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SIMILARITY THEORY OF DIFFUSION AND THE OBSERVED VERTICAL SPREAD IN THE DIABATIC SURFACE LAYER

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Abstract. The extension of Lagrangian similarity theory of diffusion to stratified flow is examined, to improve its prediction of the vertical spread of a passive substance. In the basic equation, $\mathrm{d}\bar{Z}/\mathrm{d}t = bu_*\phi(\bar{Z}/L)$ where $\bar{Z}$ is the average height of a cluster of particles, u_* is the friction velocity and L is Monin-Obukhov length. It is shown theoretically, under the assumption of an equivalence between the diffusivities of heat and matter, that the unspecified function ϕ is the reciprocal of a more familiar meteorological parameter ϕ_h, the dimensionless temperature gradient. The universal constant b is found to be approximately equal to von Karman's constant κ for various stability conditions. The predicted effect of stability on vertical spread shows excellent agreement with that of the published data from the O'Neill experiments.

1. Introduction

Ever since Batchelor (1959) introduced the idea of Lagrangian similarity to study diffusion of smoke released at or near the surface under neutral conditions, there have been various attempts at extending this approach to the case of stratified flows. Pasquill (1966) put these formulations (Gifford, 1962; Cermak, 1963; Panofsky and Prasad, 1965) to test by comparing the predicted values of vertical spread at a distance of 100 m from the source with those observed during the O'Neill experiments. It was found that the prediction of the effect of stability variation on the vertical spread was considerably less severe than that shown by experimental data. Pasquill concluded that the representativeness of the functions introduced arbitrarily in these formulations to describe the effect of stratification on the vertical displacement was questionable. The dimensional arguments can lead to the correct non-dimensional groups involved in a problem, yet not their functional form. It is the purpose of this paper to derive the similarity relationship from the appropriate diffusion equation and identify the correct functional form of the similarity parameters. This approach also enables one to estimate the errors involved in the assumptions made in the previous formulations. The method used is an extension of the techniques introduced by Chatwin (1968) in which the average position of a cluster of particles can be expressed in terms of concentration distribution.

2. Background

Batchelor (1964) hypothesized that in the constant stress region, 'the statistical properties of the velocity of a marked fluid particle at time t after release depend upon

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only u_* (friction velocity) and t'. If $\bar{X}(t)$, $\bar{Y}(t)$ and $\bar{Z}(t)$ denote average position of a particle, then according to this hypothesis, the average rate of vertical displacement of the particle released at the ground can be written as

$$\mathrm{d}\bar{Z}/\mathrm{d}t = bu_*, \tag{1}$$

where b is a universal constant. This result was supplemented with an assumption that the average horizontal velocity of a particle equals the average horizontal velocity of the fluid evaluated at the average height of the particles, $\bar{Z}$, multiplied by a constant c, i.e.,

$$\mathrm{d}\bar{X}/\mathrm{d}t = \bar{u}(c\bar{Z}). \tag{2}$$

A relation between $\bar{X}$ and $\bar{Z}$ for the neutral flow can be obtained from Equations (1) and (2) after eliminating t.

In diabatic situations, a different set of parameters determines the turbulent state and should be incorporated in any extension of the above treatment. According to Monin and Obukhov's (1954) hypothesis of Eulerian similarity, when the turbulence is homogeneous in the horizontal, the turbulent regime is completely determined by the friction velocity u_* and the length L defined as

$$L = \frac{u_*^3}{\kappa (g/T)(-H/\varrho c_p)}, \tag{3}$$

where u_* and vertical heat flux H are independent of height and T is the average temperature. All dimensionless variables are expected to be functions only of the dimensionless height $\zeta = z/L$. In particular, the average wind profiles can be represented by

$$\bar{u}(z) = \frac{u_*}{\kappa}[f(\zeta) - f(\zeta_0)], \tag{4}$$

where $\zeta_0 = z_0/L$, z_0 being the roughness length and $f(\zeta)$ a universal function.

