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The Influence of Heating on the Stability of Laminar Boundary Layers Along Concave Curved Walls

This paper considers the effect of heating on Taylor-Görtler vortices in laminar boundary layers. The effect of higher-order terms and the normal velocity component of the primary flow in the stability calculations has been demonstrated. The findings indicate that terms involving the higher-order effects of curvature as well as the normal component of the primary flow become increasingly important at small wave numbers. The effect of heating is to stabilize the flow to disturbances of long lateral wavelength but has a destabilizing effect on short wavelength disturbances.

This paper considers the buoyancy effects of wall heating on the formation of Taylor-Görtler vortices along concave walls. Previous work has concentrated almost exclusively on the stability problem under isothermal conditions.

Three-dimensional instabilities on concave curved walls was first studied under isothermal conditions by Görtler [1].[1] Earlier, Taylor [2] had noted the same variety of instability in Couette flow between rotating concentric cylinders. For the boundary-layer case, Görtler established the fact that this type of instability could occur only on walls with concave curvature and derived the relevant stability parameter $N_G = R_d\sqrt{k\delta}$ where R_d is the Reynold's number based on the boundary-layer thickness δ, and k is the reciprocal of the radius of curvature of the plate. In his calculations, Görtler examined the stability of the Blasius profile on a curved plate but ignored the quasi-parallel nature of the flow, i.e., the velocity component of the primary flow perpendicular to the wall was assumed to have a negligible effect on the calculations.

Meksyn [3] solved Görtler's stability equations using a different asymptotic method of integration. Hämmerlin [4] also resolved Görtler's stability equations by different methods and obtained very accurate solutions. He first established the peculiar result that the minimum Görtler modulus for neutral stability occurs at a lateral wave number of zero, thus implying vortex disturbances of infinite wavelength.

Smith [5] reconsidered the stability problem in detail, and concluded that certain terms involving the velocity normal to the surface in a growing boundary layer, as well as certain higher-order terms approximately describing the effect of curvature on the disturbances could not be neglected. He also argued that a spacewise representation of the disturbances was more consistent with physical reality, rather than the timewise representation that had been employed earlier. Accordingly, he recalculated the neutral stability curves as well as curves of constant spatial amplification for a growing Blasius boundary layer. His results differed slightly from those of Görtler [1] and more strongly from those of Hämmerlin [4] in that he obtained a finite value for the critical wave number, different from zero. His method of solution was via the Galerkin approximation technique.

More recently, Chang and Sartory [6] have solved the hydromagnetic Görtler stability problem using numerical integration methods. In the absence of the magnetohydrodynamic term, their differential equations reduce to those of Hämmerlin [4] for neutral disturbances except for the inclusion of the terms involving the vertical velocity component of the primary flow. The results of their computations for the Blasius boundary layer in the absence of a magnetic field indicate that the critical Görtler number as well as the critical wave number tends toward zero when the normal flow terms of the primary velocity profile are included in the calculation. They therefore surmise that at the lower wave numbers, the effect of the higher-order curvature terms neglected in their calculations could play an important role in limiting the degree of instability, and size of the vortices.

The present analysis, considers the separate effects of surface heating and homogenous suction on the stability of a boundary layer

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[1] Numbers in brackets designate References at end of paper.

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along a concave wall. The flow has been assumed incompressible with constant fluid properties, the effect of heating being represented by the appearance of an additional destabilizing buoyancy force in accordance with the Boussinesq approximation. Such situations where gravity as well as centrifugal effects are present occur in the blades in gas turbine engines and in certain heat exchangers. Indeed, McCormack, et al. [7] have shown that Taylor-Görtler vortices increase the rate of heat transfer through boundary layers and have proposed exploitation of this phenomenon in the design of high efficiency curved parallel plate heat exchangers. The work reported herein, assumes that the gravity vector is instantaneously coincident with the direction normal to the plate. Other inclinations in the vertical plane may be approximately accounted for. Previous work on Taylor-Görtler vortices has been reported by Di Prima and Dunn [8] in the flow of a liquid along curved heated walls, by Hämmerlin [9] who considered compressible flows, and Kirchgässner [10] who studied the stability of Couette-type flows along curved heated walls. In the first two analyses, the effect of variable fluid properties was studied, rather than the effect of the destabilizing buoyancy forces caused by the temperature stratification. Thus Di Prima and Dunn [8] as well as Hämmerlin [9] concluded that heating of the lower boundary results in increased stability, while Kirchgässner [10] arrived at the opposite result.

Theory

The coordinate system is defined in Fig. 1. The flow of a laminar boundary layer along a concave heated wall is considered. The standard method of linear stability theory consists in a perturbation of the Navier-Stokes equations about a mean solution, and consideration of the resulting "zeroth" and "first-order" differential equations obtained. Thus the instantaneous coordinate velocities, temperature, and pressure are represented by $U = \overline{U} + \tilde{u}$, $V = \overline{V} + \tilde{v}$, etc., where the overbars signify mean and the tildas signify the fluctuating components.

