Improved SNR Evolution for OFDM-IDMA Systems

Jian Dang, Liuqing Yang, and Zaichen Zhang

***Abstract*—The bit error rate performance of interleave division multiple access (IDMA) based systems can be predicted by signal-to-noise ratio (SNR) evolution which tracks the average symbol SNR at each iteration and provides a faster solution than brute-force simulations. As the desired SNR in the evolution procedure is hard to obtain, approximate SNR updating formula has been widely adopted in the literature. In this paper a revised SNR updating formula is proposed for orthogonal frequency division multiplexing interleave division multiple access (OFDM-IDMA) systems in Rayleigh fading channels. Theoretical analysis shows that the new updating formula provides a tighter lower bound of the expected SNR in the evolution procedure compared with the existing one. We then verified this by simulations.**

***Index Terms*—OFDM-IDMA, SNR evolution, lower bound.**

I. Introduction

ORTHOGONAL frequency division multiplexing interleave division multiple access (OFDM-IDMA) is a promising multiple access scheme for uplink wireless communications due to its potential of achieving high spectral efficiency and low decoding complexity [1], [2], [3], [4]. In OFDM-IDMA systems, users are separated by distinct interleavers and the receiver removes the multiple access interference (MAI) for each user using an iterative chip-by-chip detection algorithm. To facilitate the study on the performance of IDMA based systems, the signal-to-noise ratio (SNR) evolution method (see, *e.g.*, [5], [6], [3]) has been proposed to assess the bit error rate (BER) performance much more rapidly than brute-force simulations which are time-consuming. The key idea of SNR evolution is to treat the MAI as noise such that the BER performance in a multiuser scenario is approximated by a single user scenario with a specific SNR which is updated at each iteration. As it is difficult to obtain the exact value of the SNR in the evolution process, an approximate SNR updating formula has been proposed [5], [6], [3]. The existing SNR updating formula for OFDM-IDMA systems provides a lower bound of the expected SNR [6]. In this paper, a revised SNR updating formula is proposed. Theoretical analysis shows that the SNR obtained by the new updating formula is a tighter lower bound of the expected SNR of the evolution compared with the existing approach. Therefore, SNR evolution with our proposed updating formula is expected to have improved BER prediction performance. Such improvement is verified by simulations.

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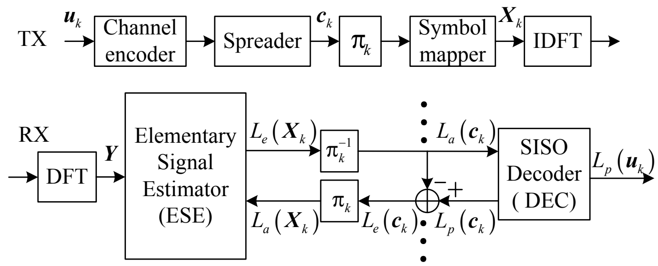

Fig. 1. OFDM-IDMA transmitter for user-k and receiver.

Notations: $\mathrm{E}\{\cdot\}$ ($\mathrm{E}\{\cdot|\cdot\}$), $\mathrm{var}\{\cdot\}$ denote the (conditional) expectation and variance of a random variable. $\Re\{\cdot\}$, $\Im\{\cdot\}$ denote the real and imaginary parts of a complex number. $(\cdot)^{\mathcal{T}}$ denotes the transpose of a vector. $\binom{n}{k}$ is the number of combinations of n things taken k at a time.

