Computers and Fluids, 1976, Vol. 4, pp. 93–107. Pergamon Press. Printed in Great Britain

AN ALGEBRAIC STRESS MODEL FOR STRATIFIED TURBULENT SHEAR FLOWS

ROBERT N. MERONEY

Department of Civil Engineering, Engineering Research Center, Colorado State University, Fort Collins, CO 80523, U.S.A.

(*Received in revised form* 28 *January* 1976)

Abstract—Utilizing a simple time dependent one dimensional example as a test case this paper discusses a solution which represents the important characteristics of a bouyancy dominated shear flow by solving four partial differential equations in addition to the mean equations of motion. This suggested model solves equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ϵ, and thermal eddy dissipation, ϵ_t. Three separate versions of this model are discussed—an algebraic length scale version, a Prandtl–Kolmogorov eddy viscosity version, and an algebraic stress and heat flux model. The final version (requiring six partial differential equations) manages to replicate results for a much more complicated version (requiring ten partial differential equations). The advantages for two and three dimensional problems are even greater.

NOMENCLATURE

A_1, A_2, A_3 C_D, C_H C_{E1}, C_{E2} C_{Et1}, C_{Et2}, F C_{P1}, C_{P2}, C_{P3}	constants, see Table 13
g	gravitational constant
l	mixing length or length scale
k	turbulent kinetic energy $(\overline{u_i'u_i'})/2$
k_t	turbulent temperature fluctuations $(\overline{T'^2})/2$
p	pressure
Pr	Prandtl number (ν/α)
Re	Reynolds number $(u_{max}L/\nu)$
Ri	Richardson number $(g/T)[\Delta T_{max}L/(u_{max})^2]$
t	time
T	temperature
u, v, w	velocity components
x, y, z	coordinate directions
X	body force
α_T	eddy diffusivity
ϵ	eddy dissipation of turbulent kinetic energy
ϵ_t	eddy dissipation of turbulent temperature fluctuations
λ, Λ	length scales: Donaldson
ν_T	eddy viscosity
ρ	density
$\sigma_k, \sigma_{k_t}, \sigma_\epsilon, \sigma_{\epsilon_t}$	effective Prandtl number for k, k_t, ϵ, ϵ_t
ϕ	gravitational potential

Subscripts

i, j, k	direction indices

Superscripts

$'$	fluctuating quantity

1. INTRODUCTION

Experimental data invariably show that for positive values of the Richardson number all turbulent fluctuating properties are suppressed by the action of the buoyancy force, while for negative values of the Richardson number turbulent fluctuating properties are accentuated[1]. However there is a vast difference between the behavior of $\overline{w'T'}$, $\overline{u'T'}$ and $\overline{u'w'}$ when $\partial T/\partial z < 0$ and when $\partial T/\partial z > 0$.† In the stable case the heat flux is often very small or negligible despite finite gradients of temperature, yet momentum transport may still be finite. In the unstable case a large heat flux is established quite rapidly, momentum transport may be significantly smaller in

CEP73-74RNM53

†In the atmosphere T must be interpreted as potential temperature.

proportion, yet temperature gradients may be near zero. It appears that buoyancy-generated eddies cause relatively little momentum transport, but they are quite effective at carrying thermal energy. In other words, the rates of the associated turbulent diffusivities for heat and momentum are much larger than one, Reynolds analogy does not apply, and the idea of a gradient transport hypothesis in a stratified medium is completely wrong. Use of the diffusivity concept in calculations thus would tend to develop too rapid dissipation of inversions, and too slow a growth of turbulence in unstable situations.

These physical considerations suggest that an adequate theory for the treatment of the interaction of stratification, gravity, and a turbulent field must include transport equations for the second order correlations or their equivalent. Work by Donaldson *et al.*[2, 3], Lewellen and Teske[4] and Mellor[5] do consider the second order correlation equations including stratification effects. Lumley[6] has also proposed sets of equations closed at the third order correlations, while Lee[7] has developed a set of expressions based on analogies between turbulence and Brownian motion utilizing the Fokker–Planck equations.† The ability of such formulations to follow the effects of stratification on turbulence are impressive. Unfortunately one must simultaneously solve a set of at least nine to as many as twelve partial differential equations for even a one-dimensional incompressible flow situation. For the equivalent two- or three-dimensional cases the ranges required are from ten to thirteen and from fourteen to seventeen partial differential equations respectively.

Such methods must thus be limited to research areas for the great majority of cases. Those situations requiring planning or engineering information generally must consider many case permutations; thus they require a method which retains the essential physical characteristics but with a lower order of solution complexity. This report discusses the efficacy of three such solution techniques. These will be discussed under the titles of

(a) An algebraic lengths scale model (ALM),

(b) A differential length scale model (DLM), and

(c) An algebraic stress model (ASM).

The number of partial differential equations required are of the order of six, seven and eight for one-, two- and three-dimensional motions.

Similar thoughts toward model simplification have been expressed by Deardorff, Donaldson[3] and Mellor and Yamada[9]. Donaldson speaks of a "super-equilibrium" situation where all advective terms are neglected and gradient transport assumptions are introduced for all second order turbulence correlations. The approach did not define length scales or permit transient phenomenon. Mellor and Yamada follow a parallel approach to that found herein except that the authors chose not to solve a transport equation for dissipation.

2. TURBULENT MODELS

In any model developed for turbulent closure one would like to have the method possess width of applicability, accuracy, economy of computational time, and simplicity. In the search for these elusive features many closures for the turbulent equations of change have been proposed[10].

Reynolds[11] has suggested a morphology for classifying methods of closure. He proposes methods which make use of eddy viscosity or mixing length concepts which will be called "mean field methods", (MF) whereas methods which relate the Reynolds stress to the turbulence and hence require calculation of some aspects of the turbulence fields will be called "turbulent-field methods" (MTF). Subsequent reviewers of turbulent models have accepted this as a critical distinction Bradshaw[12, 13]. Mellor and Herring[14] suggest two subsets of the MTF group. Those which include a turbulent kinetic energy transport equation and some accommodation for length scales are termed "mean turbulent energy" closures (MTE); whereas a "mean Reynolds stress" closure (MRS) implies a closed set of equations which include equations for all nonzero components of the Reynolds stress.

As is always the case a difficult problem soon becomes muddled again even with respect to

†Hossain and Rodi[8] compiled an exhaustive review of the governing equations for turbulent buoyant flow. Their report provides a comprehensive foundation for future attempts to numerically examine the influence of buoyancy on turbulence.

categories such as the above. The recent work by Hanjalic and Launder[15], Rodi[16], Mellor and Yamada[9] and the present suggestions may lie somewhat between the MTE and MRS classifications.

2.1 *Mean turbulent energy methods* (*MTE*)

A basis for the MTE calculations began with semiheuristic models of Prandtl[17]. They suggest the use of a turbulent kinetic energy transport equation, a turbulent-energy related eddy viscosity, and a prescribed length scale function or a differential equation for length scale.

Bradshaw[12, 13] has been very critical of methods which retain an explicit algebraic relation between stresses and the mean flow. His criticisms are related to the ad hoc nature of any eddy-viscosity transport relation, the failure to provide correct results in those cases where there is finite transport and zero velocity gradient, and the basically regressive concept of going to the trouble to solve additional transport equations and then reapplying a local-equilibrium assumption to relate stress and gradient. Mellor and Herring[14] appear more optimistic, they try to show how MTE models derive logically from the MRS models and how both involve essentially the same empirical information. Launder and Spalding[10] have reviewed the results of most of the effort in this area.

2.2 *Mean Reynolds stress methods* (*MRS*)

As noted before, MRS closure implies a closed set of equations which include equations for all nonzero components of the Reynolds stress tensor. Rotta[17] laid the foundation for future efforts when he proposed the pressure-velocity correlation terms in the Reynolds stress equations be proportional to a deviation from isotropy. This assumption was of course an approximation and was subject to modification by subsequent investigators. Other terms in the Reynolds stress equations such as the dissipation and diffusion terms have also been modeled differently by various investigators.