Following Smith [5], a spacewise growth of disturbances is chosen, consistent with experimentally observed results. Accordingly the normal modes postulated are of the form

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{p} \\ \tilde{T} \end{bmatrix} = \begin{bmatrix} u_p(y) \\ v_p(y) \\ w_p(y) \\ p_p(y) \\ T_p(y) \end{bmatrix} \begin{bmatrix} \cos \alpha z \\ \sin \alpha z \end{bmatrix} e^{\beta x}$$

where x, y, z are the streamwise, vertical, and lateral coordinates, respectively, subscript p denotes perturbation amplitudes, α is the lateral wave number, and β is the amplification parameter. Substitution of the normal mode representation into the partial differential equations governing the perturbations results in the following coupled set of ordinary differential equations where all quantities have been made dimensionless

$$\frac{d^2u}{d\eta^2} - [V_0 R_d + K]\frac{du}{d\eta} + \left[B^2 - A^2 - R_d\left(BU_0 - \frac{dV_0}{d\eta}\right) + K\eta B(2B - U_0R_d)\right]u - \left[R_d\frac{dU_0}{d\eta} + K(2B - U_0R_d)\right]v = \frac{B}{2}C(1 + K\eta) \quad (1a)$$

$$\frac{d^2v}{d\eta^2} - [V_0R_d + K]\frac{dv}{d\eta} + \left[B^2 - A^2 + 2K\eta B^2 - BR_dU_0(1 + K\eta) - R_d\frac{dV_0}{d\eta}\right]v - [2K(R_dU_0 - B)]u = \frac{1}{2}\frac{dC}{d\eta} - Gr\,T \quad (1b)$$

$$\frac{d^2w}{d\eta^2} - [V_0R_d + K]\frac{dw}{d\eta} + [B^2 - A^2 + 2KB^2\eta \quad (1c)$$

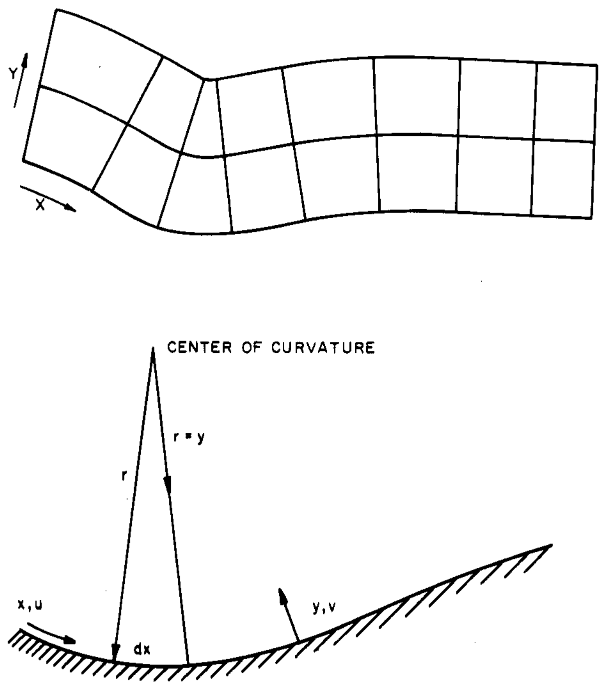

Fig. 1 Orthogonal curvilinear coordinate system

$$- BR_dU_0(1 + K\eta)]w = -\frac{A}{2}C \quad (1c) \; (Cont.)$$

$$Bu + KBu\eta + \frac{dv}{d\eta} + Aw - Kv = 0 \quad (1d)$$

$$\frac{d^2T}{d\eta^2} - [V_0 \Pr R_d + K]\frac{dT}{d\eta} + [B^2 - A^2 + 2KB^2\eta - BU_0 \Pr R_d(1 + K\eta)]\,T = \Pr v\frac{dT_0}{d\eta} \quad (1e)$$

$$\begin{aligned}
U_0 &= U/U_\infty \\
V_0 &= V/U_\infty \\
T_0 &= (T - T_\infty)/\Delta T \\
\Delta T &= T_w - T_\infty \\
u &= u_p/U_m \\
v &= v_p/U_m \\
w &= w_p/U_m \\
C &= p_p \Big/ \frac{1}{2}\rho U_m{}^2 \\
T &= T_p/\Delta T \\
\eta &= y/\delta \\
U_m &= \nu/\delta \\
R_d &= U_\infty\delta/\nu \\
K &= k\delta \\
B &= \beta\delta \\
A &= \alpha\delta \\
\Pr &= \frac{\nu}{\kappa} \text{ is the Prandtl number} \\
Gr &= \frac{g\gamma\delta^3\Delta T}{\nu^2} \text{ is the Grashof number}
\end{aligned}$$

with γ, g, κ, and ν being the bulk expansion coefficient, the gravitational acceleration, the thermal diffusivity, and the kinematic viscosity, respectively. U_∞ and δ are chosen as the free-stream velocity and characteristic boundary-layer thickness, respectively. Except for the additional term in equation (1b) representing the thermal buoyancy and the extra equation (1e) obtained from the energy equation, the foregoing set of equations are basically the same as those derived by Smith [5] for the isothermal case.

Boundary Conditions

The lower boundary is assumed to be a rigid, isothermal conducting plane. This necessitates the following "wall boundary conditions"

$$\eta = 0, \qquad u = v = w = T = 0$$

The continuity equation (1d) for the perturbations yields the auxiliary boundary condition

$$\eta = 0, \qquad \frac{dv}{d\eta} = 0$$

Since the flow is unbounded vertically, and attention focussed only on those disturbances decaying at large distances from the wall, we must have that, as $\eta \to \infty$, u, v, w, $T \to 0$. The system of equations is homogeneous with homogeneous boundary conditions and therefore constitutes an eigenvalue problem. For given values of the amplification B, the Reynold's number R_d, the Grashof number Gr, and wave number A, one seeks the lowest absolute value of K which will provide a nontrivial solution. The variation of K with A traces out amplification curves for different values of B. The calculation requires a specified variation of U_0, V_0, and T_0 (the components of the primary flow) with η. The computations were performed using the similarity solutions for a laminar boundary layer on a heated flat plate for Prandtl numbers of 0.72 and 7.0. A small number of calculations were also performed on the asymptotic suction profile in isothermal flow.