Gifford (1962) considered diffusion in stratified flow and suggested that average vertical velocity of a particle must, on dimensional grounds, be proportional to u_* times some universal function ϕ involving the stability length L, i.e.,

$$\mathrm{d}\bar{Z}/\mathrm{d}t = bu_*\phi(\bar{\zeta}), \tag{5}$$

where $\bar{\zeta} = \bar{Z}/L$. Since the relation should reduce to Equation (1) for neutral flow ($L = \infty$), if follows that $\phi(0) = 1$. In the absence of any experimental data on ϕ, a tacit assumption has been made in nearly all the works on Lagrangian similarity that $\mathrm{d}\bar{Z}/\mathrm{d}t$ is equivalent to the standard deviation of the vertical velocity fluctuations σ_w. Whereas some authors (Gifford, 1962; Cermak, 1963; Klug, 1968) employed a semi-empirical formula of Kazanski and Monin (1957) for σ_w $(u_*(1-1/f')^{1/4})$, Panofsky and Prasad (1965) used existing experimental data for this parameter. The assumption about the equivalence of $\mathrm{d}\bar{Z}/\mathrm{d}t$ and σ_w is without sufficient basis and has not been justified. These extensions of the similarity theory to the diabatic case, however, retained Batchelor's assumption expressed in Equation (2) ($c=1$) but used diabatic wind profiles.

3. Derivation of Similarity Relationships in Stratified Flow

The Lagrangian similarity hypothesis considers the problem of diffusion of a number of marked particles of fluid released from a fixed ground source at different instants of time. Each fluid particle would occupy a different position at time t after release and $\bar{X}(t)$ and $\bar{Z}(t)$ are then the average longitudinal and vertical displacements, over all such particles, at the time t. In a homogeneous and stationary flow field, such a diffusion may be likened to that of a cloud of marked particles released from a ground source at one instant (Pasquill, 1966). The latter problem is of Eulerian type and $\bar{X}(t)$ and $\bar{Z}(t)$ are now equivalent to the coordinates of the center of mass of a cloud.

The distribution of concentration $c(x, y, z, t)$ at a point x, y, z at time t in a cloud of passive substance, released in a plane homogeneous turbulent shear layer, can be described by the diffusion equation:

$$\frac{\partial c}{\partial t} + \bar{u}(z)\frac{\partial c}{\partial x} = \frac{\partial}{\partial z}\left(K_z \frac{\partial c}{\partial z}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial c}{\partial y}\right), \tag{6}$$

where the longitudinal diffusion term is neglected, being much smaller than the convective term $u(\partial c/\partial x)$ and K_y and K_z are the eddy diffusivities of matter in the y and z directions. Since the cloud is symmetrical with respect to the y direction, we can integrate out the dependence of c on y in the above equation to obtain

$$\frac{\partial c_0}{\partial t} + \bar{u}(z)\frac{\partial c_0}{\partial x} = \frac{\partial}{\partial z}\left(K_z \frac{\partial c_0}{\partial z}\right), \tag{7}$$

where $c_0(x, z, t) = \int_{-\infty}^{\infty} c(x, y, z, t)\,\mathrm{d}y$ is the concentration due to an infinite line source. If the total amount of substance in the cloud is unity, the position of its center of mass is given by

$$\bar{X}(t), \bar{Z}(t) = \int_0^{\infty}\int_{-\infty}^{\infty} x, z\, c_0(x, z, t)\,\mathrm{d}x\,\mathrm{d}z\,. \tag{8}$$

By multiplying Equation (7) by z and x, respectively, and integrating it, Chatwin (1968) obtained

$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} = \int_0^{\infty} z\frac{\partial}{\partial z}\left(K_z \frac{\partial c_{00}}{\partial z}\right)\mathrm{d}z \tag{9}$$

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = \int_0^{\infty} \bar{u}(z)\, c_{00}\,\mathrm{d}z \tag{10}$$

where $c_{00} = \int_{-\infty}^{\infty} c_0 \mathrm{d}x$ indicates the average concentration of particles over the x-y

plane at height z. An equation for the distribution of c_{00} in the vertical is obtained by integrating Equation (7) with respect to x, as

$$\frac{\partial c_{00}}{\partial t} = \frac{\partial}{\partial z}\left(K_z \frac{\partial c_{00}}{\partial z}\right). \tag{11}$$