Only a few MRS calculations have been made for comparison with experiment. To date Donaldson *et al.* [2, 3, 7] have compared results with flat plate boundary layer flows, free turbulent shear flows, transport in the atmospheric shear layer, and in vortex motions. Daly and Harlow[19], using a dissipation length scale transport equation, have made MRS calculations for a channel flow. Mellor and Herring[14] compared MRS results utilizing an algebraic length scale expression with zero and adverse pressure gradient boundary layer experiments. Comparison of numerical and experimental results suggest the models selected were of the right order. Since there were in each case a number of disposable empirical constants a certain adjustment occurred to optimize comparison.

In most respects the authors of MRS models agree on general points. Primary differences center around the use of algebraic or transport equations for dissipation rates, and the presence or absence of such terms as the mean strain rate in pressure strain. There are, however, numerous details over which they disagree with one another, especially philosophically in approach to model selection. Donaldson and his co-workers put much faith in the principle of invariant modeling to limit choices for pressure correlations, third order correlations, etc. Other investigators stress ad hoc empiricism and dimensional analysis.

2.3 *Algebraic stress models* (*ASM*)

A very novel compromise between the simplicity of the MTE approach and the universality and greater range of predictability of the MRS method, has been proposed by Launder and Ying[20]. Transport equations for turbulent fluctuational energy and eddy dissipation (or length scale) are combined with algebraic equations for each Reynolds stress. The additional algebraic stress equations are derived directly from their exact transport equation counterparts.

Following Rodi[16], one notes that it is the convection and diffusion terms in the $\overline{u_i u_j}$ equations which make them differential relationships. If such terms are eliminated from the transport equations for $\overline{u_i u_j}$ one produces a set of algebraic relations of the form

$$\overline{u_i u_j} = f\left(\overline{u_p u_q}, \frac{\partial U_l}{\partial x_m}, k, \epsilon\right). \tag{1}$$

Of course the simplest way to simulate such terms is to neglect them out of hand. This however produces inconsistances in other than equilibrium situations where production exactly balances dissipation. Rodi postulated that

$$(\text{Convection}-\text{Diffusion})_{\text{of } u_iu_j} = \frac{\bar{u}_iu_j}{k}(\text{Convection}-\text{Diffusion})_{\text{of } k}$$

$$= \frac{\overline{u_iu_j}}{k}(\text{Production}-\text{Dissipation})_{\text{of } k} \quad (2)$$

or that $\overline{u_iu_j}/k$ varies but slowly across the flow. (An assumption closely linked to the successful suggestions of Bradshaw for thin shear layers.)

Launder and Ying[20] applied an ASM formulation to turbulent flow in a rectangular channel. The method predicted the order of secondary notions found in channels with sharp corners and the distribution of lateral Reynolds stresses. Rodi[16] produced profiles of $\overline{u'_ku'_j}/k$ in plane jets and wakes which undergo both strong and weak strain, where conventional two equation models are inadequate. Launder *et al.*[21] in the NASA "free-shear flow computational olympics" compared six turbulence models and concludes MRS and ASM models produced results of comparable quality. Finally Date[22] has produced shear and heat flux results for flow in a tube containing a twisted tape by means of ASM type approximations. One concludes therefore that the algebraic stress models may combine the most important features of the MRS type (the influence of complex strain fields on the stresses) with (almost) the numerical simplicity of a MTE model.

3. A TURBULENT MODEL FOR STRATIFIED FLOW

In order to close the turbulence equation system, some of the correlations in u'_i, p' and T' must be approximated in terms of quantities that can be calculated. Model assumptions about turbulence are thereby introduced which may not be entirely realistic. These assumptions relate the chosen higher order correlations in u'_i, p' and T' to other time-averaged quantities; they are expressed in differential and/or algebraic equations which help produce a mathematically closed set.

Turbulence models have been proposed which differ greatly in physical justification, complexity and universality of application. At the level of MTE and MRS approximations, however, most models display a common thread of accepted assumptions with difference often due to philosophical taste. Since it is the intent here to primarily examine the additional effects of stratification the decision was made to follow the practice for all other terms of the team of researchers in the Department of Mechanical Engineering, Imperial College, London (i.e. Spalding, Launder, Patankar, Rodi). These investigators have tested their assumptions over a very wide set of boundary layer, free shear, and pipe or duct flow case studies; thus there is some confidence the resulting expressions have a desirable universality.

The models developed required the solution of partial differential equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ϵ, and thermal eddy dissipation, ϵ_t. Three separate versions of this model are discussed—an algebraic length scale version, a Prandtl–Kolmogorov eddy viscosity version, and an algebraic stress and heat flux model. For purposes of demonstration simple time dependent one-dimensional versions of the governing equations will be applied to a set of free shear flow test cases for which a complete MRS solution is available. Details of the model building process followed here are found in Meroney[23]. The partial differential equations and algebraic relations solved are in a dimensionless format (scales of u_{max} and L) (Table 1).

3.1 *Modeling of the equations for k, k_t, ϵ, ϵ_t*

Kolmogorov and Prandtl introduced an assumption relating shear stress to local velocity gradients through an eddy viscosity based on local effects of turbulence. $\nu_T \propto \sqrt{k}l$. This assumption has been found satisfactory for cases where stress and velocity gradients have the same sign in flow fields near local equilibrium (i.e. production of kinetic turbulent energy $\simeq$ dissipation). A number of alternative length scale relations have been studied; that

Table 1. One-dimensional equations of change

$$\frac{\partial u}{\partial t}=\underbrace{\frac{1}{Re}\frac{\partial^2 u}{\partial z^2}-\frac{\partial}{\partial z}(\overline{u'w'})}_{\text{Diffusion}}+X(t) \tag{3}$$

$$\frac{\partial T}{\partial t}=\underbrace{\frac{1}{RePr}\frac{\partial^2 T}{\partial z^2}-\frac{\partial}{\partial z}(\overline{w'T'})}_{\text{Diffusion}} \tag{4}$$

$$\frac{\partial k}{\partial t}=\underbrace{\frac{1}{Re}\frac{\partial^2 k}{\partial z^2}-\frac{\partial}{\partial z}(\overline{k'w'}+\overline{p'w'})}_{\text{Diffusion}}\underbrace{-\overline{u'w'}\frac{\partial u}{\partial z}}_{\text{Prod.}}\underbrace{-\epsilon}_{\text{Diss.}}+\underbrace{ri\overline{w'T'}}_{\text{Strat.}} \tag{5}$$

$$\frac{\partial k_t}{\partial t}=\underbrace{\frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2}-\frac{\partial}{\partial z}(\overline{k_t'w'})}_{\text{Diffusion}}\underbrace{-\overline{w'T'}\frac{\partial T}{\partial z}}_{\text{Prod.}}\underbrace{-\epsilon_t}_{\text{Diss.}} \tag{6}$$

$$\frac{\partial \epsilon}{\partial t}=\underbrace{\frac{1}{Re}\frac{\partial^2 \epsilon}{\partial z^2}-\frac{\partial}{\partial z}\left(\overline{\epsilon'w'}+\frac{1}{Re}\overline{\frac{\partial p'}{\partial z}\frac{\partial w'}{\partial z}}\right)}_{\text{Diffusion}}\underbrace{-\frac{2}{Re}\left[\frac{\partial u}{\partial z}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}}\right)+\frac{\partial^2 u}{\partial z^2}\left(\overline{w'\frac{\partial u'}{\partial z}}\right)\right]}_{\text{Production}} \tag{7}$$

$$\underbrace{-\frac{2}{Re}\left[\overline{\left(\frac{\partial u'}{\partial z}\right)^2\frac{\partial w'}{\partial z}}+\overline{\left(\frac{\partial v'}{\partial z}\right)^2\frac{\partial w'}{\partial z}}+\overline{\left(\frac{\partial w'}{\partial z}\right)^3}\right]}_{\text{Production}}\underbrace{-\frac{2}{Re}\left[\overline{\left(\frac{\partial^2 u'}{\partial z^2}\right)^2}+\overline{\left(\frac{\partial^2 w'}{\partial z^2}\right)^2}+\overline{\left(\frac{\partial^2 v'}{\partial z^2}\right)^2}\right]}_{\text{Destruction}}+\underbrace{\frac{2}{Re}Ri\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}}\right)}_{\text{Strat.}}$$