Numerical Method

The numerical method used is based on a technique described by Conte [11]. Equations [1] are rewritten as a system of nine first-order differential equations. It is possible to derive boundary conditions which may be applied at the outer edge of the boundary layer rather than at infinity, thus restricting the integration to a finite interval. This is dependent on the fact that in the free stream, the set of equations (1) reduce to a constant coefficient type since V_0, T_0, and U_0 become constants. Solutions in terms of elementary functions are thus easily obtained for the whole free-stream region and in particular at the boundary-layer edge taken here as $\eta = 9$. The outer solutions obtained are detailed in [12]. Four independent starting solutions are synthesized at the boundary-layer edge and integrated in the reverse direction toward the wall using a modified Hamming predictor-corrector fourth-order formula. At certain stages in the integration process, the independent solution vectors tend to become increasingly dependent. At this point, integration is temporarily interrupted and linear independence restored by the Gram-Schmidt orthonormalization process. At the wall, ($\eta = 0$), the boundary conditions will in general be satisfied only when the correct eigenvalue has been chosen in the computations. A modified version of the Newton iterative method was adopted that gave rapid convergence to the desired eigenvalue. The primary flow in the boundary layer was accurately represented by high-order polynomials.

Results and Discussion

The results obtained are divided into two parts: the first part deals with the calculations for an isothermal boundary layer while the second part considers the effect of wall heating. Stability results for a boundary layer on a concave wall are usually reported in terms of the Görtler number based on the momentum thickness θ of the boundary layer,

$$N_{G_\theta} = \frac{U_\infty \theta}{\nu} \sqrt{\theta / R_0}$$

where R_0 is the local radius of curvature of the wall. We also use the corresponding wave number $A_\theta = \alpha\theta$.

Fig. 2 is a plot of the neutral stability curves for the Blasius profile with and without the V_0 terms included in the calculations. The neutral stability curve for the asymptotic suction profile including the effect of the V_0 term is also presented on the same diagram. Chang and Sartory [6] found that as A_θ tended toward zero, the critical Görtler number simultaneously appeared to approach zero, when the normal flow terms were included in the analysis. The present results indicate that with the additional curvature terms of Smith [5] retained in the analysis, the value of the Görtler number approaches a critical value of zero as the wave number reaches a limiting value of 0.3. Several calculations in the vicinity of a wave number of 0.29 failed to converge to a definite positive eigenvalue, thereby indicating an absence of solutions for wave numbers below about 0.3. The discrepancy between the present results and those of Smith is probably due to his use of the Galerkin method in solving the equations. In this method the solution was assumed to have the form of a polynomial times an exponential factor which decayed far from the wall. The polynomial coefficients were calculated in solving the problem, but the exponential decay rate was assumed mainly on the basis of numerical experimentation. Chang and Sartory [6] indicate that Smith's results at the lower wave numbers could be incorrect by over an order of magnitude. At wave numbers greater than about 0.4, the present calculations are in good agreement with those of other workers. Differences, at small wave numbers, between the present results and those of Chang and Sartory's may be attributed to the retention, in the present analysis of the extra curvature terms of Smith [5]. Chang and Sartory had in fact, anticipated that the influence of these terms could be important at small wave numbers. The present results support this, although a limiting value of zero for the critical Görtler number is certainly peculiar and of no apparent physical significance.

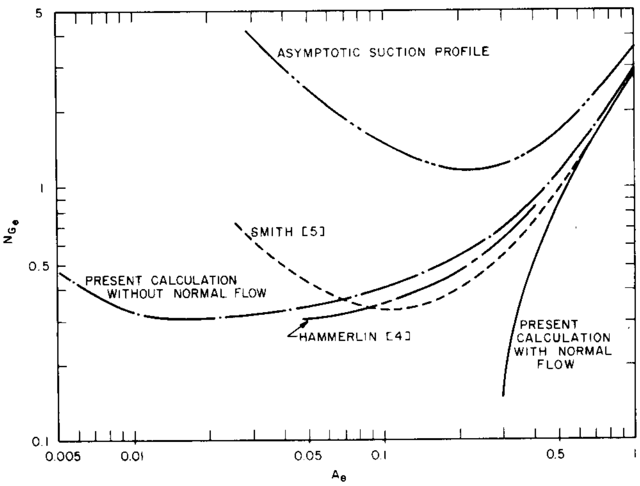

Fig. 2 Comparison of neutral amplification curves

With the exclusion of the normal flow terms, the calculations exhibit a minimum in the neutral stability curve at a wave number of approximately 0.015. The critical value of the Görtler number obtained is about 0.3 in fairly good agreement with the results of Smith [5] and of Hämmerlin [4]. Once again, the discrepancy in critical wave numbers must be attributed to the influence of the extra curvature terms in the present calculation. Some verification of this was obtained by carefully performing several computations in the vicinity of the minimum with different step sizes and convergence criteria for the eigenvalue. No changes in the eigenvalue were observed. The results demonstrate therefore, the increasing significance of the vertical velocity component of the primary flow and extra curvature terms in the computations at small wave numbers.

To better assess the effect of the normal velocity component on the stability, a small number of computations were performed on the asymptotic suction profile and the stability curve plotted in comparison. The stability of this profile has already been examined by Kobayashi [13] assuming time-varying disturbances. The critical stability of the flow is increased by a factor of about three. The dominant effect of suction throughout the boundary layer is so strong that good agreement throughout the full range of wave numbers is obtained between the present calculations and those of Kobayashi [13] who neglected the extra curvature terms of Smith. (An exami-