The relationships in Equations (9) and (10) can be evaluated for thermally stratified flow if $K_z(z)$ and $c_{00}(z, t)$ are known. Although no direct measurement of K_z is available in stratified flow, it is possible to relate it to eddy diffusivities of other entities. According to Monin and Yaglom (1965), in view of similarity of physical mechanisms governing the exchange of heat and passive substance, namely, direct mixing of air masses, their eddy diffusivities can be considered equal and should be distinguished from diffusivity of momentum which is influenced by pressure fluctuations in addition to turbulent mixing. Following this suggestion and using the Monin-Obukhov similarity hypothesis, we have

$$K_z = \frac{\kappa u_* z}{\phi_h(\zeta)}, \tag{12}$$

where $\phi_h(\zeta) = (z/T_*)(\partial T/\partial z)$ is the dimensionless temperature gradient and $T_* = (1/ku_*)(-H/\varrho C_p)$ is the scaling temperature. A number of forms for the function $\phi_h(\zeta)$ have been suggested in the literature. Since it is the form of function ϕ in Equation (5) which is of interest, we will use a power-law representation

$$\phi_h(\zeta) = A\zeta^p \tag{13}$$

in order to be able to evaluate Equation (9) and (10) analytically. Thus

$$K_z = K_1 z^{1-p}, \tag{14}$$

where $K_1 = \kappa u_* |L|^p / A$. p may have different values in different ranges of stability and may thus be regarded as an index of stability.

The solution of Equation (11) with K_z as in Equation (14), subject to the usual boundary conditions for an infinite line source at the ground, is given by Monin and Yaglom (1965) as

$$c_{00}(z, t) = \frac{\alpha}{\Gamma(1/\alpha)\,(\alpha^2 K_1 t)^{1/\alpha}} \exp\left[-\frac{z^\alpha}{\alpha^2 K_1 t}\right], \tag{15}$$

where $\alpha = 1 + p$.

By substituting c_{00} and K_z and integrating, Equation (9) reduces to

$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} = K_1 \alpha (\alpha^2 K_1 t)^{1/\alpha - 1} \frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)}. \tag{16}$$

On integrating Equation (16), a solution for $\bar{Z}(t)$ is obtained as,

$$\bar{Z}(t) = \frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)} (\alpha^2 K_1 t)^{1/\alpha}. \tag{17}$$

To express $\mathrm{d}\bar{Z}/\mathrm{d}t$ in Equation (16) in terms of $\bar{Z}(t)$, we eliminate t with the help of Equation (17) to obtain

$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} = K_1\alpha\left[\frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)}\right]^{\alpha}(\bar{Z})^{1-\alpha} \tag{18}$$

which can now be written in terms of the original parameters as,

$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} = \kappa(1+p)\left[\Gamma\left(\frac{2}{1+p}\right)/\Gamma\left(\frac{1}{1+p}\right)\right]^{1+p} u_*\phi_h^{-1}(\bar{\zeta}). \tag{19}$$

This equation is a parallel of Equation (5) which was obtained through dimensional arguments. The universal function $\phi(\zeta)$ in Equation (5) is identified as the reciprocal of a more familiar meteorological parameter $\phi_h(\zeta)$ which incorporates the effect of density stratification and, in addition, draws the distinction between matter and momentum transport. The expression $\kappa(1+p)[\Gamma(2/(1+p))/\Gamma(1/(1+p))]^{1+p}$ which replaces Batchelor's constant b, differs only slightly from κ within the practical range of p (Figure 1). Thus the present practice of regarding b as a universal constant equal to κ (Gifford, 1962; Klug, 1968; Pasquill, 1966), can be continued, that is,

$$\frac{\mathrm{d}\bar{Z}}{\mathrm{d}t} \approx \kappa u_*\phi_h^{-1}(\bar{\zeta}). \tag{20}$$

The extent of approximation involved in assuming an equality between $\mathrm{d}\bar{X}/\mathrm{d}t$ and $\bar{u}(\bar{Z})$ (i.e., $c=1$) was found by Chatwin (1968) to be $1.442u_*$ for neutral conditions of flow (i.e., for the logarithmic velocity variation). It is useful to investigate this approximation in non-neutral conditions. A velocity profile based on a log-linear form of $f(\zeta)$ (i.e., $f(\zeta)=\ln\zeta+\beta\zeta$) when substituted in Equation (10) gives after integration