$$\frac{\partial \epsilon_t}{\partial t}=\underbrace{\frac{1}{RePr}\frac{\partial^2 \epsilon_t}{\partial z^2}-\frac{\partial}{\partial z}(\overline{\epsilon_t'w'})}_{\text{Diffusion}}\underbrace{-\frac{2}{Pr}\left[\frac{\partial u}{\partial z}\overline{\left(\frac{\partial T'}{\partial z}\right)^2}+\frac{\partial T}{\partial z}\overline{\left(\frac{\partial T'}{\partial z}\right)\left(\frac{\partial u'}{\partial z}\right)}+\frac{\partial^2 T}{\partial z^2}\left(\overline{w'\frac{\partial T'}{\partial z}}\right)\right]}_{\text{Production}}\underbrace{-\frac{2}{Pr}\left[\overline{\frac{\partial w'}{\partial z}\left(\frac{\partial T'}{\partial z}\right)^2}\right]}_{\text{Production}}\underbrace{-\frac{2}{RePr}\overline{\left(\frac{\partial^2 T'}{\partial z^2}\right)^2}}_{\text{Destruction}} \tag{8}$$

$$\underbrace{\frac{\partial \overline{u'w'}}{\partial t}=\frac{1}{Re}\frac{\partial^2 \overline{u'w'}}{\partial z^2}-\frac{\partial}{\partial z}(\overline{u'w'^2}+\overline{u'p'})}_{\text{Diffusion}}\underbrace{-\overline{w'^2}\frac{\partial u}{\partial z}}_{\text{Prod.}}\underbrace{-\overline{p'\left(\frac{\partial u'}{\partial z}\right)}}_{\text{Press-Strain}}\underbrace{-\frac{2}{Re}\overline{\left(\frac{\partial u'}{\partial z}\right)\left(\frac{\partial w'}{\partial z}\right)}}_{\text{Dissipation}}+\underbrace{Ri(\overline{u'T'})}_{\text{strat.}} \tag{9}$$

$$\frac{\partial \overline{w'T'}}{\partial t}=\underbrace{\frac{1}{RePr}\frac{\partial^2 \overline{w'T'}}{\partial z^2}-\frac{\partial}{\partial z}(\overline{w'^2T'}+\overline{p'T'})}_{\text{Diffusion}}\underbrace{-\overline{w'^2}\frac{\partial T}{\partial z}}_{\text{Prod.}}+\underbrace{\overline{p'\frac{\partial T'}{\partial z}}}_{\text{Press-Strain}}\underbrace{-\frac{2}{RePr}\left(\overline{\frac{\partial w'}{\partial z}\frac{\partial T'}{\partial z}}\right)}_{\text{Dissipation}}+\underbrace{2\overline{Rik_t}}_{\text{Strat.}} \tag{10}$$

$$\frac{\partial \overline{u'T'}}{\partial t}=\underbrace{\frac{1}{RePr}\frac{\partial^2 \overline{u'T'}}{\partial z^2}-\frac{\partial}{\partial z}(\overline{w'u'T'})}_{\text{Diffusion}}\underbrace{-(\overline{u'w'})\frac{\partial T}{\partial z}-(\overline{w'T'})\frac{\partial u}{\partial z}}_{\text{Production}}\underbrace{-\frac{2}{RePr}\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial T'}{\partial z}}\right)}_{\text{Dissipation}} \tag{11}$$

$$\frac{\partial \overline{w'^2}}{\partial t}=\underbrace{\frac{1}{Re}\frac{\partial^2 \overline{w'^2}}{\partial z^2}-\frac{\partial}{\partial z}(\overline{w'w'^2}+\overline{2p'w'})}_{\text{Diffusion}}+\underbrace{2\overline{p'\left(\frac{\partial w'}{\partial z}\right)}}_{\text{Press-Strain}}\underbrace{-\frac{2}{Re}\overline{\left(\frac{\partial w'}{\partial z}\right)^2}}_{\text{Dissipation}}+\underbrace{2Ri(\overline{w'T'})}_{\text{Strat.}} \tag{12}$$

where

$$k=\frac{1}{2}(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}) \quad \text{and} \quad \epsilon_t\equiv\frac{1}{Re}\left[\overline{\left(\frac{\partial w'}{\partial z}\right)^2}+\overline{\left(\frac{\partial v'}{\partial z}\right)^2}+\overline{\left(\frac{\partial w'}{\partial z}\right)^2}\right]$$

$$k_t=\frac{\overline{T'^2}}{2} \qquad \epsilon_t\equiv\frac{1}{RePr}\overline{\left(\frac{\partial T'}{\partial z}\right)^2}$$

chosen here $l = k^{3/2}/\epsilon$ was originally proposed by Harlow and Nakayama in 1967. The final expression is then

$$\nu_T \cong C_D k^2/\epsilon. \tag{13}$$

A series of parallel arguments applied to the vertical heat flux equation will yield an expression for eddy diffusivity

$$\alpha_T \equiv C_H k k_T/\epsilon_T. \tag{14}$$

The eddy viscosity-diffusivity approximations above permit closure of diffusion and production terms in the transport equations of Table 1 for the ALD, DLM and ASM methods. The only additional terms deserving further comment are the stratification effects on eddy dissipation.

Lumley [6] fails to retain any effect of stratification upon ϵ; however if one pursues the exact relationships as suggested by Daly and Harlow [19] the term as shown in Table 1 appears. Order of magnitude arguments would suggest its inclusion is critical in order to track the effects of stratification on k. Daly and Harlow [19] suggest that the stratification term might be modeled as

$$2Ri\overline{\left(\frac{T'}{z}-\frac{w'}{z}\right)} \propto Ri\overline{w'^2}f(\xi)\frac{\partial T}{\partial z} \tag{15}$$

where $\xi = S(2k)^{1/2}/\nu$ is a turbulence Reynolds number and $f(\xi) \sim 0(1)$. To remain consistent with the earlier choices made for the transport of k it is proposed to use

$$2Ri\overline{\left(\frac{T'}{z}-\frac{w'}{z}\right)} = +FRi\overline{w'T'}\left(\frac{\epsilon}{k}\right). \tag{16}$$

When this is evaluated for the DLM case one finds

$$= -FRiC_H k_t\left(\frac{\epsilon}{\epsilon_t}\right)\left(\frac{\partial T}{\partial z}\right) \tag{17}$$

which resembles the Daly–Harlow suggestion. Final relationships for the ALM and DLM closures are found in Table 2.

3.2 *The algebraic stress and heat flux relations*

As indicated before closure for the ASM model requires algebraic expressions for $\overline{u'w'}$, $\overline{u'T'}$ and $\overline{w'T'}$. These new expressions are dependent upon an accurate representation of the

Table 2. Turbulent model equations (ALM) and (DLM)

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial z}\left(C_D\frac{k^2}{\epsilon}\frac{\partial u}{\partial z}\right) + X(t) \tag{18}$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\frac{\partial^2 T}{\partial z^2} + \frac{\partial}{\partial z}\left(C_H\frac{kk_t}{\epsilon_t}\frac{\partial T}{\partial z}\right) \tag{19}$$

$$\frac{\partial k}{\partial t} = \frac{1}{Re}\frac{\partial^2 k}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_k}\frac{k^2}{\epsilon}\frac{\partial k}{\partial z}\right) + C_D\frac{k^2}{\epsilon}\left(\frac{\partial u}{\partial z}\right)^2 - \epsilon + RiC_H\frac{kk_t}{\epsilon_t}\left(\frac{\partial T}{\partial z}\right) \tag{20}$$

$$\frac{\partial k_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{kt}}\frac{kk_t}{\epsilon_t}\right) + C_H\frac{kk_t}{\epsilon_t}\left(\frac{\partial T}{\partial z}\right)^2 - \epsilon_t \tag{21}$$

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{Re}\frac{\partial^2 \epsilon}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_\epsilon}\frac{k^2}{\epsilon}\frac{\partial \epsilon}{\partial z}\right) + C_{E1}C_D k\left(\frac{\partial u}{\partial z}\right)^2 - C_{E2}\frac{\epsilon^2}{k} - F\cdot RiC_H k_t\frac{\epsilon}{\epsilon_t}\left(\frac{\partial T}{\partial z}\right) \tag{22}$$

$$\frac{\partial \epsilon_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 \epsilon_t}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{\epsilon t}}\frac{kk_t}{\epsilon_t}\frac{\partial \epsilon_t}{\partial z}\right) + C_{Et1}C_H k_t\frac{\epsilon}{\epsilon_t}\left(\frac{\partial T}{\partial z}\right)^2 - C_{Et2}\frac{\epsilon\epsilon_t}{k} \tag{23}$$

production-dissipation terms in their respective exact relationships. Stress-equation models have been recommended by Rotta[17], Donaldson[18], Daly and Harlow[19], Rodi[16] and Launder *et al.*[21].

