Figure 1 satisfies both* (FF1) *and* (FF2)*.*

Proof. If the set of vectors $\{w_{i+km} : k = 0, \ldots, N-1\}$ is pairwise orthogonal for each $i = 1, \ldots, N$, then (FF1) and (FF2) follow automatically. Now fix $i \in \{1, \ldots, N\}$. By construction, it is sufficient to show that, for each $0 \leq k \leq N-2$, the vectors w_{i+km} and $w_{i+(k+1)m}$ are orthogonal. Again by construction, the only possibility for this to fail is that there exists some $j_0 \in \{1, \ldots, M\}$ satisfying $\lambda_{i_0} > N$. But this was excluded by the hypothesis. □

FFCIE (Fusion Frame Construction for Integer Eigenvalues)

Parameters:

- Dimension $M \in \mathbb{N}$.
- Integer eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of subspaces N, and dimension of subspaces m satisfying (Fac).

Algorithm:

1) For $j = 1, \ldots, M$ do
2) Set $k := 1$.
3) Repeat
4) $w_k := e_j$.
5) $k := k + 1$.
6) $\lambda_j := \lambda_j - 1$.
7) until $\lambda_j = 0$.
8) end.

Output:

- Fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with $\mathcal{W}_i := \mathrm{span}\{w_{i+km} : k = 0, \ldots, N-1\}$.

Figure 1. The FFCIE Algorithm for constructing a fusion frame with a fusion frame operator with prescribed integer eigenvalues.

The algorithm outlined in Figure 1 shuffles the intended eigenvalues in terms of associated unit vectors $e_1, \ldots, e_M \in \mathbb{R}^M$ as basis vectors into the subspaces of the fusion frame to be constructed. Considering a matrix $W \in \mathbb{R}^{Nm \times M}$ with the vectors $w_1, \ldots, w_{Nm}$ as rows, intuitively (FFCIE) fills this matrix up from top to bottom, row by row in such a way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j, $j = 1, \ldots, M$, and the columns are orthogonal. The vectors w_k are then assigned to subspaces in such a way that the vectors assigned to each subspace forms an orthonormal system. We note that the generated vectors w_k, $k = 1 \ldots, Nm$ are as sparse as possible, providing optimal fast computation abilities.

We wish to note that the condition $N \geq \lambda_1$ is necessary for (FFCIE). The question whether or not this is necessary in general is much more involved and will not be discussed here.

3.2 The General Case

We now discuss the general case where the desired eigenvalues for the fusion frame operator are real positive values that satisfy (Fac).

THEOREM 3.1. *Suppose the real values* $\lambda_1 \geq \cdots \geq \lambda_M$, $N \in \mathbb{N}$, *and* $m \in \mathbb{N}$ *satisfy* (FAC) *as well as the following conditions.*

(i) $\lambda_M \geq 2$.

(ii) If j_0 *is the first integer in* $\{1, \ldots, M\}$, *for which* λ_{j_0} *is not an integer, then* $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.

Then the fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ *constructed by the* (FFCRE) *algorithm outlined in Fig. 2 fulfills* (FF1) *and* (FF2).

Proof. See.[6]

FFCRE (FUSION FRAME CONSTRUCTION FOR REAL EIGENVALUES)

Parameters:

- Dimension $M \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of subspaces N, and dimension of subspaces m satisfying (FAC).

Algorithm:

For $j = 1, \ldots, M$ do
    Set $k := 1$.
    Repeat
        If $\lambda_j < 2$ and $\lambda_j \neq 1$ then
            $w_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $w_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
            $k := k + 2$.
            $\lambda_j := 0$.
            $\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
        else
            $w_k := e_j$.
            $k := k + 1$.
            $\lambda_j := \lambda_j - 1$.
        end;
    until $\lambda_j = 0$.
end;

Output:

- Fusion frame $\{\mathcal{W}_i\}_{i=1}^N$ with $\mathcal{W}_i := \text{span}\{w_{i+km} : k = 0, \ldots, N\}$.

Figure 2. The FFCRE algorithm for constructing a fusion frame with a desired fusion frame operator.

The principle for constructing the row vectors w_k which generate the subspaces $\mathcal{W}_i$ of the fusion frame is similar to that in (FFCIE), that is, again the matrix W which contains the vectors w_k, $k = 1 \ldots, Nm$ as rows is filled up from top to bottom, row by row in such as way that the ℓ_2 norm of the rows is 1, the ℓ_2 norm of column j is λ_j, $j = 1, \ldots, M$, and the columns are orthogonal. The vectors w_k are then assigned to subspaces in such a way that the vectors assigned to each subspace form an orthonormal system. However, here the task is more delicate since the λ_j's are not all integers. This forces the introduction of (2×2)-submatrices of the type

$$\begin{pmatrix} \sqrt{\frac{\lambda_j}{2}} & \sqrt{1-\frac{\lambda_j}{2}} \\ \sqrt{\frac{\lambda_j}{2}} & -\sqrt{1-\frac{\lambda_j}{2}} \end{pmatrix}.$$

These submatrices have orthogonal columns and unit norm (ℓ_2 norm) rows and allow us to handle non-integer eigenvalues. This construction was introduced in[8] for constructing tight fusion frames.

Theorem 3.1 is now applied to generate a tight fusion frame from a given fusion frame, satisfying some mild conditions.

THEOREM 3.2. *Let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for $\mathbb{R}^M$ with $\dim \mathcal{W}_i = m < M$ for all $i = 1, \ldots, N$, and let S be the associated fusion frame operator with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M$ and eigenvectors $\{e_j\}_{j=1}^M$. Further, let A be the smallest positive integer, which satisfies the following conditions.*

(i) $\lambda_1 + 2 \leq A$.