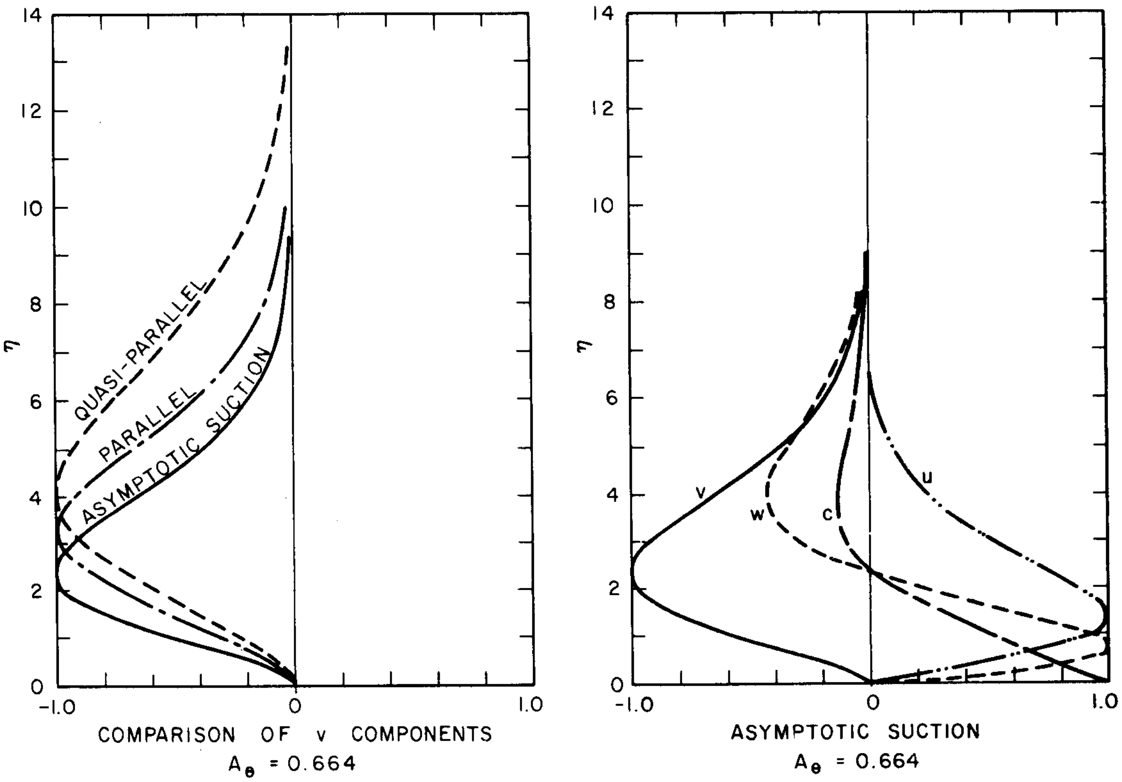

Fig. 3 Eigenfunctions

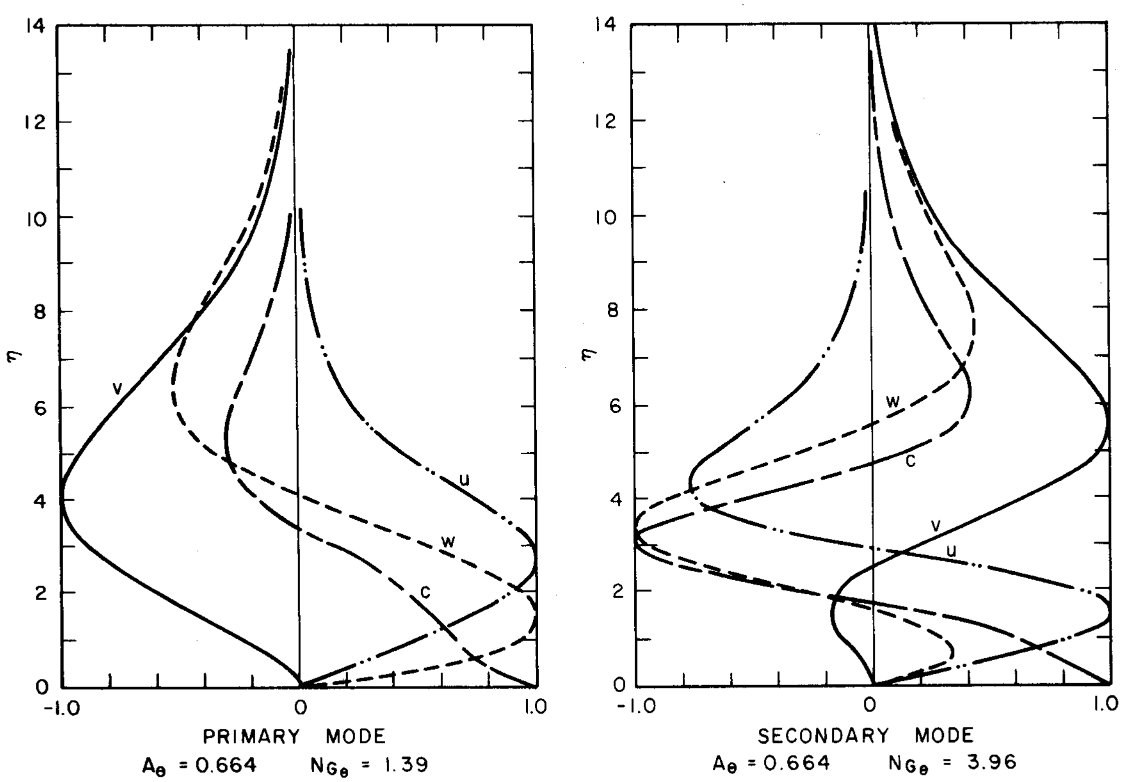

Fig. 4 Selected eigenfunctions

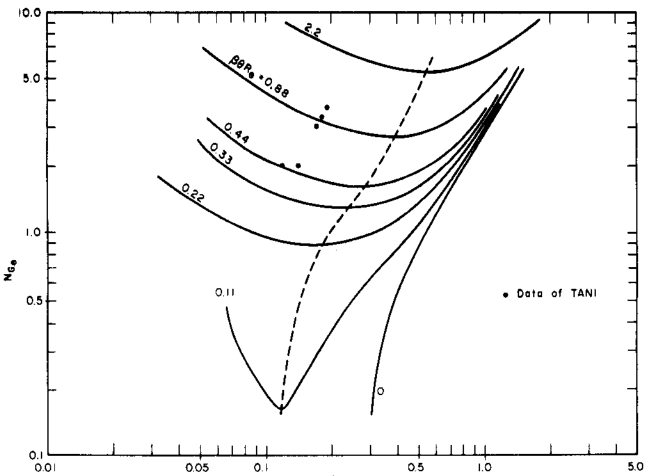

Fig. 5 Amplification curves for the Blasius profile

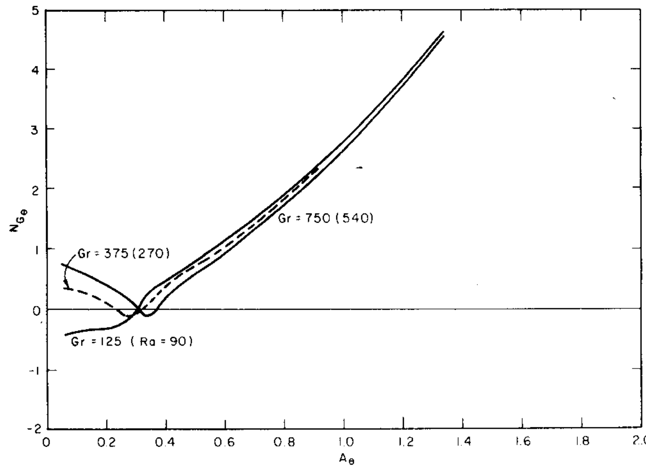

Fig. 7 Critical curves of Görtler number versus wave number for three Grashof numbers (quasi-parallel flow); Prandtl number = 0.72

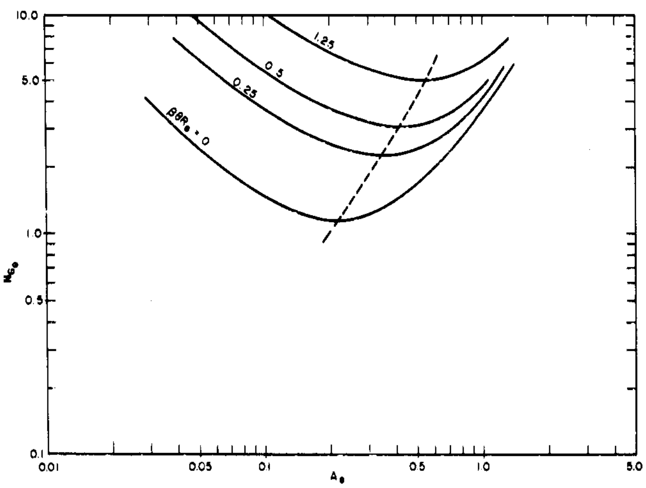

Fig. 6 Amplification curves for the asymptotic suction profile

nation of the coefficients in the set of equations (1) clearly indicates that the terms involving V_0, which is constant throughout the boundary layer with asymptotic suction, will at all times dominate the extra curvature terms.)

The role of the vertical velocity component of the primary flow is to alter the extent of free-stream penetration of the vortices. This in turn alters the dissipative influence of viscosity (which is restricted to the boundary-layer region) on the perturbations. Verification of this is provided in Fig. 3, where the vertical (v) component of the eigenfunctions for the Blasius profile with and without the normal component of the primary flow, and for the asymptotic suction profile have been plotted in comparison at a wave number of $A_\theta = 0.664$. The curves clearly indicate the altered extent of free-stream penetration.