II. OFDM-IDMA System Model

In this section a brief introduction to the OFDM-IDMA system model is given. Consider the uplink of an OFDM-IDMA system with K users and N subcarriers. Fig. 1 shows the transmitter for user-k and the receiver. At the transmitter, information bits $\boldsymbol{u}_k$ are first encoded and then spread by a length-S spreader. The bits $\boldsymbol{c}_k$ (± 1) after spreading are referred to as *chips*. The chips are interleaved by a random *user-specific* interleaver π_k and then modulated using quadrature phase shift keying (QPSK), giving rise to the modulated symbols $\boldsymbol{X}_k$ ($\pm 1 \pm j$) which are finally transmitted on the N subcarriers via an inverse discrete Fourier transform (IDFT) module. The receiver mainly consists of two modules, *i.e.*, the elementary signal estimator (ESE) for all users and the soft-input soft-output decoders (SISO DECs) for each and every user [5], [2]. ESE and SISO DECs are connected by interleavers and de-interleavers (de-interleaver for user-k is represented by π_k^{-1} in Fig. 1). Both ESE and SISO DEC modules refine the soft estimates of the chips generated by each other, based on the channel input/output (I/O) relationship and the code structure, respectively. $L_a(\cdot)$, $L_p(\cdot)$ and $L_e(\cdot)$ denote the *a priori*, the *a posteriori* and the *extrinsic* log-likelihood ratios (LLRs), respectively. Estimations of $\boldsymbol{u}_k$ are obtained according to the signs of $L_p(\boldsymbol{u}_k)$ after the final iteration. More detailed description of OFDM-IDMA transceiver principles can be found in [1], [2], [3].

III. SNR Evolution and Problem Formulation

The I/O relationship on subcarrier-n is given by:

$$Y(n) = H_k(n)X_k(n) + T_k(n), \tag{1}$$

where $H_k(n)$ is the channel frequency response of user-k on subcarrier-n, $Z(n)$ is the additive white Gaussian noise (AWGN) with zero mean and variance σ^2, $T_k(n) =$

$\sum_{l=1,\ l\neq k}^{K} H_l(n)X_l(n) + Z(n)$ is the total interference term, compromising of MAI and noise, for user-k on subcarrier-n. At each iteration for user-k, the soft estimate of the interference (mean value of the interference) is reconstructed and subtracted from the received signal. The power of the residual interference for user-k, which is equal to the variance of the random interference term $T_k(n)$, is given by

$$I_k(n) = \sum_{l=1,\ l\neq k}^{K} |H_l(n)|^2 \mathrm{var}\{X_l(n)\} + \sigma^2, \tag{2}$$

where (see [6], [5])

$$\begin{aligned}\mathrm{var}\{X_l(n)\} &= \mathrm{var}\{\Re\{X_l(n)\}\} + \mathrm{var}\{\Im\{X_l(n)\}\} \\ &= [1 - \tanh^2(L_a(\Re\{X_l(n)\})/2)] \\ &\quad +[1 - \tanh^2(L_a(\Im\{X_l(n)\})/2)],\end{aligned} \tag{3}$$

and $L_a(\Re\{X_l(n)\})$,$L_a(\Im\{X_l(n)\})$ are generated by the SISO DECs at the previous iteration. The values of $L_a(\Re\{X_l(n)\})$ and $L_a(\Im\{X_l(n)\})$ are channel and noise dependent. At the first iteration, $L_a(\Re\{X_l(n)\})$ and $L_a(\Im\{X_l(n)\})$ are set to zero. Therefore, the updated SNR for user-k on subcarrier-n after soft interference cancelation is given by [7]

$$\begin{aligned}\gamma_k(n) &= \frac{|X_k(n)H_k(n)|^2}{I_k(n)} \\ &= \frac{2|H_k(n)|^2}{\sum_{l=1,\ l\neq k}^{K} |H_l(n)|^2 \mathrm{var}\{X_l(n)\} + \sigma^2}.\end{aligned} \tag{4}$$

In the SNR evolution method, the BER performance of user-k after this iteration is approximated by that of a single user system with zero-mean AWGN at an equivalent average symbol SNR given by:

$$\gamma_k = \mathrm{E}\{\gamma_k(n)\}, \tag{5}$$

where the expectation is taken over all possible channel and noise realizations. Therefore, the predicted BER is a function of the equivalent symbol SNR γ, *i.e.*, BER $= g_k(\gamma)$ and $g_k(\gamma)$ can be obtained by simulating a single user system with much lower complexity than the multi-user case.