Following the experience of Rodi[16] the production (P), dissipation (D), pressure strain (PS) and stratification effects (S) are identified and modeled. Since the diffusion terms are eliminated through the arguments presented previously they are not considered here.

The Reynolds stress equations. For high Reynolds number situations

$$D_{ij} = \frac{2}{3}\epsilon\delta_{ij}. \tag{24}$$

Production terms are exact as

$$P_{ij} = -\overline{u'_i u'_k}\frac{\partial u_j}{\partial x_k} - \overline{u'_j u'_k}\frac{\partial u_i}{\partial x_k}. \tag{25}$$

$$P = \frac{1}{2}P_{ii} = -\overline{u'_k u'_l}\frac{\partial u_k}{\partial x_l}.$$

Pressure strain terms can as yet not be measured in laboratory flows. Unfortunately they are also of great importance since they are roughly equal and opposite to the production terms in the shear-stress equation; and in the normal-stress equations they redistribute energy to various direction components. Pressure-strain consists of two parts: one due to the interaction of the various fluctuating velocities, and the second originates from the interaction of mean and fluctuating flow. Rotta[17] proposed to take the first part proportional to the anisotropy of turbulence. Naot *et al.*[25] and Reynolds[26] proposed the second part should be proportional to the anisotropy of the production of turbulence, thus

$$PS_{ij} = -C_{P1}\frac{\epsilon}{k}\left(\overline{u_i u_j} - \delta_{ij}\frac{2}{3}k\right) - A_1\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right). \tag{26}$$

Stratification effects are directly expressed as

$$S_{ij} = Ri\left(\frac{\partial\phi}{\partial x_i}\overline{u'_j T'} + \frac{\partial\phi}{\partial x_j}\overline{u'_i T'}\right) \tag{27}$$

where ϕ is the gravitational potential.

When these terms are substituted into the exact relationships the following expressions are obtained:

$$(\text{Conv} - \text{Diff})_{\overline{u'w'}} = -(1 - A_1)\overline{w'^2}\frac{\partial u}{\partial z} - C_{P1}\frac{\epsilon}{k}\overline{u'w'} + Ri\overline{u'T'} \tag{28}$$

$$(\text{Conv} - \text{Diff})_{\overline{w'^2}} = -\frac{2}{3}A_1\overline{u'w'}\frac{\partial u}{\partial z} + \frac{2}{3}(C_{P1} - 1)\epsilon - C_{P1}\frac{\epsilon}{k}\overline{w'^2} + 2Ri\overline{w'T'}. \tag{29}$$

As noted previously the above expressions have been prepared to substitute into the expression below

$$\begin{aligned}(\text{Conv} - \text{Diff})_{\text{of } \overline{u_i u_j}} &= \frac{\overline{u_i u_j}}{k}(\text{Conv} - \text{Diff})_{\text{of } k} \\ &= \frac{\overline{u_i u_j}}{k}(P - \epsilon + S)_{\text{of } k} \\ &= (P + PS - \epsilon + S)_{\text{of } \overline{u_i u_j}}.\end{aligned} \tag{30}$$

The appropriate substitution for the k terms are found on the right side of eqn (35) (Table 3).

When (28) is introduced into (30):

$$\overline{u'w'} = \frac{-(1-A_1)\overline{w'^2}\dfrac{\partial u}{\partial z} + Ri\overline{u'T'}}{C_{P1}\dfrac{\epsilon}{k} + \dfrac{1}{k}\left(-\overline{u'w'}\dfrac{\partial u}{\partial z} - \epsilon + Ri\overline{w'T'}\right)}. \tag{31}$$

Similarly if (26) is introduced into (27):

$$\overline{w'^2} = \frac{\dfrac{2}{3}(A_1(-\overline{u'w'})\dfrac{\partial u}{\partial z} + \epsilon(C_{P1}-1) + 3Ri\overline{w'T'})}{C_{P1}\dfrac{\epsilon}{k} + \dfrac{1}{k}\left(-\overline{u'w'}\dfrac{\partial u}{\partial z} - \epsilon + Ri\overline{w'T'}\right)}. \tag{32}$$

Elimination of $\overline{w'^2}$ in eqn (31) by means of eqn (32) yields eqn (39) (Table 3).

The heat flux equations. In the $\overline{u'T'}$ and $\overline{w'T'}$ relations at high Reynolds numbers it is again appropriate to suggest that the dissipation items are

$$D_{\overline{u_iT'}} \simeq 0. \tag{42}$$

The production and stratification terms may be treated exactly, i.e.

$$P_{\overline{u_iT'}} = -\overline{u_i'u_k'}\frac{\partial T}{\partial x_k} - \overline{u_k'T'}\frac{\partial u_i}{\partial x_k} \tag{43}$$

$$S_{\overline{u_iT'}} = +2Ri\left[\frac{\partial \phi}{\partial x_i}k_T\right]. \tag{44}$$

The pressure scrambling terms may be postulated by analogy to the Reynolds stress result:

$$PS_{\overline{u_iT'}} = -C_{P2\text{ or }3}\frac{\epsilon}{k}(\overline{u_i'T'}) - A_{2\text{ or }3}P_{\overline{u_iT'}}. \tag{45}$$

One might argue that the first term should be $-C_{P2\text{ or }3}(\epsilon_t/k_t)(\overline{u_iT'})$ in order that the final relation reduce to the Prandtl–Kolmogorov formulation in equilibrium flows. Unfortunately such a choice eliminates the production of $\overline{w'T'}$ in unstable regions where k_T has been set zero as an initial condition. The modeled relations do not contain the physics of thermal instability within themselves.

If one now postulates that

$$\begin{aligned}(\text{Conv}-\text{Diff})_{\text{of }\overline{u_iT'}} &= \frac{\overline{u_i'T'}}{k}(\text{Conv}-\text{Diff})_{\text{of }k} \\ &= \frac{\overline{u_i'T'}}{k}(P + PS - D + S)_{\text{of }\overline{u_iT'}}\end{aligned} \tag{46}$$

and substitutes eqn (42), (43), (44) and (45) into eqn (46) there results new expressions:

$$\overline{w'T'} = \frac{-(1-A_2)\overline{w'^2}\dfrac{\partial T}{\partial z} + 2Rik_t}{C_{P2}\dfrac{\epsilon}{k} + \dfrac{1}{k}\left(-\overline{u'w'}\dfrac{\partial u}{\partial z} - \epsilon + Ri\overline{w'T'}\right)} \tag{47}$$

$$\overline{u'T'} = \frac{(1-A_3)\left[-\overline{u'w'}\dfrac{\partial T}{\partial z} - \overline{w'T'}\dfrac{\partial u}{\partial z}\right]}{C_{P3}\dfrac{\epsilon}{k} + \dfrac{1}{k}(-\overline{u'w'}\dfrac{\partial u}{\partial z} - \epsilon + Ri\overline{w'T'})}. \tag{48}$$