Some calculated eigenfunctions are presented in Fig. 4. It is interesting that the pressure fluctuation is the only nonzero component at the wall and is in fact a maximum there. No boundary conditions have been assumed for the pressure in the present calculations, although it may be shown that it must decay exponentially like the other components in the free stream. With amplification of the vortices beyond the stage covered by linear theory, the pressure fluctuations induce a significant redistribution of momentum within the primary flow of the boundary layer causing it to exhibit three-dimensional effects. The maximum of the pressure fluctuation occurring at the wall suggests that if the wall were distensible, a drastic lowering of the critical Görtler number may be expected. This reduction in the stability of the flow, could be speculated upon as being due to higher-order coupling effects between the wall and the fluid layer so that at some range of wave numbers the energy exchange between the two could result in strong amplification of the disturbances.

The eigenfunctions for the second mode were calculated at $A_\theta = 0.664$ and are included in Fig. 4. They indicate a system of two counterrotating vortices stacked one above the other, the lower vortex being the smaller of the two. Presumably, if the Görtler number is raised to a sufficiently high value, these additional modes could be excited since the linearized theory permits an independent existence between all modes and wave numbers. For an actual observation of these higher-order fluctuations however, it is necessary that they have a higher amplification rate than the normal modes. This type of behavior, although not observed in the Görtler instability, has been experimentally observed by Whitehead [14] in the analogous case of thermal convection.

