Optimum Resource Allocation for Amplify-and-Forward Relay Networks With Differential Modulation

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***Abstract*—The optimum resource allocation is an important method to improve the error performance and energy efficiency of wireless relay networks. In this paper, we consider the resource allocation as a two-dimensional optimization problem; that is, the energy optimization and the location optimization. Differential modulation which bypasses the channel estimation at the receiver is investigated using the amplify-and-forward protocol for a relay system with arbitrary number of relays. At high signal-to-noise ratio (SNR), we first derive the average symbol error rate for systems with and without a direct link. Then, the optimum resource allocation schemes which minimize the system error are developed. The comparisons based on analytical and simulated results confirm that the optimized systems provide considerable improvement over unoptimized ones, and that the minimum error can be achieved via the joint energy-location optimization.**

***Index Terms*—Amplify-and-forward, differential modulation, relay networks, resource optimization, symbol error rate.**

I. Introduction

RELAY networks allow a source node to communicate with a destination node via a number of relay nodes. By forming virtual antenna arrays in a cooperative manner, the spatial diversity gain can be achieved without imposing antenna packing limitations [14], [15], [18]. Most previous work on relay networks focused on the coherent modulation assuming the availability of the channel state information (CSI) [4], [12], [14], [18]. To reduce hardware complexity and communication overhead, modulation schemes bypassing CSI have been recently adopted in relay systems. These systems employ frequency-shift keying (FSK), differential phase-shift keying (DPSK) [6], [8], [13], [19], [22] or space-time coding (STC) techniques [7], [20].

To improve the performance of relay systems, the optimum resource allocation has been studied for both the coherent and noncoherent systems [2], [4], [9], [11], [13]. These works deal with the power allocation based on various protocols [amplify-and-forward (AF), decode-and-forward (DF), and STC-based] and different levels of CSI (full or partial CSI). In [16] and [21], it has been noticed that the relay location is a critical factor influencing the system performance. However, the relay location problem is treated as a special case of the energy allocation problem [16], and the effect of general location optimization has not been thoroughly investigated. By formulating the relay location optimization as a parallel problem to the energy optimization, the joint energy and location optimization for the relay network with DF protocol was introduced in [8].

In this paper, we will address the joint resource allocation issue in a relay network employing the AF protocol and differential (de-)modulation. In addition to the "no direct link" scenario in [8], where the direct transmission is blocked by obstacles, we also consider the "with a direct link" scenario. Also different from [8] where an error bound is used for the optimization, here we derive an approximation of the system error performance. This approximation is shown to be tight for arbitrary number of relays. Based on this closed-form quantitative error expression, the resource allocation problem is investigated in a two-dimensional optimization manner: energy optimization and location optimization. The joint energy and location optimization is also studied. We show that the minimum system error can be achieved by jointly optimizing the energy distribution and the relay location under the constraints of a constant total transmit energy and the fixed source-destination and source-relay-destination distances.

The rest of this paper is organized as follows. The system model is introduced in Section II. In Section III, the approximated system error performance is established for arbitrary number of relays. Energy optimization, location optimization and joint optimization are analyzed in Section IV. Simulations and discussions are presented in Section V, and summarizing remarks of our work are given in Section VI.

Notation: We use $(\cdot)^*$ for conjugate, $\Re\{\cdot\}$ for the real part, and $\mathcal{CN}(\mu,\sigma^2)$ for the complex Gaussian distribution with mean μ and variance σ^2. The abbreviations "DL" and "ND" stand for "with a direct link" and "no direct link" between the source and the destination, respectively.

II. System Model

Consider a network setup with one source node s, L relay nodes $\{r_k\}_{k=1}^{L}$ and one destination node d. We will consider the time-division multiplexing (TDM) for presentation convenience. However, the presented analysis and results are readily applicable to other multiplexing schemes.