In general, it is difficult to formulate a closed-form expression of (5), as $H_k(n)$ and $\mathrm{var}\{X_k(n)\}$ are random variables for all k and the probability density function (PDF) of $\mathrm{var}\{X_k(n)\}$ may not have a closed-form expression. Nevertheless, one can use Monte-Carlo simulations to numerically evaluate the exact value of (5). In other words, one can calculate $\gamma_k(n)$ for different n (both $|H_l(n)|^2$ and $\mathrm{var}\{X_l(n)\}$ vary with index n) and average them out over sufficient number of channel and noise realizations. For each channel and noise realization for user-l, $\mathrm{var}\{X_l(n)\}$, $\forall n$, are generated by simulating a single user system with zero-mean AWGN at symbol SNR γ_l which is given by the previous iteration. At the first iteration, $\mathrm{var}\{X_l(n)\} = 2$, $\forall l, n$. We refer to the evolution method with simulated γ_k, $\forall k$, as *simulation-based* SNR evolution.

Denote $\gamma_k^{(q)}$ as the symbol SNR of user-k after the q-th iteration and Q the total number of iterations. Given the channel model and $g_k(\gamma)$ of all users, the simulation-based SNR evolution is summarized as follows.

Simulation-based SNR evolution

1) ***Initialization*****:** $q = 1, \gamma_k^{(q)} = \mathrm{E}\{\gamma_k(n)\}, \forall k,$ **where** $\mathrm{var}\{X_k(n)\} = 2, \forall k, n.$
2) ***SNR updating*****: For** $q = 2, 3, \ldots, Q$, $\gamma_k^{(q)} = \mathrm{E}\{\gamma_k(n)\}$, $\forall k$, **where** $\mathrm{var}\{X_k(n)\}$ **are obtained at SNR** $\gamma_k^{(q-1)}$.
3) ***Termination*****: BER of user-**k **after** Q**-th iteration is given by** $g_k(\gamma_k^{(Q)})$.

Note that numerically evaluating (5) is computationally demanding and, as a consequence, contradicts the purpose of introducing SNR evolution as a fast BER assessment method. To make SNR evolution practical, it is more desirable to update the average SNR using some closed-form expression, as will be shown in the next section.

IV. SNR Updating Formulas

A. Existing Updating Formula

The existing updating formula proposed by J. Tong, *et al.*, is as follows [3]. Denote $\eta_k = \mathrm{E}\{|H_k(n)|^2\}$ as the average channel power for user-k and $f_k(\gamma)$ the average value of $\mathrm{var}\{X_k(n)\}$ over index n at symbol SNR γ. $f_k(\gamma)$ is obtained together with $g_k(\gamma)$ by simulating a single user system. At the first iteration, $f_k(\gamma) = 2$. The average SNR is approximated by

$$\gamma_k^{(q)} \approx \frac{2\eta_k}{\sum_{l=1,\ l\neq k}^{K} \eta_l f_l\left(\gamma_l^{(q-1)}\right) + \sigma^2}. \tag{6}$$

Note that in this updating formula, the expectations of both $|H_l(n)|^2$ and $\mathrm{var}\{X_l(n)\}$ are taken before the division. Intuitively, this may lead to large approximation error, which motivates us to propose a revised updating formula as detailed in the following.

B. Proposed Updating Formula

In this subsection a revised updating formula is proposed. Similar to the existing updating formula, the expectations of the variances are taken before the division. However, channel responses are treated in a different manner such that the expectations of $|H_k(n)|^2$, $\forall k$, are taken after the division. Specifically, denote $U_k = |H_k(n)|^2$ and $V_k = \sum_{l=1,\ l\neq k}^{K} |H_l(n)|^2 f_l\left(\gamma_l^{(q-1)}\right)$ as two random variables with PDFs $p_U(u)$ and $p_V(v)$, respectively. Then at the q-th iteration, (5) can be approximated by

$$\gamma_k^{(q)} \approx \int_0^\infty \int_0^\infty \frac{2u}{v + \sigma^2} p_U(u) p_V(v) du dv. \tag{7}$$

It is possible to get a closed-form expression of (7) for commonly used channel models. We consider Rayleigh fading channels where $H_k(n)$ is a complex Gaussian random variable with zero mean and variance η_k.