Fig. 5 presents the amplification curves for the Blasius profile while Fig. 6 is the corresponding plot for the asymptotic suction profile. By eliminating the characteristic thickness θ from the definition of the Görtler number and the dimensionless wave number $A_\theta (= \alpha\theta = 2\pi\theta/\lambda$ where λ is the lateral wavelength) we obtain $N_{G_\theta}{}^2/A_\theta{}^3 = U_\infty{}^2\lambda^3/8\pi^2\nu^2R_0$ where R_0 is the radius of curvature of the plate. Following the argument of Kobayashi [13], if U_∞, R_0, and λ remain constant with increasing x, then $N_{G_\theta}{}^2/A_\theta{}^3$ becomes a constant C_0. This may be represented by a line of gradient 3/2 on the log-log plots of Figs. 5 and 6. Experiments by Tani [15] and McCormack, et al. [7], in a laminar boundary layer indicate that the wavelengths of the vortices are relatively unaffected as they amplify in the streamwise direction. The experimental results of Tani [15] taken in a laminar boundary layer have been plotted in Fig. 5. The data points lie in the amplifying region of the curves and fall approximately along a line of gradient 3/2. The dotted line drawn in Fig. 5 through the minima of the amplification curves has approximately a slope of 3/2 for values of the amplification parameter $\beta\theta R_\theta$ greater than about 0.22. At lower values of the amplification parameter, the slope becomes much steeper indicating that in this region, amplification does not necessarily occur at constant wavelength. The corresponding curve for the asymptotic suction profile has a gradient much closer to 3/2 as Fig. 6 indicates. Kobayashi [13] did not obtain as good an agreement with his amplification curves, probably due to his assumption of temporally amplifying disturbances.

We now turn to the results for the cases which include the effects of heating. The validity of the results are restricted to those fluids which have only a minor dependence of their viscosity and thermal diffusivity on temperature, or alternatively, where temperature differences are small, but not small enough that the buoyancy term from the Boussinesq approximation may be neglected. In the results presented here, the Grashof number has been calculated using the definition of

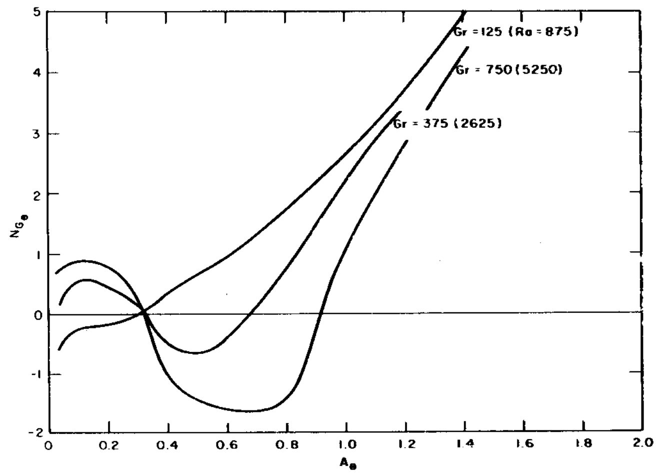

Fig. 8 Critical curves of Görtler number versus wave number for three Grashof numbers (quasi-parallel flow); Prandtl number = 7

$$\delta = 5.0\sqrt{\frac{\nu x}{U_\infty}}$$

given be Schlichting [16] for a laminar Blasius layer. This definition of δ is the height at which the boundary-layer streamwise velocity reaches 99 percent of it's free-stream value. Although it may appear that this use of δ in the definition of the Grashof number is not strictly relevant, δ is in fact related to δ_T the thermal boundary-layer thickness through the Prandtl number. Based on the similarity variable therefore, the values of the Grashof and Rayleigh numbers presented would be reduced by a factor of 5^3 or 125. The other parameters are presented using the momentum thickness as before. Calculations have been performed for Prandtl numbers of 0.72 and 7.0, respectively. Neutral stability curves for the lower Prandtl number are presented in Fig. 7 at three different Grashof numbers. The figures in brackets are corresponding values of the Rayleigh number (= Gr × Pr). The effect of wall heating is to reduce the stability of the flow to three-dimensional disturbances. Within certain intervals of the wave number, the curves enter the negative quadrant indicating that convex curvature would be required to maintain stability. In order to make certain that convergence to the desired eigenvalues was being obtained, the eigenfunctions were inspected at several computed eigenvalues.

An interesting observation is that for wave numbers below about 0.3, the effect of heating results in an increased stability of the flow. The same type of behavior is evident in Fig. 8 where neutral stability curves for a Prandtl number of 7 are presented. This stabilizing effect of heating may be due to a "phase shift" type interaction between the velocity and temperature fluctuations. This means that at these lower wave numbers, the inertial and thermal mechanisms strive to create disturbances of different vertical magnitude. If the temperature fluctuations are such that at certain regions in the boundary layer, the vertical velocity fluctuation would have to do work against the gravity vector, this would constitute a damping force, thus resulting in increased stability.

Typical eigenfunctions for the two Prandtl numbers are presented in Fig. 9. Those at the higher Prandtl number are seen to display considerable distortion from their unheated values.

Conclusions

The study reported here was undertaken to provide further information on the combined effect of body forces in laminar transition. The results indicate that the parallel flow assumption as well as neglection of higher-order curvature terms may lead to erroneous results in the computations of Görtler instability. The effect of the vertical velocity component of the primary flow is to stabilize the flow if directed toward the wall and reduce it if directed away from the wall. The effect of heating if assumed small enough, are to destabilize the flow to disturbances of small wavelength but result in increased stability to those of long wavelength.

The results should provide useful information in developing nonlinear theories of boundary-layer transition involving body forces.

