Efficient Large-Domain 2-D FEM Solution of Arbitrary Waveguides Using p-Refinement on Generalized Quadrilaterals

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***Abstract*—An efficient and accurate large-domain higher order two-dimensional (2-D) Galerkin-type technique based on the finite-element method (FEM) is proposed for analysis of arbitrary electromagnetic waveguides. The geometry of a waveguide cross section is approximated by a mesh of large Lagrangian generalized curvilinear quadrilateral patches of arbitrary geometrical orders (large domains). The fields over the elements are approximated by a set of hierarchical 2-D polynomial curl-conforming vector basis functions of arbitrarily high field-approximation orders. When compared to the conventional small-domain 2-D FEM techniques, the large-domain technique requires considerably fewer unknowns for the same (or higher) accuracy and offers a significantly faster convergence when the number of unknowns is increased. A comparative analysis of solutions using p- and h-refinements shows that the p-refinement represents a better choice for higher accuracy with lesser computation cost. In addition to increasing the field-approximation orders, the geometrical orders of elements (where needed) should also be set high for the improved accuracy of the solution without subdividing the elements. However, in general, an arbitrarily high accuracy cannot be achieved by performing the p-refinement in arbitrarily coarse meshes alone; instead, a combined hp-refinement should be utilized in order to obtain an optimal modeling performance.**

***Index Terms*—Computer-aided analysis, electromagnetic analysis, finite-element methods (FEMs), waveguides.**

I. Introduction

TWO-DIMENSIONAL (2-D) vector full-wave computation based on the finite-element method (FEM) is an important general tool for analysis and design of electromagnetic waveguides of arbitrary cross sections [1]–[9]. Accurate and efficient 2-D FEM evaluation of waveguide modes is important both on its own, for predicting the propagation characteristics (e.g., cutoff wavenumbers and propagation constants) of arbitrary waveguides [2]–[9], and as a part of three-dimensional (3-D) FEM–modal-expansion techniques for 3-D characterization (e.g., computation of scattering matrices) of multiport waveguide structures (3-D passive microwave devices) with arbitrary discontinuities [10], [11].

Although several recent breakthroughs in 2-D FEM modeling in high-frequency electromagnetics, such as tangential vector finite elements [2]–[5], [12], covariant-projection elements [6], and higher order field approximations [6]–[9], have led to more accurate and efficient FEM solutions of electromagnetic waveguiding structures, there are still challenges and need for further improvements in this area. Previous studies show that higher order basis functions enable more efficient FEM solutions with better convergence properties as compared to low-order solutions. However, almost none of the reported results demonstrate effectiveness of the large-domain (or entire-domain) modeling of arbitrary 2-D waveguides. The large-domain approach implies using a relatively small number of electrically large elements that are on the order of λ in each dimension, λ being the wavelength in the medium, with basis functions of sufficiently high orders for the approximation of the fields in the waveguide cross section. The goal is to obtain an "optimal" numerical model for a waveguide that would ensure a high level of accuracy of the results with a minimum number of unknowns and minimum computation cost (CPU time and memory usage). Instead, practically all existing 2-D FEM techniques for analysis of arbitrary waveguides are small-domain (subdomain) techniques—the waveguide cross section is modeled by a large number of electrically very small finite elements, most commonly triangles, which are on the order of $\lambda/10$ in each dimension. An exception is found in [9], where a two-element solution for a rectangular waveguide is demonstrated.

This paper proposes an efficient and accurate large-domain higher order finite-element technique for 2-D eigenvalue analysis of waveguides of arbitrarily shaped cross sections and with arbitrary inhomogeneous material loads. The surface planar elements proposed for the approximation of geometry of an arbitrary waveguide cross section are generalized Lagrangian curvilinear parametric quadrilaterals of arbitrary geometrical orders. The 2-D basis functions proposed for the approximation of fields over the elements are hierarchical polynomial curl-conforming vector basis functions of arbitrarily high field-approximation orders. These functions are very suitable for p-refinement, where the field-approximation orders over elements (where needed) are increased to improve the accuracy of the solution without subdividing the elements. The same quadrilateral (in a generally nonplanar form) and similar

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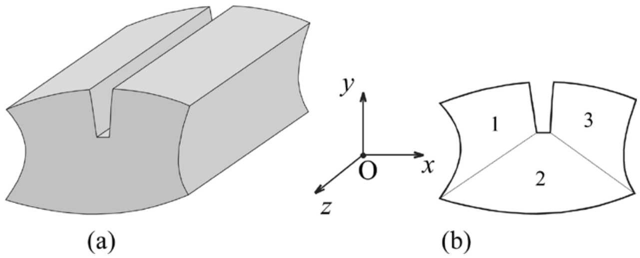

Fig. 1. 2-D FEM analysis of arbitrary electromagnetic waveguides. (a) Waveguide geometry. (b) Large-domain 2-D mesh using generalized curvilinear parametric quadrilaterals of higher geometrical orders.