For the general case in which η_k are different for different k, $f_k\left(\gamma_k^{(q-1)}\right)$ are also different. In this case, U_k is a chi-square random variable with two degrees of freedom and V_k

is a generalized chi-square random variable. The PDF of V_k is given by [8]

$$p_V(v) = \sum_{l=1,l\neq k}^{K} c_{k,l} e^{-\frac{v}{d_l}},\ v \geq 0, \tag{8}$$

where $c_{k,l} = \left[d_l \prod_{j=1,j\neq l,k}^{K} (1 - d_j/d_l)\right]^{-1}$, $d_l = \eta_l f_l\left(\gamma_l^{(q-1)}\right)$. After some mathematical manipulations, (7) can be written as

$$\gamma_k^{(q)} \approx 2\eta_k \sum_{l=1,l\neq k}^{K} c_{k,l} e^{\frac{\sigma^2}{d_l}} E_1\left(\frac{\sigma^2}{d_l}\right), \tag{9}$$

where $E_1(x) = \int_x^\infty \frac{1}{t} e^{-t} dt$ is the exponential integral [9].

For the special case where $\eta_k = \eta$, $f_k(\cdot) = f(\cdot)$, $\forall k$, V_k is a chi-square random variable with $2(K-1)$ degrees of freedom. The PDF of V_k is given by [7]

$$p_V(v) = \frac{1}{d^{K-1}\Gamma(K-1)} v^{K-2} e^{-v/d}, \tag{10}$$

where $d = \eta f(\gamma_k^{(q-1)})$ and $\Gamma(x)$ is the Gamma function. Then (7) can be calculated as

$$\gamma_k^{(q)} \approx b \sum_{m=0}^{K-2} \binom{K-2}{m} (-\sigma^2)^{K-2-m} d^m \Gamma\left(m, \frac{\sigma^2}{d}\right), \tag{11}$$

where $b = \frac{2\eta e^{\sigma^2/d}}{d^{K-1}\Gamma(K-1)}$ and $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$ is the incomplete Gamma function [9].

By plugging (6) and (9) (or (11)) into step 1 and 2 of the simulation-based SNR evolution process, the predicted BER performance can be obtained accordingly. More computation is needed for the proposed formula compared to the existing one as integrals are involved. However, its complexity is still much lower than simulation-based updating. Even only a single realization (much less than sufficient) of channels in simulation-based evolution for $K = 8$ requires more computation time than the proposed formula in MATLAB. In the following section we will prove theoretically that the proposed updating formula has higher approximation accuracy compared with the existing one.

V. SNR Comparison for Different Formulas

Denote $W_k = \mathrm{var}\{X_k(n)\}$. It is apparent that $U_k, k = 1, 2, \ldots, K$, are mutually independent and so are W_ks. On the other hand, W_k is calculated based on the channel frequency response of $2S$ subcarriers determined by the specific interleaver π_k. Therefore whether W_k is independent of U_k is determined by the dependence among those $2S$ subcarriers and subcarrier-n for a given channel and interleaver. Due to random interleaving, the correlation coefficients between $H_k(n)$ and the channel response of the aforementioned $2S$ subcarriers are likely small. For simplicity, we assume W_k is independent of U_k as a result of random interleaving.

The expectation of $\gamma_k(n)$ in (5) is given by

$$\gamma_k = 2\mathrm{E}\{U_k\}\mathrm{E}\left\{\frac{1}{\sum_{l\neq k} U_l W_l + \sigma^2}\right\}. \tag{12}$$

For the existing updating formula, the SNR is essentially

$$\gamma_{k,1} = \frac{2\mathrm{E}\{U_k\}}{\sum_{l\neq k} \mathrm{E}\{U_l\}\mathrm{E}\{W_l\} + \sigma^2}, \tag{13}$$

and for the proposed updating formula, the SNR is

$$\gamma_{k,2} = 2\mathrm{E}\{U_k\}\mathrm{E}\left\{\frac{1}{\sum_{l\neq k} U_l \mathrm{E}\{W_l\} + \sigma^2}\right\}. \tag{14}$$