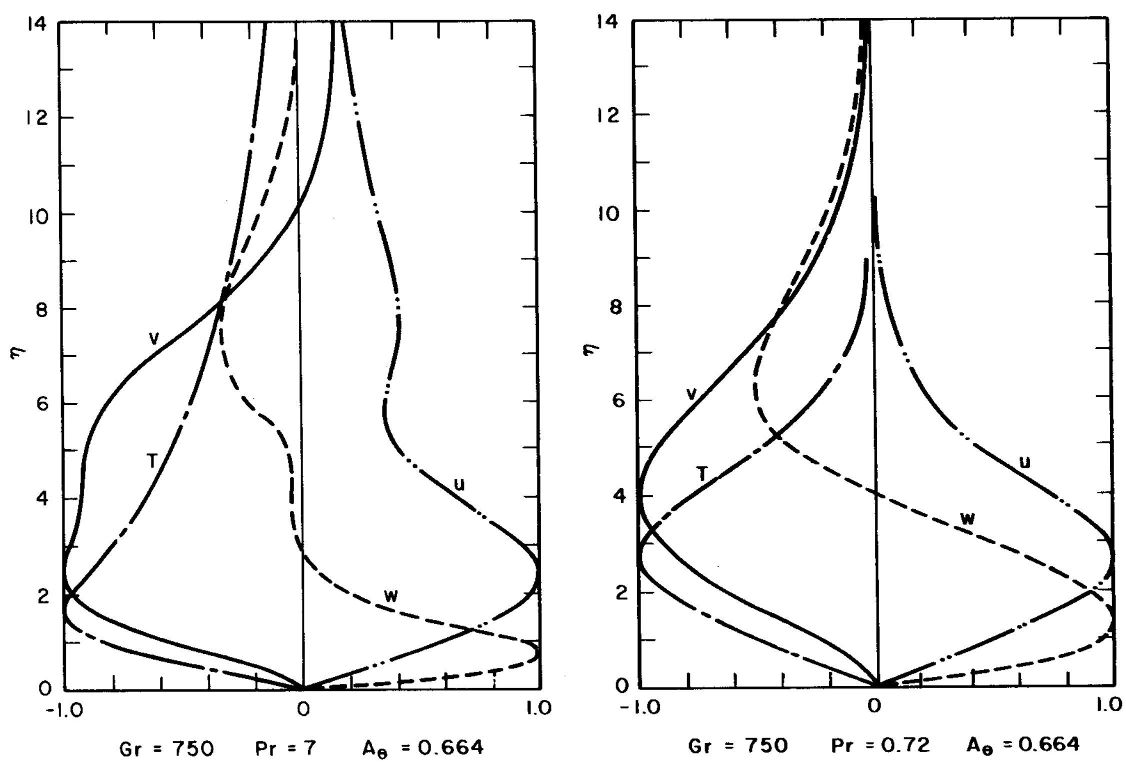

Fig. 9 Selected eigenfunctions for curved heated Blasius profile

Acknowledgments

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References

1 Görtler, H., "Über eine dreidimensionale Instabilitat laminarer Grenzschichten on Konkaven Wanden," *Nachr. Akad. Wiss. Goetlingen Math-Physik Kl. IIa*, Math-Physik-Chem., Abt. 2, 1–26 1940.

2 Taylor, G. I., "Stability of a Viscous Liquid Contained Between Two Rotating Cylinders," *Philosophical Transactions of the Royal Society*, London, Series A, Vol. 223, 1923, p. 289.

3 Meksyn, D., "Stability of Viscous Flow Over Concave Cylindrical Surfaces," *Proceedings of the Royal Society*, London, Series A, Vol. 203, 1950, p. 253.

4 Hämmerlin, G., "Über das Eigenwertproblem der dreidimensionalen Instabilitat laminarer Grenzschichten an Konkaven Wanden," *Journal of Rational Mechanics and Analysis*, Vol. 4, 1955, p. 279.

5 Smith, A. M. O., "On the Growth of Taylor-Görtler Vortices Along Highly Concave Walls," *Quarterly of Applied Mathematics*, Vol. 13, 1955, p. 223.

6 Chang, T. S., and Sartory, W. K., "Hydromagnetic Görtler Instability in a Boundary Layer on a Concave Wall," *Developments in Theoretical and Applied Mechanics*, Vol. 4, 1968, ed., D. Frederick.

7 McCormack, P. D., Welker, H., and Kelleher, M., "Taylor-Görtler Vortices and Their Effect on Heat Transfer," ASME-AICHE Heat Transfer Conference, Minneapolis, Minn., 1969, ASME Paper No. 69-HT-3.

8 Di Prima, R. C., and Dunn, D. W., "The Effect of Heating and Cooling on the Stability of the Boundary-Layer Flow of a Liquid Over a Curved Surface," *Journal of the Aeronautical Sciences*, Vol. 23, No. 10, Oct. 1956, p. 913.

9 Hämmerlin, G., "Über die Stabilitat einer Kompressiblen Stromung Langs einer Konkaven Wand bei Verschiedenen Wandtemperaturverhaltnissen," *Bericht der Deutschen Versuchsanstalt fur Luftfahrt*, E. V. Nr. 176, 1961.

10 Kirchgässnor, K., "Stromungen an konkaven und erwarmten Wanden," *Ingenieur-Archiv.*, Vol. 31, 1962, p. 115.

11 Conte, S. D., "The Numerical Solution of Linear Boundary-Value Problems," *SIAM Review*, Vol. 8, No. 3, 1966, p. 309.

12 Kahawita, R. A., PhD thesis, 1973.

13 Kobayashi, R., "Note on the Stability of a Boundary Layer on a Concave Wall With Suction," *Journal of Fluid Mechanics*, Vol. 52, Part 2, 1972, p. 269.

14 Whitehead, J. A., "Cellular Convection," *American Scientist*, Vol. 59, 1971, p. 444.

15 Tani, I., "Production of Longitudinal Vortices in the Boundary Layer Along a Concave Wall," *Journal of Geophysical Research*, Vol. 67, No. 8, 1962.