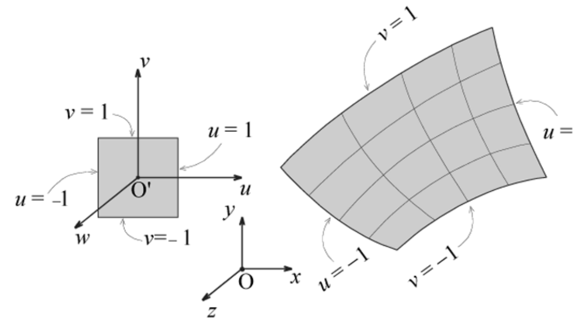

Fig. 2. Square-to-generalized quadrilateral mapping defined by (1) and (2).

twofold higher order polynomial basis functions in a divergence-conforming version have been used in a large-domain method of moments (MoM) technique for solving surface integral equations [13]. These proposed 2-D finite elements, on the other side, are developed as a 2-D version of the large-domain higher order hexahedral finite elements from [11] and [14]. The current technique is implemented with the electric-field vector formulation of the 2-D FEM eigenvalue analysis of waveguides based on transversal and axial (longitudinal) electric-field wave equations, which are discretized using the weighted residual (Galerkin) testing procedure.

The performance of the new technique has been tested in four characteristic examples of 2-D waveguides. The results obtained by the large-domain FEM are validated and evaluated in comparisons with exact solutions (where applicable) and the numerical results obtained by the existing higher order FEM techniques, as well as other results. A detailed numerical study is presented of convergence properties of large-domain elements, including a comparative analysis of solutions using p-refinement and h-refinement (mesh refinement) in terms of accuracy and computation costs.

II. Theory and Implementation

Consider a waveguide with an arbitrarily shaped cross section, as shown in Fig. 1(a). Let the z-axis of the global Cartesian coordinate system be in the longitudinal direction of the waveguide. In our analysis method, the waveguide cross section is first tessellated using generalized Lagrangian curvilinear parametric quadrilaterals of higher (theoretically arbitrary) geometrical orders, as indicated in Fig. 1(b). A generalized quadrilateral is analytically described as

$$\boldsymbol{r}(u,v)=\sum_{i=1}^{M}\boldsymbol{r}_i\hat{L}_i^{K_u,K_v}(u,v), \qquad -1\le u,v\le 1 \tag{1}$$

where $\boldsymbol{r}_1,\boldsymbol{r}_2\ldots,\boldsymbol{r}_M$ are the position vectors of the interpolation nodes. The polynomials $\hat{L}_i^{K_u,K_v}(u,v)$ are defined as

$$\begin{aligned}\hat{L}_i^{K_u,K_v}(u,v)=L_m^{K_u}(u)L_n^{K_v}(v),&\\ i=1+m+n(K_u+1);&\\ 0\le m\le K_u;&\\ 0\le n\le K_v;&\\ 1\le i\le M=(K_u+1)(K_v+1)&\end{aligned} \tag{2}$$

where $L_m^{K_u}(u)$ represents Lagrange interpolating polynomials with uniformly spaced interpolation nodes along an interval $-1\le u\le 1$, and similarly for $L_n^{K_v}(v)$. Equations (1) and (2) define a mapping from a square parent domain to the generalized quadrilateral, as illustrated in Fig. 2. Although this mapping, in principle, leads to generalized quadrilaterals with nonplanar surfaces in 3-D [13], only planar (2-D) curvilinear quadrilaterals are used in our model (all the interpolation nodes belong to the waveguide cross section considered, i.e., to the x-y $(z=0)$ plane in Fig. 1). Note that the covariant-projection element used in [6] can be obtained as a special case of the generalized quadrilateral in Fig. 2 by letting $K_u=K_v=2$ in (2).