Table 3. Turbulent model equations (ASM)

$$\frac{\partial u}{\partial t}=\frac{1}{Re}\frac{\partial^2 u}{\partial z^2}-\frac{\partial}{\partial z}(\overline{u'w'})+X(t) \tag{33}$$

$$\frac{\partial T}{\partial t}=\frac{1}{RePr}\frac{\partial^2 T}{\partial z^2}-\frac{\partial}{\partial z}(\overline{w'T'}) \tag{34}$$

$$\frac{\partial k}{\partial t}=\frac{1}{Re}\frac{\partial^2 k}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_k}\frac{k^2}{\epsilon}\frac{\partial k}{\partial z}\right)-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'} \tag{35}$$

$$\frac{\partial k_t}{\partial t}=\frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{kt}}\frac{kk_t}{\epsilon_t}\frac{\partial k_t}{\partial z}\right)-\overline{w'T'}\frac{\partial T}{\partial z}-\epsilon_t \tag{36}$$

$$\frac{\partial \epsilon}{\partial t}=\frac{1}{Re}\frac{\partial^2 \epsilon}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_\epsilon}\frac{k^2}{\epsilon}\frac{\partial \epsilon}{\partial z}\right)-C_{E1}\overline{u'w'}\left(\frac{\epsilon}{k}\right)\left(\frac{\partial u}{\partial z}\right)-C_{E2}\frac{\epsilon^2}{k}+FRi\overline{w'T'}\frac{\epsilon}{k} \tag{37}$$

$$\frac{\partial \epsilon_t}{\partial t}=\frac{1}{RePr}\frac{\partial^2 \epsilon_t}{\partial z^2}+\frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{\epsilon t}}\frac{kk_t}{\epsilon_t}\frac{\partial \epsilon_t}{\partial z}\right)-C_{Et1}\overline{w'T'}\left(\frac{\epsilon}{k}\right)\left(\frac{\partial T}{\partial z}\right)-C_{Et2}\frac{\epsilon\epsilon_t}{k} \tag{38}$$

$$\overline{u'w'}=\frac{\left\{-\frac{2}{3}(1-A_1)k^2\frac{\partial u}{\partial z}\left(C_{P1}\epsilon-\overline{u'w'}A_1\frac{\partial u}{\partial z}-\epsilon+3Ri\overline{w'T'}\right)\right\}}{\left\{C_{P1}\epsilon-u'w'\frac{\partial u}{\partial z}-\epsilon+Riw'T'\right\}^2}+\frac{Ri\overline{u'T'}k}{\left\{C_{P1}\epsilon-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'}\right\}} \tag{39}$$

$$\overline{w'T'}=\frac{\left\{-\frac{2}{3}(1-A_2)k^2\frac{\partial T}{\partial z}\left(C_{P1}\epsilon-\overline{u'w'}A_1\frac{\partial u}{\partial z}-\epsilon+3Ri\overline{w'T'}\right)\right\}}{\left\{C_{P1}\epsilon-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'}\right\}\left\{C_{P2}\epsilon-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'}\right\}}+\frac{2Rik_tk}{\left\{C_{P2}\epsilon-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'}\right\}} \tag{40}$$

$$\overline{u'T'}=\frac{\left\{(1-A_3)\left(-\overline{u'w'}\frac{\partial T}{\partial z}-\overline{w'T'}\frac{\partial u}{\partial z}\right)k\right\}}{\left\{C_{P3}\epsilon-\overline{u'w'}\frac{\partial u}{\partial z}-\epsilon+Ri\overline{w'T'}\right\}} \tag{41}$$

Elimination of $\overline{w'^2}$ in eqn (47) with eqn (32) yields: eqn (40) (Table 3).

Equations (39)–(41) are the final result of the exercise to produce ASM relations.

4. TEST CALCULATIONS FOR STRATIFIED TURBULENCE MODELS

Mellor and Herring[14] recommend that investigators evaluate MTE and MRS models in tandem in order that critical limitations of the MTE approach are identified. Bradshaw[24] has also proposed such tuning of a "simple" calculation method by a "refined" calculation technique.

4.1 *Simple case of atmospheric shear*

The method chosen for comparison with the ALM, DLM, and ASM method proposed herein was the MRS technique developed by Donaldson and Rosenbaum[18]. Their "invariant" modeling closure was applied to a hypothetical free-shear clear air turbulence case in Donaldson *et al.*[2]. This was a simple, time-dependent, one-dimensional example which characterizes the important effects of buoyancy dominated turbulent shear flows.

The atmospheric test case is assumed to have an initially 4000 ft band at turbulence that is isotropic with $\overline{u'^2}=\overline{v'^2}=\overline{w'^2}=1$ (fps)^2. The band is centered at an altitude of 20,000 ft; however the effects of altitude upon p_0, T_0 and ρ_0 are neglected for purposes of the example. The atmosphere is initially at rest, i.e. at $t=0$, $u=0$. A body force acts on the atmosphere to create a

Table 4.

Investigator	C_D	C_H	C_{E1}	C_{E2}	C_{Et1}	C_{Et2}	F	σ_k	σ_ϵ	σ_{kt}	$\sigma_{\epsilon t}$	A_1	A_2	A_3	C_{P1}	C_{P2}	C_{P3}
Donaldson *et al.*[2]	0.1	0.1	1.5	1.5	1.0–1.5	1.5	1.5	0.1		0.2		0.0	0.0	0.0	5.0	5.0	5.0
Meroney[23]																	
ALM	0.1	0.1						0.1		0.2							
DLM	0.1	0.1	1.5	1.59	1.5	1.59	1.5	0.1	0.1	0.2	0.2						
ASM	0.1	0.1	1.5	1.55	1.5	1.5	1.5	0.1	0.1	0.2	0.2	0.0	0.0	0.0	5.0	5.0	5.0

mean motion. The dimensionless driving function and initial temperature distributions imposed are shown in Fig. 1.

The algebraic length method of the MTE type approach suggested here still requires an expression to specify ϵ and ϵ_t in the governing partial differential equations. The algebraic relation formulated by Donaldson *et al.* [2] can be re-expressed in terms of equivalent values of dissipation; hence for large Re_Λ one can find

$$\epsilon = 5.52k^{1.5}/z_\delta \tag{49}$$

$$\epsilon_T = 5.52k_tk^{0.5}/z_\delta. \tag{50}$$

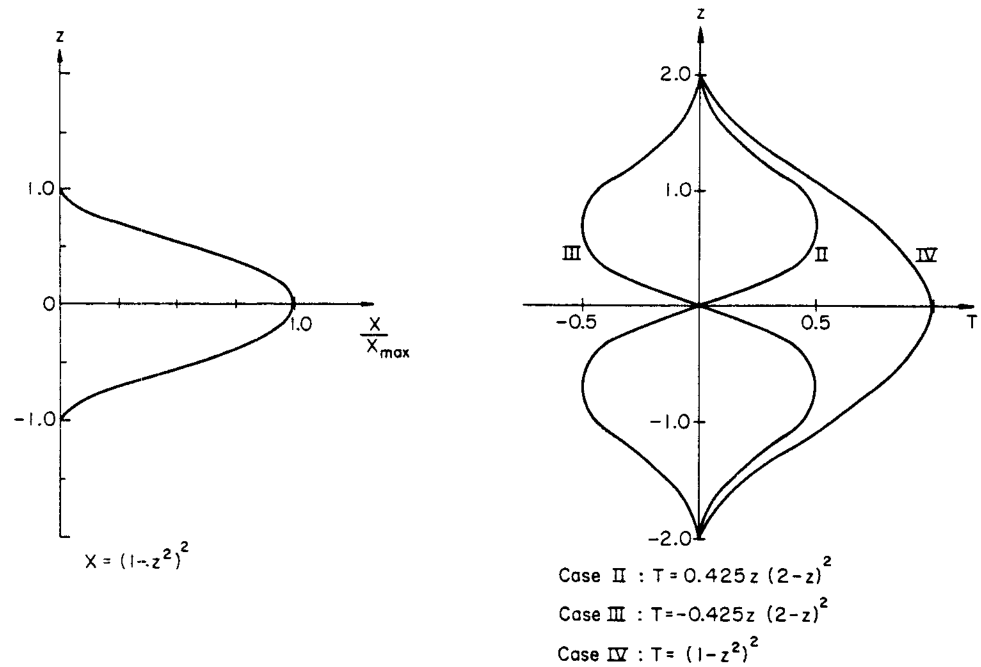

Fig. 1. Driving force function and initial temperature conditions clear-air turbulence model.