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = \frac{u_*}{\kappa}\left[\frac{1}{\alpha\Gamma(1/\alpha)}\int_0^{\infty}\ln t_1 e^{-t_1}t_1^{1/\alpha-1}\,\mathrm{d}t_1 + \ln(\alpha^2K_1t)^{1/\alpha} + \right.$$
$$\left. + \beta(\alpha^2K_1t)^{1/\alpha}\frac{\Gamma(2/\alpha)}{\Gamma(1/\alpha)} - (\ln z_0 + \beta\zeta_0)\right], \tag{21}$$

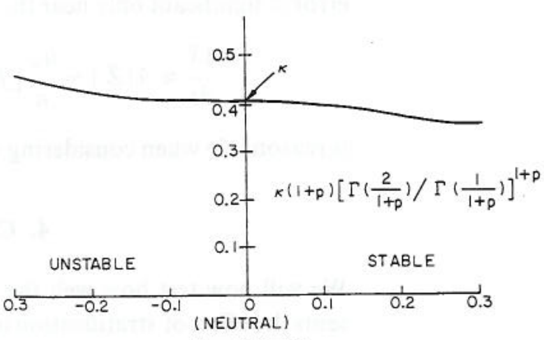

Fig. 1. Variation of $\kappa(1+p)[\Gamma(2/(1+p))/\Gamma(1/(1+p))]^{1+p}$ with p.

where t_1, is a dummy variable. The integral on the right-hand side is the first derivative of the Gamma function and may be written as

$$\int_0^\infty \ln t_1 e^{-t_1} t_1^{1/\alpha - 1} \, \mathrm{d}t_1 = \Gamma(1/\alpha)\, \psi(1/\alpha), \tag{22}$$

where $\psi(1/\alpha)$ is Gauss' ψ-function. On substituting this and the expression for $\bar{Z}$ from Equation (17) in Equation (21), we have

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} = \bar{u}(\bar{Z}) + \frac{u_*}{\kappa} \ln\left\{\frac{\Gamma(1/1+p)}{\Gamma(2/1+p)} \exp\left[\frac{1}{1+p}\, \psi\left(\frac{1}{1+p}\right)\right]\right\}. \tag{23}$$

The second term on the right-hand side represents the error involved in assuming an equality between the average horizontal velocity of a particle and the average wind velocity at its mean position $\bar{Z}$. The latter is always in excess by an amount which lies between u_* and $2u_*$ for extreme values of p (Figure 2). Interestingly, the error has a

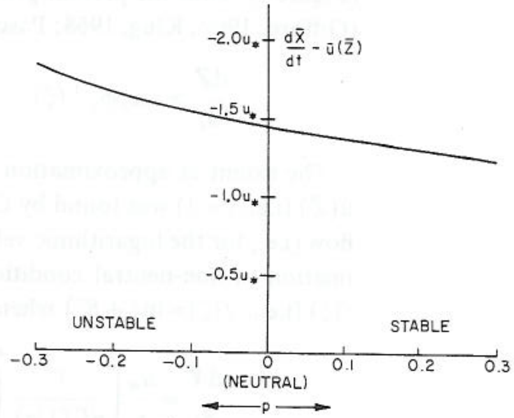

Fig. 2. $\mathrm{d}\bar{X}/\mathrm{d}t - \bar{u}(\bar{Z})$ as a function of p.

tendency to increase with increasing instability. Because of its small magnitude, the error is significant only near the point of release and hence the assumption

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}t} \approx \bar{u}(\bar{Z}) = \frac{u_*}{\kappa}\left[f(\bar{\zeta}) - f(\zeta_0)\right] \tag{24}$$

is reasonable when considering diffusion at large distances.

4. Comparison with Observation

We will now test how well the function $\phi_h(\zeta)$, suggested in the Equation (20), represents the effect of stratification on the average vertical spread of particle $\bar{Z}$ by comparison with the experimental data used earlier by Pasquill (1966). An approximate relation between $\bar{X}$ and $\bar{Z}$ is obtained from Equations (20) and (24) by eliminating t:

$$\bar{X} = \frac{L}{\kappa^2} \int_{\zeta_0}^{\bar{\zeta}} [f(\bar{\zeta}) - f(\zeta_0)]\, \phi_h(\bar{\zeta})\, \mathrm{d}\bar{\zeta}. \tag{25}$$