After the transversal and longitudinal components of the electric-field intensity vector in the cross section of the waveguide are transformed as in [8, eqs. (3) and (4)], the following expansions are used over each of the quadrilateral patches in the model:

$$\begin{aligned}\boldsymbol{e}_t&=\sum_{i=0}^{N_u-1}\sum_{j=0}^{N_v}\alpha_{uij}\boldsymbol{f}_{uij}+\sum_{i=0}^{N_u}\sum_{j=0}^{N_v-1}\alpha_{vij}\boldsymbol{f}_{vij}\\ e_z&=\sum_{i=0}^{N_u}\sum_{j=0}^{N_v}\alpha_{zij}f_{zij}\end{aligned} \tag{3}$$

where $\boldsymbol{f}$ are curl-conforming hierarchical-type vector basis functions defined as

$$\begin{aligned}\boldsymbol{f}_{uij}&=u^iP_j(v)\boldsymbol{a}'_u\\ \boldsymbol{f}_{vij}&=P_i(u)v^j\boldsymbol{a}'_v\\ f_{zij}&=P_i(u)P_j(v).\end{aligned} \tag{4}$$

The P-functions are the polynomials defined in [14, eq. (4)], N_u and N_v are the adopted degrees of the polynomial approximation for fields, which are entirely independent from the element geometrical orders K_u and K_v, and α_{uij}, α_{vij}, and α_{zij} are unknown field-distribution coefficients. The reciprocal unitary vectors $\boldsymbol{a}'_u$ and $\boldsymbol{a}'_v$ in (4) are obtained as

$$\boldsymbol{a}'_u=\frac{(\boldsymbol{a}_v\times\boldsymbol{a}_z)}{J}\quad \boldsymbol{a}'_v=\frac{(\boldsymbol{a}_z\times\boldsymbol{a}_u)}{J}\quad J=|\boldsymbol{a}_u\times\boldsymbol{a}_v| \tag{5}$$

where J is the Jacobian of the covariant transformation, $\boldsymbol{a}_u=\partial\boldsymbol{r}/\partial u$ and $\boldsymbol{a}_v=\partial\boldsymbol{r}/\partial v$ are local unitary vectors, with $\boldsymbol{r}$ given in (1), and $\boldsymbol{a}_z$ is the unit vector along the global z-axis.

Polynomial degrees N_u and N_v in (3) can be high so that electrically large quadrilateral patches (large domains) that are up to 2λ in each transversal dimension ($\sim 4\lambda^2$ in area) can be used, thus fully exploiting the accuracy, efficiency, and convergence properties of the higher order FEM. Moreover, basis functions in (4) are hierarchical functions—each lower order set of functions is a subset of all higher order sets. With this, different orders of field approximation, along with different geometrical orders, in different elements, as well as in different transversal directions within each element, can be used in a simulation model. Hierarchical basis functions, on the other hand, generally have poor orthogonality properties, which results in FEM matrices with large condition numbers. However, if needed, the orthogonality and conditioning properties of functions in (4) can be improved by modifying them as in [15].

A symmetric Galerkin-type weak-form discretization of the coupled electric-field vector wave equations for the transversal and longitudinal fields for the general 2-D waveguide problem in Fig. 1 yields [8]

$$\int_S \Big\{ \mu_r^{-1}(\nabla_t \times \boldsymbol{f}_{t\hat{i}\hat{j}}) \cdot (\nabla_t \times \boldsymbol{e}_t) - \varepsilon_r(\boldsymbol{f}_{t\hat{i}\hat{j}} - \nabla_t f_{z\hat{i}\hat{j}}) \cdot (\boldsymbol{e}_t - \nabla_t e_z) - \gamma^2 \left(\mu_r^{-1} \boldsymbol{f}_{t\hat{i}\hat{j}} \cdot \boldsymbol{e}_t - \varepsilon_r f_{z\hat{i}\hat{j}} e_z \right) \Big\} dS = 0 \quad (6)$$

where ϵ_r and μ_r are the complex relative permittivity and permeability, respectively, of the inhomogeneous (possibly lossy) medium inside the waveguide, γ is the propagation constant along the waveguide ($\gamma = j\beta$ for lossless media), S is the surface of a generalized quadrilateral in the model, and $\boldsymbol{f}_{t\hat{i}\hat{j}}$ stands for any of the testing (weighting) functions $\boldsymbol{f}_{u\hat{i}\hat{j}}$ or $\boldsymbol{f}_{v\hat{i}\hat{j}}$ [testing functions are the same as basis functions in (4)]. Substituting the field expansions (3) into (6) leads to a generalized eigensystem, which is solved using the constrained Lanczos algorithm [8].