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A. Relaying Protocol and Channel Modeling

The AF relaying protocol is considered here, in which the relays amplify the signal from the source and then forward it to the destination. With TDM, the relay transmission consists of two phases. In phase I, the source broadcasts a symbol to all relays. In phase II, each relay transmits the amplified signal to the destination during their distinct time slots. Generally speaking, the L source-to-relay $(s-r)$ links share a common channel, while the L relay-to-destination $(r-d)$ links are mutually orthogonal.

In order to bypass channel estimation and to cope with time variation of the channel, differential modulation is employed at the source node. Specifically, with the nth phase-shift keying (PSK) symbol being $s_n = e^{j2\pi c_n/M}$, $c_n \in \{0,1,\ldots,M-1\}$, the transmitted signal from the source is

$$x_n^s = \begin{cases} x_{n-1}^s s_n, & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{1}$$

In phase I, the encoded signal is broadcast via a common channel. The received signal at the kth relay and the destination are given by

$$\begin{aligned} y_n^{r_k,s} &= \sqrt{\mathcal{E}_s} h_n^{r_k,s} x_n^s + z_n^{r_k,s}, \quad k = 1,2,\ldots,L, \\ y_n^{d,s} &= \sqrt{\mathcal{E}_s} h_n^{d,s} x_n^s + z_n^{d,s} \end{aligned} \tag{2}$$

where $\mathcal{E}_s$ is the energy per symbol at the source. We denote the fading coefficients of channels $s-r_k$ and $s-d$ during the nth symbol duration as $h_n^{r_k,s} \sim \mathcal{CN}(0,\sigma_{r_k,s}^2)$ and $h_n^{d,s} \sim \mathcal{CN}(0,\sigma_{d,s}^2)$, and the corresponding noise components as $z_n^{r_k,s} \sim \mathcal{CN}(0,\mathcal{N}_{r_k,s})$ and $z_n^{d,s} \sim \mathcal{CN}(0,\mathcal{N}_{d,s})$, respectively.

In phase II, the received signal at each of the relays is amplified to generate

$$x_n^{r_k} = A_{r_k} y_n^{r_k,s}, \quad k = 1,2,\ldots,L \tag{3}$$

where A_{r_k} is the amplification factor. To maintain a constant average power at the relay output, the amplification factor is

$$A_{r_k} = \sqrt{\frac{1}{\sigma_{r_k,s}^2 \mathcal{E}_s + \mathcal{N}_0}}, \quad k = 1,2,\ldots,L. \tag{4}$$

This factor is reasonable for both differential and noncoherent modulations, since $\sigma_{r_k,s}^2$ can be estimated by averaging the received signals without knowing the instantaneous CSI [5], [22].

The amplified signals are then retransmitted to the destination, and the received signal corresponding to each relay is given by

$$y_n^{d,r_k} = \sqrt{\mathcal{E}_{r_k}} h_n^{d,r_k} x_n^{r_k} + z_n^{d,r_k}, \quad k = 1,2,\ldots,L \tag{5}$$

where $\mathcal{E}_{r_k}$ is the energy per symbol. The fading coefficients of the $r_k - d$ channels and the noise components at the destination are $h_n^{d,r_k} \sim \mathcal{CN}(0,\sigma_{d,r_k}^2)$ and $z_n^{d,r_k} \sim \mathcal{CN}(0,\mathcal{N}_{d,r_k})$.

Throughout this paper, all fading coefficients are assumed to be independent. Without loss of generality, we also assume that all noise components are independent and identically distributed (i.i.d.) with $\mathcal{N}_{i,j} = \mathcal{N}_0$, $i,j \in \{s,r_k,d\}$. Accordingly, we can find the received instantaneous signal-to-noise ratio (SNR) between the transmitter j and the receiver i as

$$\gamma_{i,j} = \frac{|h_n^{i,j}|^2 \mathcal{E}_j}{\mathcal{N}_0}, \quad i,j \in \{s,r_k,d\}.$$

It then follows that the average received SNR is $\bar{\gamma}_{i,j} = \left(\sigma_{i,j}^2 \mathcal{E}_j\right)/\mathcal{N}_0$.