(ii) $AM = N_0 m$ *for some* $N_0 \in \mathbb{N}$.

(iii) $A \leq \lambda_1 + N_0 - (N+3)$.

Then there exists a fusion frame $\{\mathcal{V}_i\}_{i=1}^{N_0-N}$ for $\mathbb{R}^M$ with dim $\mathcal{V}_i = m$ for all $i \in \{1, \ldots, N_0 - N\}$ so that

$$\{\mathcal{W}_i\}_{i=1}^N \cup \{\mathcal{V}_i\}_{i=1}^{N_0-N}$$

is an A-tight fusion frame.

Proof. See.[6]

The number of m-dimensional subspaces added in Theorem 3.2 to force a fusion frame to become tight is in fact the smallest number that can be added in general. For this, let $\{\mathcal{W}_i\}_{i=1}^N$ be a fusion frame for $\mathbb{R}^M$ with fusion frame operator S having eigenvalues $\{\lambda_j\}_{j=1}^M$. Suppose $\{\mathcal{V}_i\}_{i=1}^{N_1}$ is any family of m-dimensional subspaces with fusion frame operator S_1, say, and so that the union of these two families is an A-tight fusion frame for $\mathbb{R}^M$. Thus

$$S + S_1 = AI,$$

which implies that the eigenvalues $\{\mu_j\}_{j=1}^M$ of S_1 satisfy

$$\mu_j = A - \lambda_j \quad \text{for all } j = 1, \ldots, M,$$

and

$$\sum_{j=1}^M \mu_j = \sum_{j=1}^M (A - \lambda_j) = AM - Nm = N_1 m.$$

In particular,

$$AM = (N_1 - N)m = N_0 m.$$

Thus, in general, fusion frames with the above properties of S_1 cannot be constructed unless the hypotheses of Theorem 3.2 are satisfied. This shows that the smallest constant satisfying this theorem is in general the smallest number of subspaces we can add to obtain a tight fusion frame.

3.2.1 Frames with Arbitrary Frame Operators

When $m = 1$ a fusion frame reduces to a frame. In such a case the (FFCRE) simplifies to an algorithm for constructing frames with desired frame operators, as Corollary 3.2 shows. This algorithm, which we refer to as (FCRE), is outlined in Figure 3.

Corollary 3.2. *Suppose the real values* $\lambda_1 \geq \cdots \geq \lambda_M$ *and* $N \in \mathbb{N}$ *satisfy*

$$\sum_{j=1}^{M} \lambda_j = N$$

as well as the following conditions.

(i) $\lambda_M \geq 2$.

(ii) If j_0 *is the first integer in* $\{1, \ldots, M\}$*, for which* λ_{j_0} *is not an integer, then* $\lfloor \lambda_{j_0} \rfloor \leq N - 3$.

Then the eigenvalues of the frame operator of the frame $\{w_k\}_{k=1}^N$ *constructed by* (FFCRE) *are* $\{\lambda_j\}_{j=1}^M$.

Proof. This result follows directly from Theorem 3.1 by choosing $m = 1$. □

FCRE (Frame Construction for Real Eigenvalues)

Parameters:

- Dimension $M \in \mathbb{N}$.
- Eigenvalues $\lambda_1 \geq \ldots \geq \lambda_M > 0$, number of frame vectors N satisfying (Fac) with $m = 1$.

Algorithm:

For $j = 1, \ldots, M$ do
Set $k := 1$.
Repeat
If $\lambda_j < 2$ and $\lambda_j \neq 1$ then
$w_k := \sqrt{\frac{\lambda_j}{2}} \cdot e_j + \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
$w_{k+1} := \sqrt{\frac{\lambda_j}{2}} \cdot e_j - \sqrt{1 - \frac{\lambda_j}{2}} \cdot e_{j+1}$.
$k := k + 2$.
$\lambda_j := 0$.
$\lambda_{j+1} := \lambda_{j+1} - (2 - \lambda_j)$.
else
$w_k := e_j$.
$k := k + 1$.
$\lambda_j := \lambda_j - 1$.
end.
until $\lambda_j = 0$.
end.

Output:

- Frame $\{w_k\}_{k=1}^N$.

Figure 3. The FCRE algorithm for constructing a frame with a desired frame operator.

We now present an example to demonstrate the application of (FCRE) as a special case of (FFCRE).

EXAMPLE 3.3. Let $M = 3$, $m = 1$ (special case of frame construction), $N = 8$, and $\lambda_1 = \frac{11}{4}$, $\lambda_2 = \frac{11}{4}$, $\lambda_3 = \frac{10}{4}$. Then, the algorithm constructs the following matrix W.

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \sqrt{3/8} & \sqrt{5/8} & 0 \\ \sqrt{3/8} & -\sqrt{5/8} & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{1/4} & \sqrt{3/4} \\ 0 & \sqrt{1/4} & -\sqrt{3/4} \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that indeed the ℓ_2 norm of the rows is 1, the ℓ_2 norm of the column j is λ_j, $j = 1, \ldots, M$, and the columns are orthogonal. The eigenvalues of the frame operator of the constructed frame $\{w_{k,\cdot}\}_{k=1}^{8}$ are indeed $\frac{11}{4}$, $\frac{11}{4}$, and $\frac{10}{4}$ as a simple computation shows. This also follows from Theorem 3.1 or Corollary 3.2 presented later in this subsection.

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