III. Results and Discussion

As the first example, consider an air-filled rectangular waveguide. Its cross section, shown in the inset in Fig. 3, is modeled by a single bilinear ($K_u = K_v = 1$) quadrilateral element (which, in this case, reduces to a rectangle). Note that this is literally an entire-domain 2-D FEM model (an entire computational domain is represented by a single finite element). Fig. 3 shows the relative error for the computed effective relative dielectric constant $\varepsilon_{r_{\text{eff}}} = \beta^2/k_0^2$, k_0 being the free-space wavenumber, for the dominant (TE_{10}) mode of the waveguide $\delta\varepsilon_{r_{\text{eff}}} = |1 - \varepsilon_{r_{\text{eff}}}/\varepsilon_{r_{\text{eff}}\text{exact}}|$ plotted against the total number of FEM unknowns. The results obtained by the current method are compared with FEM results from [8], where higher order small-domain triangular elements with orthogonal basis functions are used. For both FEM techniques, the convergence rate is presented for both h- and p-refinements. In the current analysis, the points on the h-refinement (or mesh refinement) curve are obtained using 2, 8, 64, and 256 regularly distributed quadrilateral elements (squares), respectively, with $N_u = N_v = 1$. The points on the p-refinement curve correspond to an entire-domain model (single quadrilateral element) with $N_v = N_u/2$ being varied from 1 to 4. It can be concluded based on this figure that both methods yield results of similar accuracy with h-refinement. However, when p-refinement is carried out, the large-domain approach presented in this paper yields the results of a considerably superior accuracy-to-number of unknowns ratio, as compared to the small-domain approach in [8]. In addition, we note that the results in Fig. 3 are consistent with the theoretical prediction for the convergence rate for FEM regions with smooth fields [9, eq. (38)], as well as that the p-refinement curve obtained by the current method can actually be considered as an extension of the series of curves shown in [9, Fig. 4] toward the left-hand (better efficiency) side of the diagram.

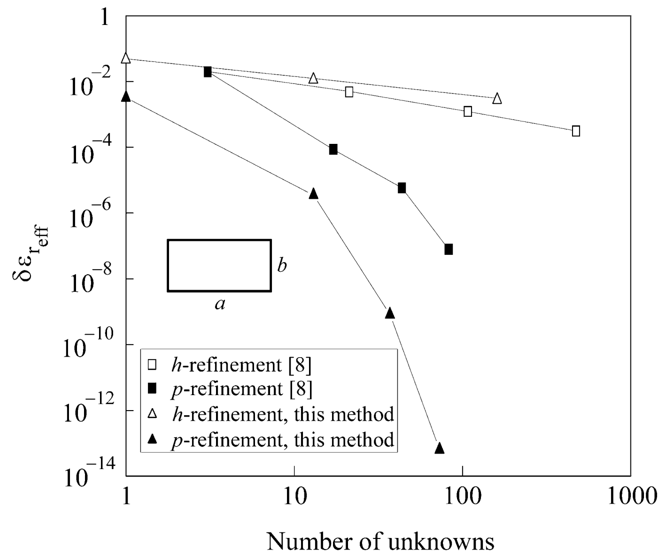

Fig. 3. Relative error in calculating the effective dielectric constant for the dominant mode of an air-filled rectangular waveguide ($a = 16$ mm, $b = 8$ mm) at 20 GHz against the number of unknowns with h- and p-refinement, respectively. Comparison of the results obtained by the current method and the higher order small-domain FEM [8].

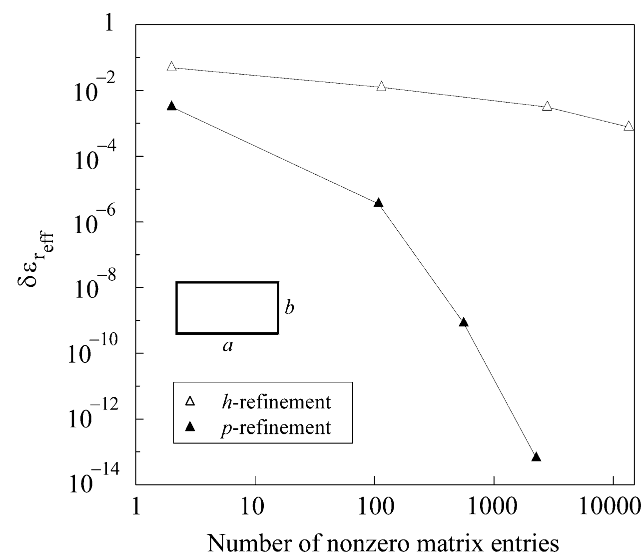

Fig. 4. Relative error for the effective dielectric constant for the dominant mode of the rectangular waveguide (see Fig. 3) using the current method against the number of nonzero matrix elements.