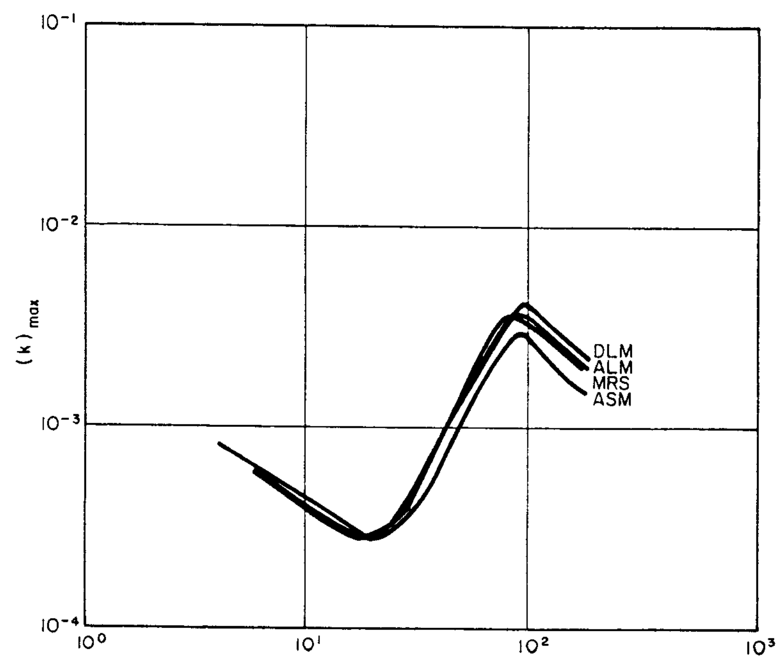

Fig. 2. Maxima of velocity correlations: Case I.

where $z_\delta \equiv \delta^*/L^*$ and δ^* is the breadth of the mean shear layer at the half-maximum velocity points.

Although the MRS method compared to here did not require specification of the series of constants used in the ALM, DLM, and ASM models, consideration of the algebraic length scale equation plus asymptotic forms of the governing equations in those regions where production or dissipation dominate allows one to specify the equivalent constant values. After consideration of the results obtained from the original constants it was only found necessary to adjust C_{E2} and $C_{E\theta 2}$ slightly to arrive at a reasonable comparison to the Donaldson *et al.* [2] results.

5. RESULTS

5.1 *Case I: neutrally stable atmosphere*

For each model proposed this neutral case was used to adjust C_{E2} to obtain optimum time behavior of k when compared to the MRS results. At very small times, $t < 20$, there is still very little turbulence created by the mean action and most turbulence is left over from the decaying initial turbulent layer. When the forcing function, X, is removed at $t = 90$, the turbulent kinetic energy is a maximum. The influence of the value $\partial u/\partial z = 0$ at $z = 0$ on the production of new turbulence is evident. Finally at $t = 180$ one sees the effect after decay has set in.

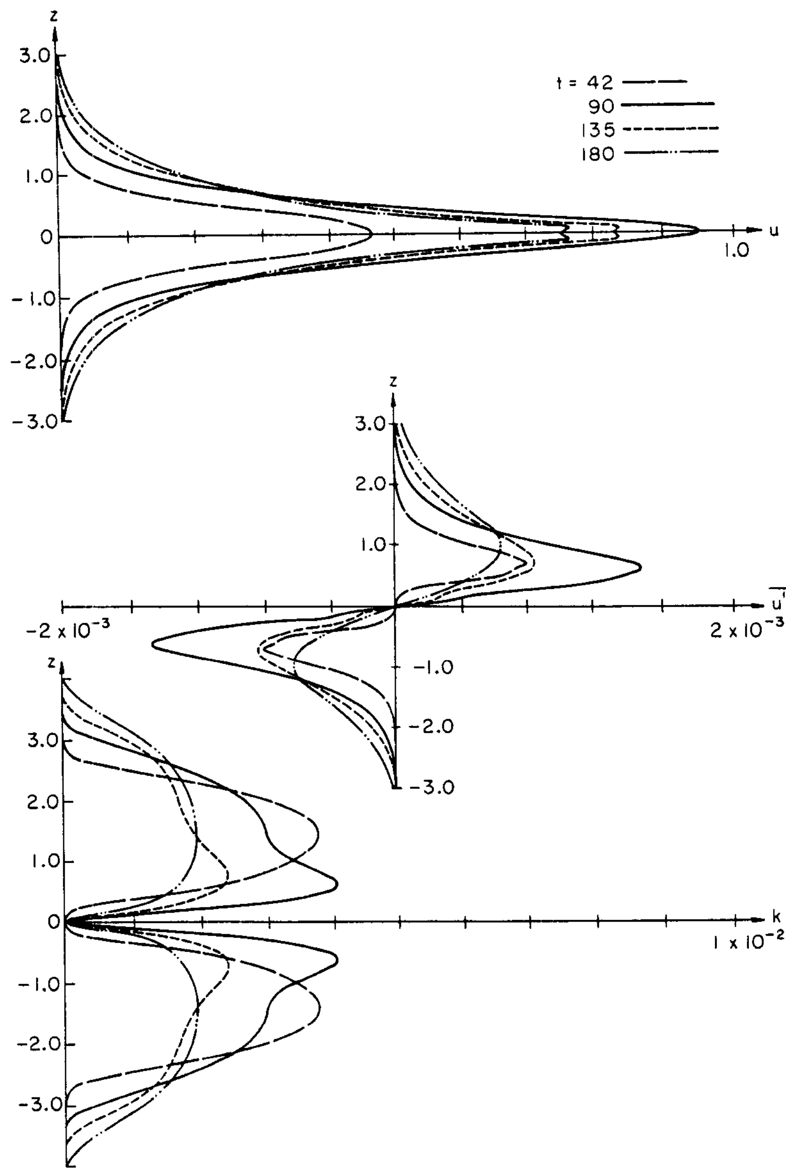

Fig. 3. (a) Case II: ASM model.

By plotting the maxima of k for the various models versus time in Fig. 2 one can perceive the degree of agreement between the closure trials. One perceives the initial decay of turbulence, followed by the production of the correlations as a result of the interaction of turbulence and mean flow, and finally the decay after the forcing function is removed.

5.2 *Case II*: *unstable–stable–unstable atmosphere*

The mean velocity and shear are changed somewhat but not drastically by the effect of the temperature profile. In the thermally unstable regions the development of turbulence is accelerated. At large times the production by $\partial u/\partial z$ becomes significant and a maxima in k occurs at $t = 90$. There is a large influence on k, k_T and $\overline{w'T'}$ in the regions where $\partial T/\partial z < 0$ vs those regions where $\partial T/\partial z > 0$. If one insists upon an eddy diffusivity model a large variation in turbulent Prandtl number with stability is indicated. In regions of high stability the existence of a single length scale in the ALM method is not sufficient to develop the necessary degrees of temperature fluctuation dissipation. Thus k_T is excessively large near $z = 0$.