16 Schlichting, H., *Boundary-Layer Theory*, 6th ed., 1968, p. 130.

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21 BRIDGE SQUARE, WESTPORT, CN 06880

DATE : JUNE 18-20, 1979 — LOCATION: BUFFALO, NEW YORK
TITLE: 1979 APPLIED MECHANICS CONFERENCE
SPONS: ASME,CSME — CONTACT:

DATE : MID-OCTOBER, 1979 — LOCATION: CINCINNATI-DAYTON, OH
TITLE: SESA FALL MEETING
SPONS: SESA — CONTACT: MR. B.E. ROSSI
21 BRIDGE SQUARE, WESTPORT, CN 06880

DATE : DECEMBER 2-7, 1979 — LOCATION: NEW YORK
TITLE: ASME WINTER ANNUAL
SPONS: ASME — CONTACT:

DATE : MID-MAY, 1980 — LOCATION: BOSTON, MA
TITLE: 4TH SESA INTERNATIONAL CONF. ON EXPERIMENTAL MECHANICS
SPONS: SESA — CONTACT: MR. B.E. ROSSI
21 BRIDGE SQUARE, WESTPORT, CN 06880

DATE : MID-OCTOBER 1980 — LOCATION: ST PALM BEACH, FL
TITLE: SESA FALL MEETING
SPONS: SESA — CONTACT: MR. B.E. ROSSI
21 BRIDGE SQUARE, WESTPORT, CN. 06880

DATE : NOVEMBER 16-21, 1980 — LOCATION: CHICAGO, ILL.
TITLE: ASME WINTER ANNUAL MEETING
SPONS: ASME — CONTACT:

DATE : NOVEMBER 15-21, 1981 — LOCATION: NEW YORK (TENTATIVE)
TITLE: ASME WINTER ANNUAL MEETING
SPONS: ASME — CONTACT:

DATE : NOVEMBER 14-18, 1982 — LOCATION: LOS ANGELES (TENTATIVE)
TITLE: ASME WINTER ANNUAL MEETING
SPONS: ASME — CONTACT:

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Stability of Poiseuille Flow in Elastic Tubes[1]

This paper presents a theoretical analysis of the temporal and spatial stability of Poiseuille flow in elastic tubes to infinitesimal axisymmetric disturbances. A cylindrical shell model which includes the effects of transverse shear and rotatory inertia is employed for the tube wall. The characteristic equation of the system is solved numerically and two sets of modes are obtained; one set has eigenvalues that are independent of the properties and dimensions of the tube wall, while the other set has eigenvalues that depend on the tube parameters. One mode of the "tube-dependent" set is shown to have a critical Reynolds number that depends on the elastic properties and dimensions of the tube and either wave number or frequency of the disturbance.

Introduction

Previous analyses of the linear stability of Poiseuille flow in a circular tube have assumed that the tube is rigid; these studies by Gill [1],[2] Garg and Rouleau [2], and Salwen and Grosh [3], have demonstrated conclusively that the flow is temporally and spatially stable to infinitesimal axisymmetric and nonaxisymmetric disturbances for all Reynolds numbers. Experimental confirmation has been obtained by Leite [4]. That tube elasticity might be a factor in linear stability of viscous flow was suggested by recent work of DeArmond and Rouleau [5], which showed that certain modes of wave propagation in a viscous compressible liquid confined in an elastic tube were strongly dependent on the elastic properties of the tube. This effect upon stability has since been confirmed by Midvidy and Rouleau [6].

In this present paper the rigid tube assumption is removed, and the ensuing analysis shows that when the tube is elastic, a critical Reynolds number exists, above which the flow is temporally and spatially unstable to infinitesimal axisymmetric disturbances.

Spatial stability is emphasized because this corresponds most closely to physical reality [2]; in the spatial analysis an infinitesimal disturbance is imposed at a specific location in the fluid, and its growth or decay with increasing downstream distance provides the criterion for stability. Temporal stability is also analyzed; in this type of analysis an infinitesimal disturbance is initially applied everywhere in the fluid, and its subsequent growth or decay with increasing time determines stability.

The elastic tube is modeled as a cylindrical shell, including transverse shear and rotatory inertia. Compressibility of the liquid is taken into account. Values of the critical Reynolds number are calculated for typical sets of parameters.

Basic Equations

The motion of a compressible Newtonian liquid subjected to infinitesimal disturbances is described by the continuity and Navier-Stokes equations, linearized by neglecting the product of disturbances (perturbations) with themselves or their derivatives, and by the equation of state

$$\frac{\partial p}{\partial \rho} = c_f{}^2 \qquad (1)$$

where p, ρ are the liquid pressure and density, respectively, and c_f is the speed of sound in the liquid. These equations are made dimensionless with respect to the inner radius of the tube r_p, the maximum velocity of the mean flow $\mathcal{V}$ and the mean density of the fluid ρ_0.

The infinitesimal arbitrary disturbances are synthesized by a Fourier series; in view of the linearity of the governing equations it is sufficient to examine each Fourier component separately in the nondimensional form

$$\phi(r, z, t) = \mathrm{Re}\,[\hat{\phi}(r, z, t)] = \mathrm{Re}\,[\overline{\phi}(r) \exp{(kz - i\omega t)}] \qquad (2)$$

where ϕ is a disturbance quantity, r, z, and t are the nondimensional radial, axial, and time coordinates, respectively, Re denotes the real part of a complex function, $\hat{\phi}$ is a preliminary complex solution, $\overline{\phi}(r)$ is a complex eigenfunction, $\omega = \omega_r + i\omega_i$ is the complex frequency, $k = k_r + ik_i$ is the complex axial propagation constant and $i \equiv \sqrt{-1}$. For a spatial stability analysis, ω is specified real and k is the complex eigenvalue to be found; depending on whether k_r is positive, negative, or zero, the flow is unstable, stable, or neutrally stable, respectively. For a temporal stability analysis, k is specified imaginary and ω is the

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[2] Numbers in brackets designate References at end of paper.

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