Shown in Figs. 4 and 5 are relative errors for the effective dielectric constant for the dominant mode of the rectangular waveguide using the current method with the h- and p-refinements

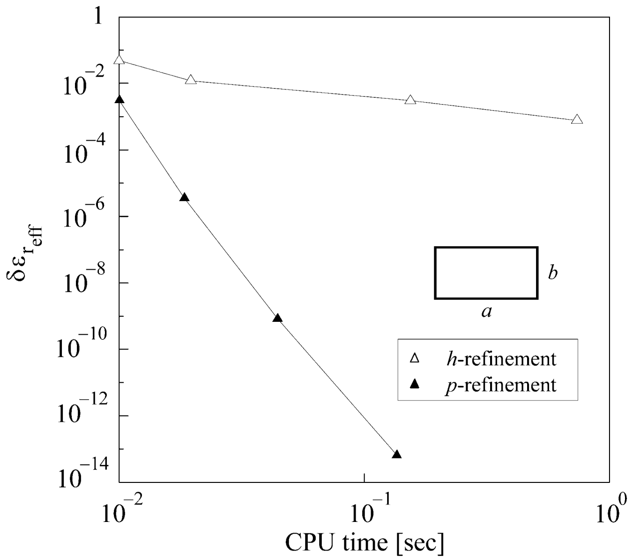

Fig. 5. Relative error from Fig. 4 plotted against the CPU time.

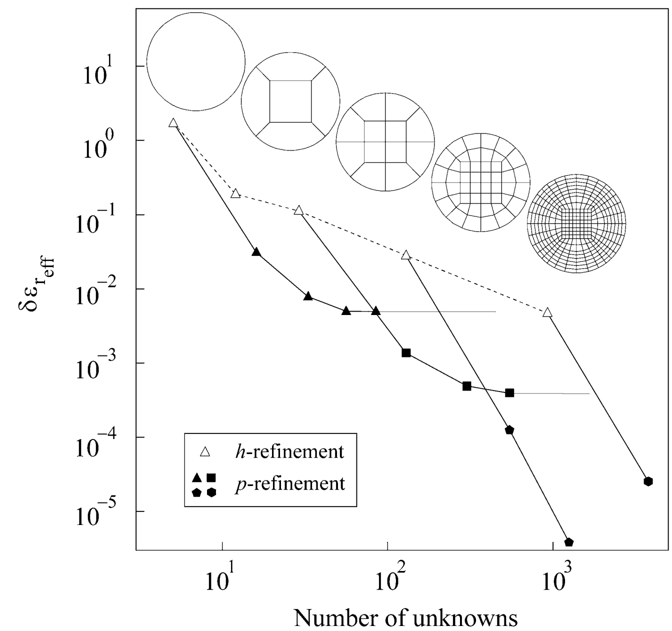

Fig. 7. Relative error in calculating the effective dielectric constant averaged for the four dominant modes of an air-filled circular waveguide 2 cm in diameter at 20 GHz against the number of unknowns with h- and p-refinement, respectively.

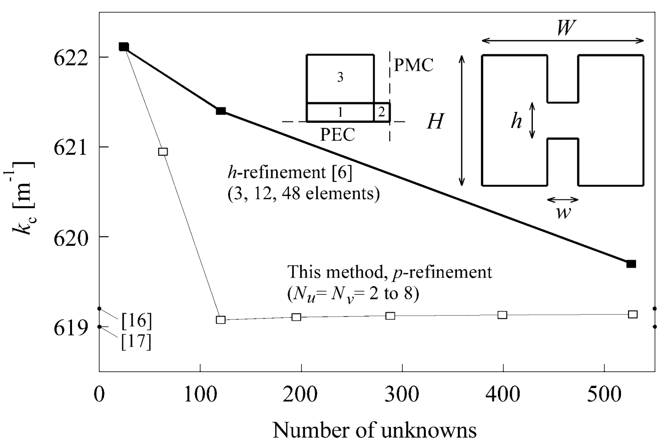

Fig. 6. Cutoff wavenumber of a double-ridged waveguide ($W = 12.7$ mm, $H = 10.16$ mm, $w = 2.54$ mm, and $h = 2.794$ mm) for the second mode. Comparison of a large-domain solution using a three-element mesh shown in the inset and p-refinement and an alternative FEM solution with h-refinement [6]. The results from [16] and [17] (indicated on the k_c-axis) are given as reference (true) values.

plotted against the total number of nonzero elements in system matrices and CPU time, respectively. A PC with 3-GHz CPU and 1 GB of RAM is used for computations in this and all the following examples. The results demonstrate that, in this case, it takes much less computational storage and time to obtain a certain level of accuracy by using the entire-domain model and p-refinement than with the h-refinement.