Now we compare the relationship between γ_k, $\gamma_{k,1}$ and $\gamma_{k,2}$. Denote random vectors $\boldsymbol{U}_k = [U_1, \ldots, U_{k-1}, U_{k+1}, \ldots, U_K]^{\mathcal{T}}$ and $\boldsymbol{W}_k = [W_1, \ldots, W_{k-1}, W_{k+1}, \ldots, W_K]^{\mathcal{T}}$ and their function $\varphi(\boldsymbol{U}_k, \boldsymbol{W}_k) = 1/(\boldsymbol{U}_k^{\mathcal{T}}\boldsymbol{W}_k + \sigma^2)$. It can be readily verified that $\varphi(\cdot)$ is a convex function of both $\boldsymbol{U}_k$ and $\boldsymbol{W}_k$. Using Jensen's inequality [10], for a given $\boldsymbol{U}_k$, we have

$$\varphi(\boldsymbol{U}_k, \mathrm{E}\{\boldsymbol{W}_k\}) \leq \mathrm{E}\{\varphi(\boldsymbol{U}_k, \boldsymbol{W}_k)|\boldsymbol{U}_k\}. \tag{15}$$

Taking the expectation with respect to $\boldsymbol{U}_k$ on both sides of (15) yields

$$\begin{aligned}\gamma_{k,2}/(2\mathrm{E}\{U_k\}) &= \mathrm{E}\{\varphi(\boldsymbol{U}_k, \mathrm{E}\{\boldsymbol{W}_k\})\} \\ &\leq \mathrm{E}\{\mathrm{E}\{\varphi(\boldsymbol{U}_k, \boldsymbol{W}_k)|\boldsymbol{U}_k\}\} = \gamma_k/(2\mathrm{E}\{U_k\}).\end{aligned} \tag{16}$$

Applying Jensen's inequality to the convex function of $\varphi(\boldsymbol{U}_k, \mathrm{E}\{\boldsymbol{W}_k\})$ with respect to $\boldsymbol{U}_k$, we have

$$\begin{aligned}\gamma_{k,1}/(2\mathrm{E}\{U_k\}) &= \varphi(\mathrm{E}\{\boldsymbol{U}_k\}, \mathrm{E}\{\boldsymbol{W}_k\}) \\ &\leq \mathrm{E}\{\varphi(\boldsymbol{U}_k, \mathrm{E}\{\boldsymbol{W}_k\})\} = \gamma_{k,2}/(2\mathrm{E}\{U_k\}).\end{aligned} \tag{17}$$

Combining (16) and (17) leads to the relationship

$$\gamma_{k,1} \leq \gamma_{k,2} \leq \gamma_k. \tag{18}$$

Therefore, compared with the existing updating formula, the proposed one provides a tighter lower bound of the SNR in simulation-based evolution. Hence better prediction performance can be expected.

VI. Simulations

In this section we present simulations to show the superiority of our proposed updating formula over the existing one. Specifically, we compare the coded BER performance obtained by four means: SNR evolution using existing updating formula (BER-1); SNR evolution using proposed updating formula (BER-2); SNR evolution using exact SNR, *i.e.*, simulation-based SNR evolution (BER-s); brute-force simulation of the system (BER-t, the true BER). System parameters are set as follows: channel code is a rate-1/2 convolutional code with generator $(23, 35)_8$, spreading length $S = 4$, number of subcarriers $N = 512$, the channels of all users are assumed to be Rayleigh fading channels and have the same correlation matrix as $\boldsymbol{I}_L/L$ with channel length $L = 8$. In this case, $f_k(\cdot) = f(\cdot)$ and $g_k(\cdot) = g(\cdot)$ for all k.

Fig. 2 depicts the average BER performance with $K = 6$ users after $Q = 2, 3$ iterations. Fig. 3 illustrates the BER performance after $Q = 5$ iterations. Fig. 4 and 5 show the BER performance with $K = 8$ users after $Q = 5$ and 7 iterations, respectively. Clearly, as a consequence of the relationship in (18), our proposed SNR evolution method (BER-2) does

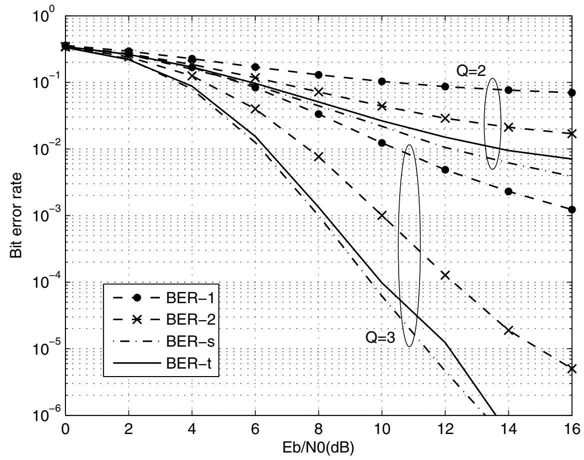

Fig. 2. BER performance comparison with 2 and 3 iterations. $K = 6$.