The ALM and DLM models fail to keep pace with the changes recorded by the ASM and MRS relations. In addition the ALM and DLM models inherently require $\overline{w'T'}$, $\overline{u'T'}$ and $\overline{u'w'}$ be identically zero where the respective gradients, $\partial u/\partial z$ and $\partial T/\partial z$, are zero. It appears that failure to follow the time rate of change in k and other correlations is due to the inherent assumption that Production equals Dissipation in the ALM and DLM formulations. This effect is most marked in

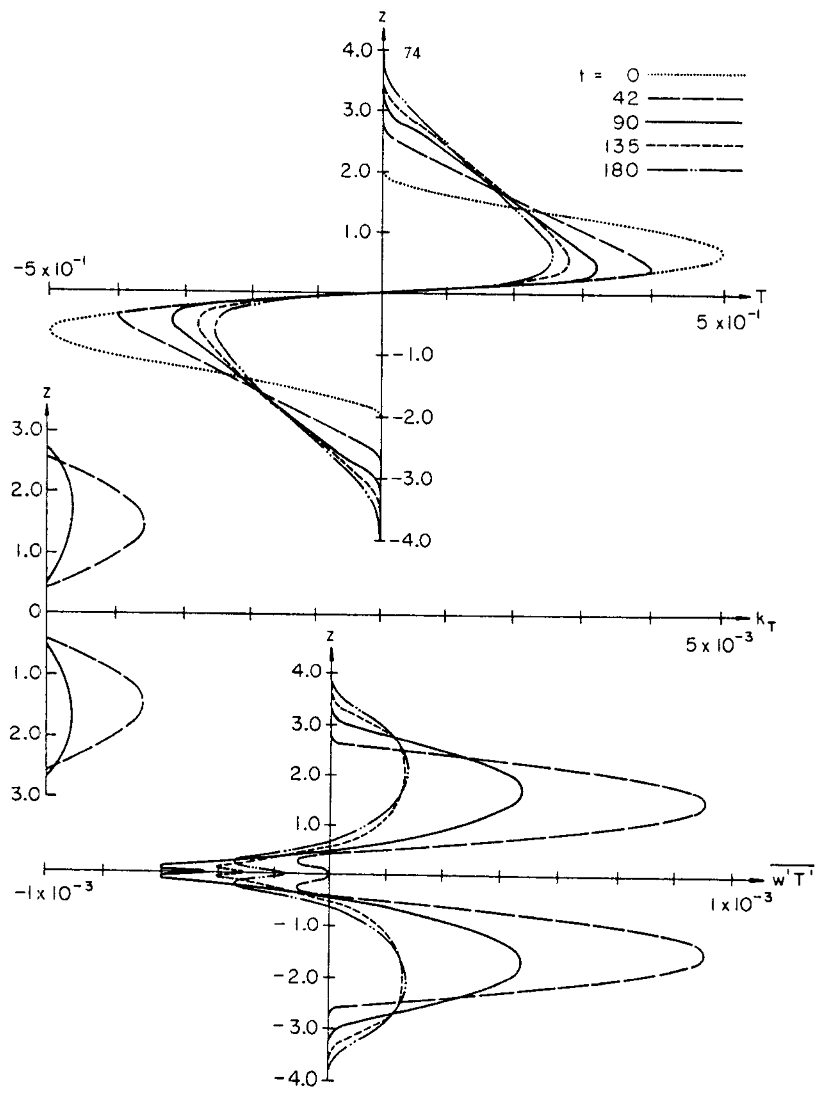

Fig. 3. (b) Case II: ASM model.

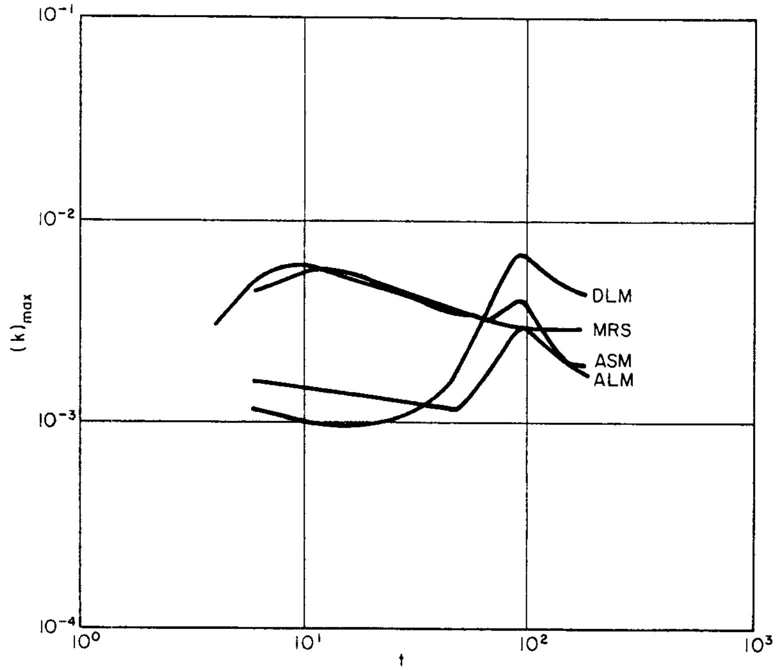

Fig. 4. Maxima of velocity correlations: Case II.

the logarithmic plot of the maxima of the turbulent kinetic energy as shown in Fig. 4. At small times the major production of turbulence is by thermal instability. Near the turbulence maxima there is finite contribution due to mean shear, $\partial u/\partial z$; but, after $t = 90$, the thermal instabilities in the outer regions dominate.

5.3 *Case III*: *stable–unstable–stable atmosphere*

This case was utilized to specify the magnitude of constant C_{Et2}. In Cases II and IV the constants were unchanged for all models. Again we do not detect major influence of stability upon the production of k, k_T and $\overline{w'T'}$ in the center of the unstable region. Again only the ASM method can track the nonequilibrium behavior displayed by the MRS relations. The outlying stable regions limit the vertical dispersion of this intense turbulence. The temperature inversion is extremely persistent at large $\pm z$. In the time dependent plot of k (Fig. 5), we detect the initial production of turbulence by thermal instability, the modification of this by shear generated turbulence near $t = 90$, and the final phase where production of turbulence by thermal instability just about balances dissipation.

5.4 *Case IV*: *stable–unstable atmosphere*

This example is marked by its asymmetric appearance, the persistence of the inversion region, $\partial T/\partial z > 0$, and the rapid vertical diffusion of energy into the neutral upper regions where initially ∂T—$\partial z = 0$. No existing eddy diffusivity model would be expected to perform adequately. Figure 6 records the time rate of change of k maxima.

5.5 *Discussion of results*

Of course one of the most interesting aspects of these results is the duplication of Donaldson *et al.*[2] conclusion that there is a radically different behavior of the heat flux correlation $\overline{w'T'}$ depending on whether $\partial T/\partial z$ is greater or less than zero. In fact, the ASM eqns (39)–(41) indicate that for small values of $\partial T/\partial z$ and $\partial u/\partial z$ there may be transport of heat and momentum up the gradients due to finite values of k_T or $\overline{u'T'}$. This effect has often been observed by experimentalists in atmospheric transport. In addition if one considers only production terms and neglects pressure scrambling and dissipation terms in equations for k, k_T and $\overline{w'T'}$ it is not difficult to show that when $\partial T/\partial z < 0$, there is an exponential development of $\overline{w'T'}$. However when the atmosphere is stable, i.e. when $\partial T/\partial z > 0$, the heat flux correlation $\overline{w'T'}$ is oscillatory about the Braunt–Väisälä frequency.

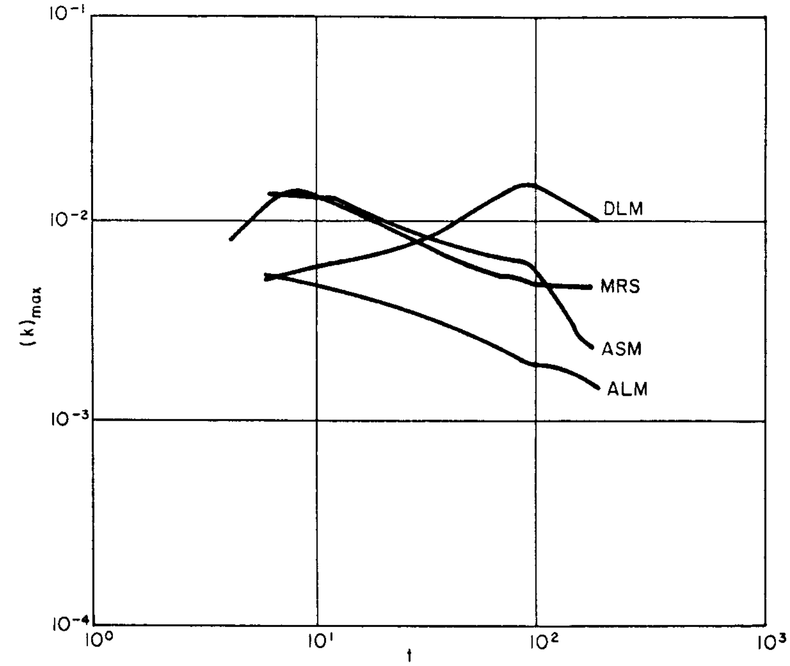

Fig. 5. Maxima of velocity correlations: Case III.