The second example is a double-ridged air-filled waveguide shown in the inset in Fig. 6. The analysis is performed over a quarter of the waveguide cross section only by introducing a perfect electric conductor (PEC) and perfect magnetic conductor (PMC) in the symmetry planes. In this example only, the E-field version of the H-field formulation that solves for the free-space wavenumber given in [7, eqs. (6)–(8)] is used. In Fig. 6, the results for the cutoff wavenumber k_c for the second waveguide mode obtained using p-refinement on a large-domain three-element quadrilateral mesh (shown in the inset) are compared with those obtained by an alternative higher order FEM technique [6] (E-field formulation version), where second-order covariant-projection elements and h-refinement are employed. Since there is no exact (analytical) solution for this geometry, two additional independent results, from [16] (FEM) and [17] (Ritz–Galerkin solution to an integral eigenvalue equation), are also included in this figure, as reference (true) values. We observe from this figure that the higher order large-domain model with p-refinement exhibits much better accuracy and convergence rate (toward the reference values from [16] and [17]) when compared with the second-order modeling and h-refinement from [6].

Next, consider a circular air-filled waveguide. Shown in Figs. 7–9 are the relative errors of the computed effective dielectric constant of the waveguide averaged for the four dominant modes and plotted against the number of FEM unknowns, number of nonzero matrix elements, and CPU time, respectively. The points on the broken line in each of these figures are obtained by using the meshes with 1, 5, 12, 48, and 320 elements (h-refinement), respectively, as indicated in the inset in Fig. 7, and the second-order field-approximation functions ($N_u = N_v = 2$) in the single-element model and first-order field-approximation functions ($N_u = N_v = 1$) in all other models. The points on the solid lines in these figures are obtained by p-refining the respective initial models, i.e., by incrementing both N_u and N_v (in all elements) by one for each additional point in the graph (the results of the p-refinement of the five-element model are similar to those for the single-element model and are, therefore, omitted for the clearness of these figures). In any given mesh shown in Fig. 7, a layer of quadrilaterals with geometrical orders $K_u = 2K_v = 2$ is placed along the edge of the waveguide (these elements are as conformal as possible to the circular boundary), while bilinear elements ($K_u = K_v = 1$) are used for modeling of the interior of the waveguide cross section. It can be observed

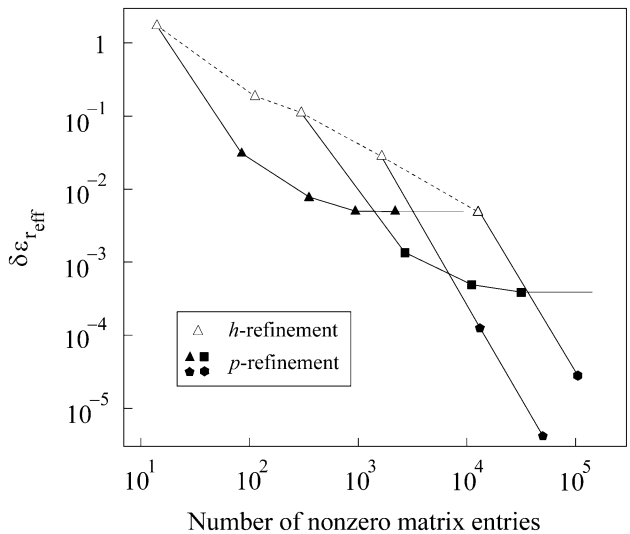

Fig. 8. Relative error from Fig. 7 against the number of nonzero matrix elements.

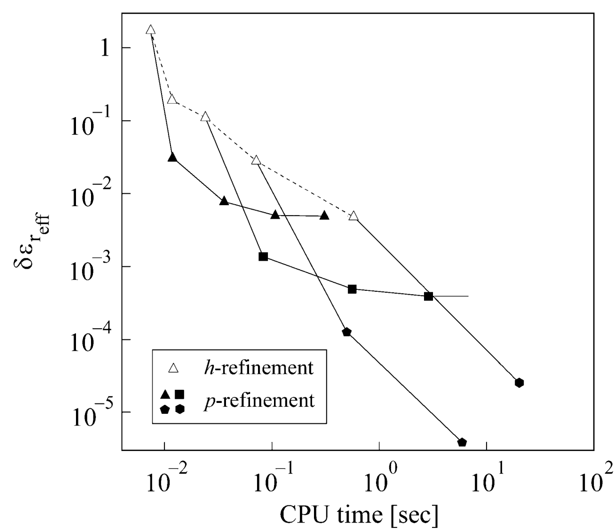

Fig. 9. Relative error from Fig. 7 against the CPU time.