In Section 2 we used an approximate form for $\phi_h(\zeta)$ in order to investigate analytically the extension of Lagrangian similarity to the case of stratified flow. However, for practical application of the results of this analysis, it is more appropriate to use observational data on $\phi_h(\zeta)$ and $f(\zeta)$. Their determination from field experiments involves measurement of vertical transport of momentum and sensible heat. In most micrometeorological studies these fluxes have been obtained indirectly from velocity and temperature profiles to test these profiles. A complete set of data on $\phi_h(\zeta)$ and $f(\zeta)$ has recently been reported by Businger *et al.* (1971) which is based on measured values of fluxes. According to their data for stable conditions

$$f(\zeta) = \ln \zeta + 4.7\,\zeta$$

$$\phi_h(\zeta) = 0.74 + 4.7\,\zeta$$

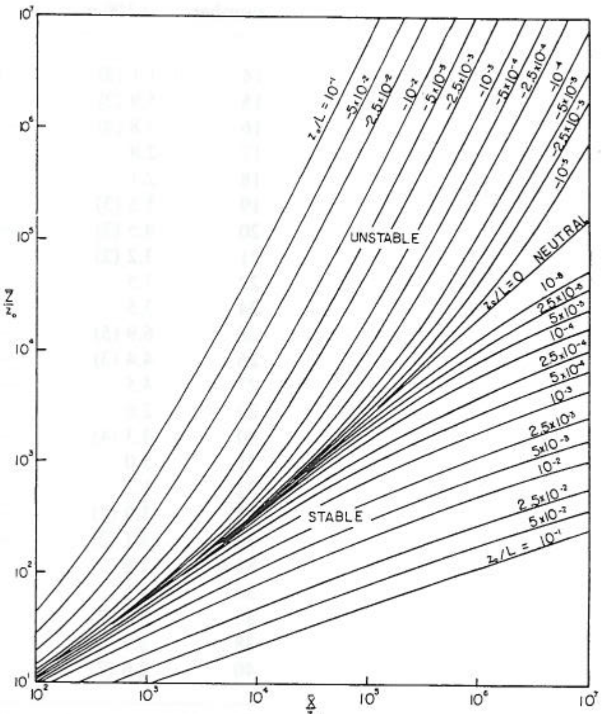

Fig. 3. Variation of dimensionless mean height $\bar{Z}/z_0$ with dimensionless distance from the release point $\bar{X}/z_0$ for various stability conditions.

and for unstable conditions

$$f(\zeta) = -\tan^{-1}(1 - 15\,\zeta)^{-1/4} - 2\tanh^{-1}(1 - 15\,\zeta)^{-1/4}$$
$$\phi_h(\zeta) = 0.74\,(1 - 9\,\zeta)^{-1/2}.$$

A value of 0.74 (and not 1.0) for $\phi_h(0)$ results from their suggestion that the value of κ is 0.35 (and not 0.41). This controversy need not concern us because Equation (25) is insensitive to this distinction. ($\phi_h(0)/\kappa^2$ in the two conventions is the same, i.e., $1/(0.41)^2 \approx 0.742/(0.35)^2$). Equation (25) has been integrated numerically using the above empirical functions and the results are presented in Figure 3 for both stable and unstable conditions of the atmosphere.

Pasquill (1966) demonstrated that $\bar{Z}$ measured at a fixed distance, i.e., $\bar{Z}(x)$, is virtually identical with $\bar{Z}(\bar{X})$ as computed above so that such data can be used to test the predictions which are in the latter form. The O'Neill experimental data on vertical spread at 100 m from the source are presented in Table I in a slightly different manner than that of Pasquill. Following Elliot's analysis of concentration distributions, $\sigma_z[=\int z^2 c(z)\,dz/\int c(z)\,dz]$ has been converted to $\bar{Z}$, and R_i has been expressed