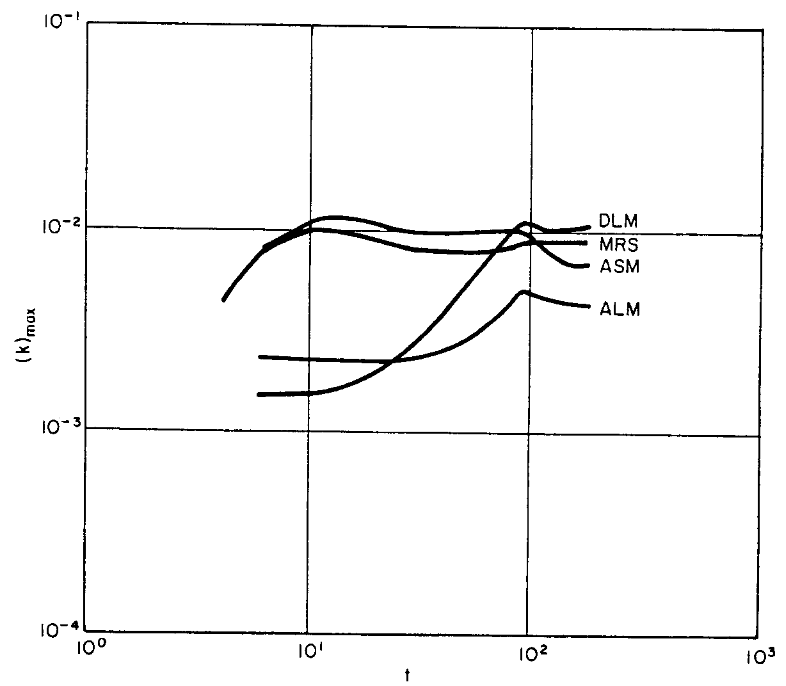

Fig. 6. Maxima of velocity correlations: Case IV.

One must conclude on the basis of these results that:

(1) An algebraic length scale version of a MTE model closure is not able to replicate the behavior of thermally stratified flow, especially in regions where production and dissipation of turbulence are not in equilibrium. A single dissipation length scale does not appear sufficient here to develop the expected degree of damping in stable regions.

(2) Addition of transport equations for length scales does not suffice to solve the above problem. Such MTE models are still inadequate.

(3) Addition of algebra relations for stress and heat flux which incorporate the influence of stability do appear to incorporate the physics of the phenomena to the extent that results are similar to the MRS test case. Thus the ASM model simplifies computation without undue penalty to results.

Acknowledgement—Support provided by the National Science Foundation (Grant Number GK33800) is gratefully acknowledged.

REFERENCES

1. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics.* MIT Press, Cambridge, Mass., 769 pp. (1971).
2. C. du P. Donaldson, R. D. Sullivan and H. Rosenbaum, A theoretical study of the generation of atmospheric—clear air turbulence. *AIAA J.* **10**(2), 162–170 (1972).
3. C. du P. Donaldson, Atmospheric turbulence and the dispersion of atmospheric pollutants. Aeronautical Research Associates of Princeton, Inc., New Jersey; Environmental Protection Agency Control Report EPA-R4-73-016a, 210 p. See also Construction of a dynamic model of the production of atmospheric turbulence and the dispersal of atmospheric pollutants. *Workshop on Micrometeorology* Boston, American Meteorological Society, pp. 313–390 (1973).
4. W. S. Lewellen and M. Teske, Prediction of the Monin–Obukhov similarity functions from an invariant model of turbulence. *J. Atmos. Sci.* **30**(10), 1340–1345 (1973).
5. G. L. Mellor, Analytic prediction of the properties of stratified planetary surface layers. *J. Atmos. Sci.* **30**(9), 1061–1069 (1973).
6. J. L. Lumley, A model for computation of stratified turbulent flows. *Int. Symposium on Stratified Flows*, Novosilursk, USSR, 9 pp. (1972).
7. R. H. C. Lee, Buoyancy effects on a turbulent shear flow. Unpublished Paper Aerospace Corporation, El Segundo, Calif., 20 pp. (1974).
8. M. S. Hossain and W. Rodi, Equations for turbulent buoyant flows and their modeling. SFB 80/T/46, 36 pp. Sonderforschungsbereich 80, Ausbreitungs-Und Transportvorgange in Stromungen, Universitat Karlsruhe (Dec. 1974).
9. G. L. Mellor and T. Yamada, A hierarchy of turbulence closure models for planetary boundary layers. *J. Atmos. Sci.* **31**(7), 1791–1806 (1974).
10. B. E. Launder and C. B. Spalding, *Lectures in Mathematical Models of Turbulence.* Academic Press, New York, 169 pp. (1972).
11. W. C. Reynolds, A morphology of the prediction methods. *Proc. Computation of Turbulent Boundary Layers—1968*, AFOSR-IFP-Stanford Conference, Vol. 1, pp. 1–15 (1968).
12. P. Bradshaw, The understanding and prediction of turbulent flow. *Aeronaut. J.* **76**(739), 403–418 (1972).
13. P. Bradshaw, Advances in turbulent shear flow. Notes for VKI Short Course, 19–23 March 1973, Dept. of Aeronautics, Imperial College of Science and Technology, London, 190 pp. (1973).
14. G. L. Mellor and H. J. Herring, A survey of the mean turbulent field closure models. *AIAA J.* **11**(5), 590–599 (1973).
15. K. Hanjalic and B. E. Launder, A Reynolds stress model of turbulence and its application to thin shear flows. *J. Fluid Mech.* **52**(4), 609–638 (1972).
16. W. Rodi, The prediction of free turbulent boundary layers by use of a two-equation model of turbulence. Ph.D. Dissertation Mechanical Engineering Department, Imperial College, London, 310 pp. (1972).
17. J. Rotta, Statistische Theorie Nichthomogener Turbulenz. *Seitsch. Fur. Physik* **124**, 547; **131**, 51 (also translated by W. Rodi (1968), Dept. of Mechanical Engineering Imperial College, London, *Rep.* TWF/TW/38 and 39 (1951).
18. C. du P. Donaldson and H. Rosenbaum, Calculation of turbulent shear flows through closure of the Reynolds equations by invariant modeling. NASA SP-216, pp. 231–253 (1968).
19. B. J. Daly and F. H. Harlow, Transport equations in turbulence. *Phys. Fluids* **13**(11), 2634–2649 (1970).
20. B. E. Launder and W. M. Ying, Fully-developed turbulent flow in ducts of square cross section. Imperial College of Science and Technology, London, Dept. of Mechanical Engineering Report TM/TN/A/11, 37 pp. (1971).
21. B. E. Launder, A. Morse, W. Rodi and D. B. Spalding, The production of free shear flows—a comparison of the performance of six turbulence models. NASA Conference on Free Shear Flows, Lengley Field, Virginia, 20–21 July 1972, 66 pp. (also Dept. of Mechanical Engineering, Imperial College, *Rep.* TM/TN/B/19 (1972).
22. A. W. Date, Prediction of friction and heat transfer characteristics of flow in a tube containing a twisted tape. Ph.D. Dissertation, Dept. of Mechanical Engineering, Imperial College of Science and Technology, London, 226 pp. (1972).
23. R. N. Meroney, Buoyancy effects on a turbulent shear flow. Colorado State University, Fort Collins, *Rep.* CER73-74RNM38 (Apr. 1974).
24. P. Bradshaw, Use of a refined boundary-layer calculation method to calibrate a simple one. Dept. of Aeronautics, Imperial College of Science and Technology, London, *IC Aero Rep.* 71-23, 13 pp. (1971).
25. D. Naot, A. Shavit and M. Wolfshtein, Interactions between components of the turbulent velocity correlation tensor. *Israel J. Tech.* **8**, (1970).
26. W. C. Reynolds, Computation of Turbulent Flow—State of the art, 1970. Stanford University Mech. Engr. Dept. Rep. MD-27 (1970).