from Figs. 7–9 that, in this example, the p-refinement represents a better choice in terms of achieving a certain level of accuracy with lesser computation cost. For instance, a 0.5% accuracy can be achieved by both the single-element model with $N_u = N_v = 5$ and 320-element model with $N_u = N_v = 1$, and the number of unknowns and the number of nonzero matrix elements are by an order of magnitude smaller in the first case. However, these figures also show that an arbitrarily high accuracy cannot be achieved by performing the p-refinement alone; instead, a combined hp-refinement should be utilized in order to obtain an optimal modeling performance. This conclusion holds generally in a sense that there exists a low-error bound beyond which a p-refinement in arbitrarily coarse meshes does not improve further the accuracy of the solution and the size of elements needs to be reduced as well [18].

To investigate the influence of the accuracy of the geometrical approximation of the circular boundary to the overall accuracy of the large-domain modeling of the circular waveguide, shown in Fig. 10 are two sets of results obtained by p-refining of

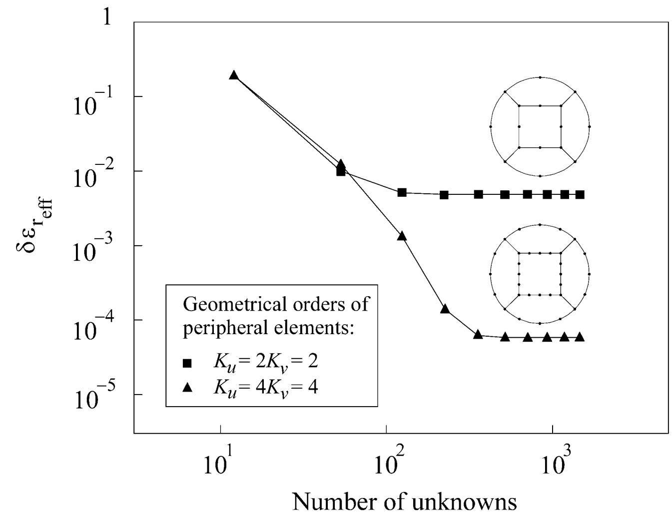

Fig. 10. Comparison of two large-domain p-refined five-element solutions for the effective dielectric constant averaged for the four dominant modes of the circular waveguide in Fig. 7 using second- and fourth-order Lagrangian geometrical elements, respectively, near the waveguide edge.

two five-element meshes of the waveguide. The geometrical orders of the curvilinear quadrilaterals on the circular boundary in the two meshes are adopted to be $K_u = 2K_v = 2$ and $K_u = 4K_v = 4$, respectively, and the two models, with indicated interpolation nodes of the Lagrangian elements, are shown in the inset in this figure. The field-approximation orders across elements $(N_u = N_v)$ are varied from 1 to 10 in both models. As can be seen from this figure, the convergence rate of the p-refinement with the fourth-order Lagrangian elements is maintained well into the low-error region with the error being by two orders of magnitude lower than with the second-order curvilinear elements.

A similar numerical experiment to that in Fig. 10 has been conducted with entire-domain (single-element) models of the circular waveguide using the second- and fourth-order geometrical approximations, respectively. In this case, however, no improvement is observed when the second-order Lagrangian element is substituted by a fourth-order element. Instead, the p-refinement of the single-element model yields unpredictable oscillating errors in the solution. This may be attributed to the fact that the entire-domain models of the circular waveguide involve severe distortions of parametric lines and introduce ill-posed, almost singular, values for the Jacobian in (5). These deviations from the rectangular shape are especially drastic at element vertices, where the unitary vectors are almost collinear, and are more pronounced in the fourth-order model. Note that such instability of the results due to the extreme distortion of the elements is not observed in the 3-D entire-domain case [14], where a spherical cavity is successfully modeled with both second- and fourth-order entire-domain (single-element) curvilinear hexahedra.