TABLE I

O'Neill data on $\bar{Z}$ for various stability conditions

Run number	$\bar{Z}$ (m) at 100 m	$1/L$	Run number	$\bar{Z}$ (m) at 100 m	$1/L$
14	1.1 (3)	0.130	41	2.6	0.021
15	5.9 (2)	−0.16	42	3.2 (2)	0.011
16	7.8 (4)	−0.355	43	5.4 (4)	−0.088
17	2.8	0.018	44	4.8 (3)	−0.036
18	2.1	0.043	45	3.5 (2)	−0.013
19	5 3 (3)	−0.076	46	2.5	0.013
20	4.5 (3)	−0.023	48	4.5 (2)	−0.021
21	3.2 (2)	0.009	49	3.5 (3)	−0.036
22	3.5	0.006	50	4.5 (2)	−0.03
24	3.5	0.005	51	3.7	−0.029
25	6.9 (5)	−0.18	52	7.7 (2)	−0.115
26	4.4 (3)	−0.083	54	3.0	0.025
27	4.5	−0.027	55	3.4	0.007
29	2.6	0.048	56	3.4	0.014
30	4.3 (4)	−0.028	57	3.8 (3)	−0.006
31	5.0	−0.016	58	1.2	0.004
32	1.2	0.043	59	1.5	0.050
33	3.9 (2)	−0.023	60	3.0	0.019
34	4.1 (2)	−0.018	61	3.7 (3)	−0.026
35-s	2.6	0.027	62	5.3 (2)	−0.026
36	1.5	0.054	65	2.6	0.018
37	3.5 (2)	0.011	66	1.6 (2)	0.031
38	2.9	0.008	67	2.5	0.011
40	2.6 (2)	0.087	68	1.5	0.036

$\bar{Z}$ and $1/L$ values obtained from data on σ_z and R_i in Table III of Pasquill (1966) as explained in text. Numbers in parentheses are numbers of vertical distributions used.

TABLE II

Predicted $\bar{Z}$ at 100 m distance for various stability conditions

$\bar{X} = 100$ m $\quad z_0 = 0.8$ cm
$\bar{X}/z_0 = 1.25 \times 10^4$

	Stable						Neutral	Unstable					
$z_0/L \times 10^3$	2.5	1.5	1.0	0.5	0.25	0.1	0	−0.1	−0.25	−0.5	−1	−1.5	−2.5
$\bar{Z}/z_0$	162	192	229	280	322	362	420	460	520	623	840	970	1570
$\bar{Z}$	1.29	1.54	1.83	2.24	2.58	2.90	3.40	3.68	4.16	4.98	6.72	7.76	12.6
$1/L$	0.312	0.1875	0.125	0.0625	0.0312	0.0125	0	−0.0125	−0.0312	−0.0625	−0.125	−0.1875	−0.312

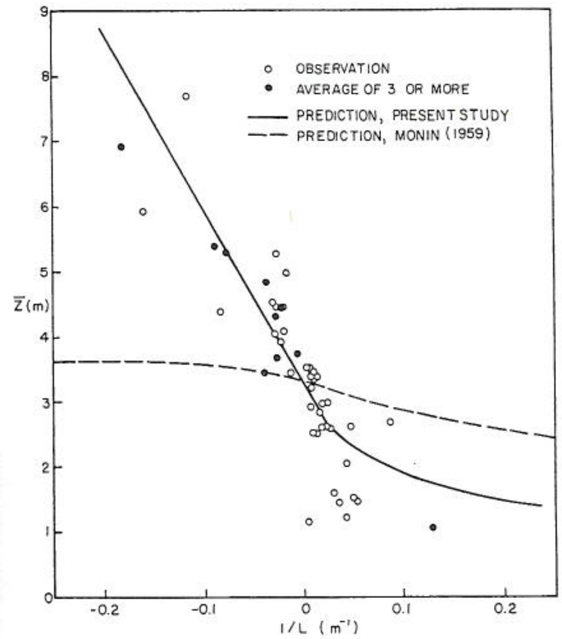

Fig. 4. Comparison of the O'Neill experimental data on mean height $\bar{Z}$ at 100 m from the source with the prediction of this paper for various stability conditions.

in terms of $1/L$ using the relationship suggested by Businger *et al.* (1971). These data have been plotted in Figure 4 along with the predictions from Figure 3 (evaluated in Table II) and from curves of Monin (1959) and Gifford (1962). The effect of stratification on the vertical spread $\bar{Z}$ predicted by the present analysis agrees with the experimental data much more closely than Monin's and Gifford's. An improved comparison on the stable side and a complete comparison on the unstable one, strengthens the conclusion reached analytically that

$$\mathrm{d}\bar{Z}/\mathrm{d}t \approx \kappa u_* \phi_h^{-1}(\bar{\zeta}).$$

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