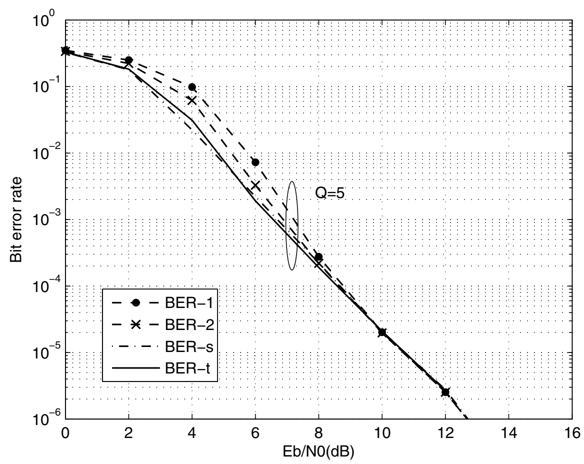

Fig. 3. BER performance comparison with 5 iterations. $K = 6$.

provide a tighter upper bound of BER-s compared with BER-1. It is of practical interest to compare BER-2 and BER-1 with respect to BER-t. Although such comparisons are not analytically tractable, simulations strongly suggest that BER-t can be approximated more accurately by BER-2 as compared with BER-1. Those figures also show that the predicted BER converges to the true BER after sufficient number of iterations and this number increases with K. Even in these converged cases, one can still observe that BER-2 has better prediction performance than BER-1 at all SNR values. Therefore, compared with the existing updating formula, our proposed one provides better prediction performance with respect to both BER-s and BER-t.

VII. Conclusions

In this paper a new SNR updating formula for SNR evolution in OFDM-IDMA systems has been proposed. Theoretical analysis showed that the SNR obtained by our new updating formula is a tighter lower bound of the SNR in simulation-based evolution compared with the existing formula in the literature. This is also verified by simulations. In addition, simulations showed that the true BER can also be predicted more accurately using our proposed formula.

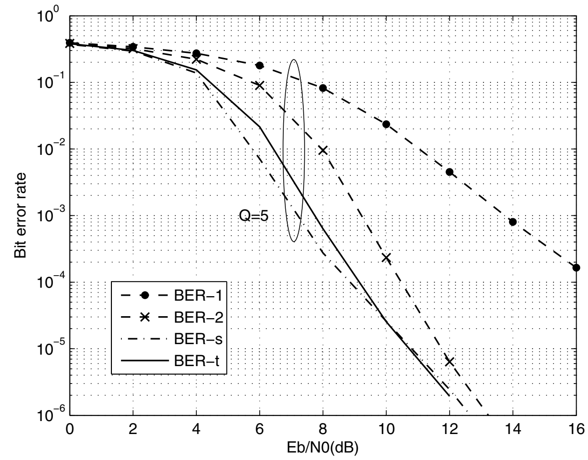

Fig. 4. BER performance comparison with 5 iterations. $K = 8$.

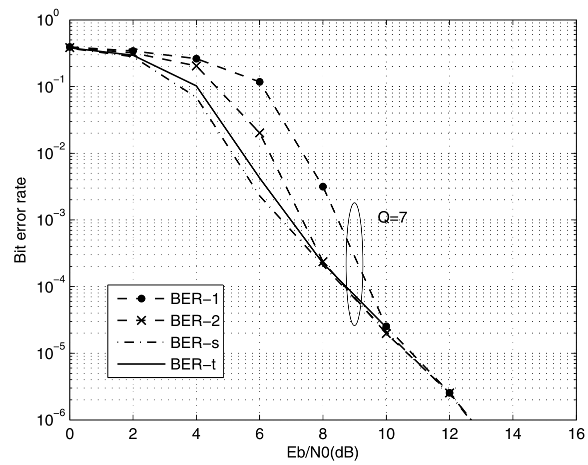

Fig. 5. BER performance comparison with 7 iterations. $K = 8$.

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