As the last example, consider two coupled microstrip lines on a cylindrical substrate shown in the inset in Fig. 11. The computational domain is halved by introducing a PEC or PMC in the symmetry plane for the analysis of odd or even modes, respectively. The FEM computation is carried out using a 14-element higher order mesh consisting of both large and small elements

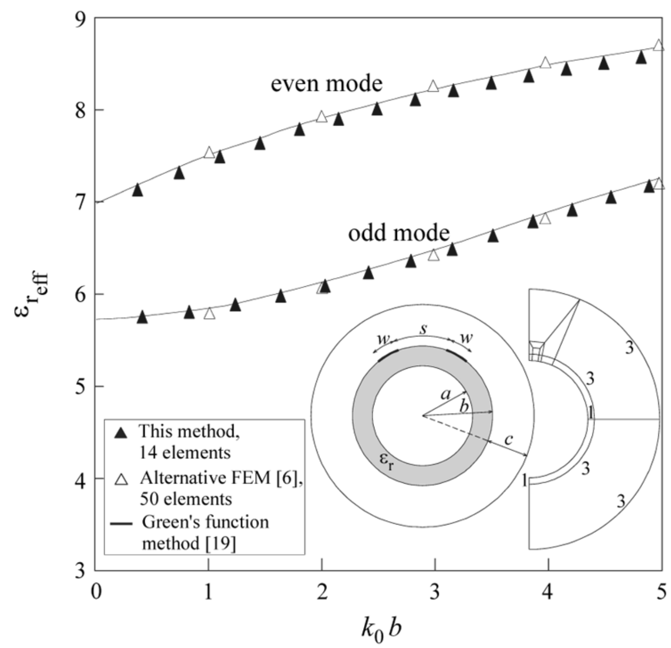

Fig. 11. Dispersion curves of two coupled microstrip lines on a cylindrical substrate, the cross section of which is shown in the inset ($a/b = 0.9$, $c/h = 10$, $s/h = 1$, $w/h = 1$, $h = b - a$, and $\varepsilon_r = 9.6$). Comparison of the large-domain higher order FEM solution using a 14-element quadrilateral mesh also shown in the inset with designated field-approximation orders along the element edges (where not shown, the adopted order is two), an alternative second-order FEM solution with covariant-projection elements [6], and a dyadic Green's function Galerkin solution for the unbounded problem [19].

of very different shapes (Fig. 11). The geometrical orders of the elements are $K_u = 2$ and $K_v = 1$. The orders of the field-approximation polynomials along some of the element edges are shown in this figure; for the edges where the numbers are not shown, the adopted polynomial orders are $N_{u \text{ or } v} = 2$. The FEM domain is closed by introducing a PEC cylinder around the structure at a distance from the strips that is ten times the substrate thickness. This model results in a total of 154 or 138 unknowns for the analysis of even or odd modes, respectively, and requires 0.06 s of CPU time per frequency. Fig. 11 shows the dispersion curves of the structure. The results obtained by the current method are compared with the results presented in [6] (FEM) and [19] (dyadic Green's function Galerkin solution for the unbounded problem), and an excellent agreement of the three sets of results is observed. The FEM results presented in [6] are obtained using 50 second-order covariant-projection elements for a half of the structure, and the number of unknowns is not reported.

IV. Conclusions

This paper has proposed an efficient and accurate large-domain higher order Galerkin-type finite-element technique for 2-D analysis of arbitrary electromagnetic waveguides. The geometry of a waveguide cross section is approximated by a mesh of generalized Lagrangian curvilinear parametric planar quadrilaterals of arbitrary geometrical orders. The fields over the elements are approximated by a set of hierarchical 2-D polynomial curl-conforming vector basis functions of arbitrary field-approximation orders, and p-refinement is used.

The results obtained by the new technique have been validated and evaluated in four characteristic examples of 2-D waveguides. The examples have demonstrated very effective large-domain meshes constructed from a very small number of generalized quadrilateral elements (large domains) with field approximations of high orders. When compared to the conventional small-domain 2-D FEM techniques, the presented large-domain technique requires considerably fewer unknowns for the same (or higher) accuracy and offers a significantly faster convergence when the number of unknowns is increased using p-refinement of solutions.

By a comparative analysis of solutions using p-refinement and h-refinement (mesh refinement) within the current method, it has been shown that the p-refinement represents a better choice in terms of achieving a certain level of accuracy with much lesser computation cost (computational storage and time). In addition to increasing the field-approximation orders, the geometrical orders of elements (where needed) should also be set high for the improved accuracy of the solution without subdividing the elements. However, in general, an arbitrarily high accuracy cannot be achieved by performing the p-refinement in arbitrarily coarse meshes alone; instead, a combined hp-refinement should be utilized in order to obtain an optimal modeling performance.

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