B. Differential Demodulation

Using the amplification factor in (4), we can rewrite the received signal at the destination corresponding to each relay node as

$$\begin{aligned} y_n^{d,r_k} &= \tilde{h}_n^{d,r_k} x_n^s + \tilde{z}_n^{d,r_k} = y_{n-1}^{d,r_k} s_n + \left(\tilde{z}_n^{d,r_k}\right)' \\ k &= 1,2,\ldots,L \end{aligned} \tag{6}$$

where $\tilde{h}_n^{d,r_k} = \sqrt{\mathcal{E}_s}\sqrt{\mathcal{E}_{r_k}} A_{r_k} h_n^{r_k,s} h_n^{d,r_k}$, $\tilde{z}_n^{d,r_k} = \sqrt{\mathcal{E}_{r_k}} A_{r_k} h_n^{d,r_k} z_n^{r_k,s} + z_n^{d,r_k}$, and $\left(\tilde{z}_n^{d,r_k}\right)' = \tilde{z}_n^{d,r_k} - \tilde{z}_{n-1}^{d,r_k} s_n$. Then, conditioned on the channels, the received signal is $y_n^{d,r_k} \sim \mathcal{CN}\left(y_{n-1}^{d,r_k} s_n, \sigma_{h_{k,eff}}^2\right)$, where the variance of the aggregate noise is given by

$$\sigma_{h_{k,eff}}^2 = 2\mathcal{N}_0 \left(\mathcal{E}_{r_k} A_{r_k}^2 \sigma_{d,r_k}^2 + 1\right), \quad k = 1,2,\ldots,L. \tag{7}$$

The received signal at the destination corresponding to the source can be likewise rewritten as $y_n^{d,s} = y_{n-1}^{d,s} s_n + (\tilde{z}_n^{d,s})'$ with $y_n^{d,s} \sim \mathcal{CN}(y_{n-1}^{d,s} s_n, 2\mathcal{N}_0)$. As a result, we obtain $(L+1)$ different log likelihood functions (LLF) corresponding to the L transmitted signals from the relays and 1 transmitted signal from the source, given that x_n^m is transmitted by the source

$$\begin{aligned} l_m^{d,r_k}(y_n) &:= \ln p_{y_n^{d,r_k}|x_n^s}\left(y_n^{d,r_k}|x_n^m\right) \\ &= \Re\left\{\left(y_n^{d,r_k}\right)^* y_{n-1}^{d,r_k} s_n^m\right\} \\ k &= 1,2,\ldots,L \\ l_m^{d,s}(y_n) &:= \ln p_{y_n^{d,s}|x_n^s}\left(y_n^{d,s}|x_n^m\right) \\ &= \Re\left\{\left(y_n^{d,s}\right)^* y_{n-1}^{d,s} s_n^m\right\} \end{aligned} \tag{8}$$

where $s_n^m = e^{j2\pi m/M}$ and $m \in \{0,1,\ldots,M-1\}$. At the destination, these $(L+1)$ signals can be combined to estimate the transmitted signal from the source. Using the multichannel communication results in [17, Ch. 12] and the above LLFs, the decision rule at the destination node can be obtained as (see [13] and [22])

$$\begin{aligned} \hat{s}_n = e^{j2\pi\hat{m}/M} : \hat{m} = \arg &\max_{m \in \{0,1,\ldots,M-1\}} \\ &\times \left[w_{d,s} l_m^{d,s}(y_n) + \sum_{k=1}^{L} w_{d,r_k} l_m^{d,r_k}(y_n)\right] \end{aligned} \tag{9}$$

where $w_{d,s}$ and w_{d,r_k} are combining weights given by $1/\mathcal{N}_0$ and $2/\sigma_{h_{k,eff}}^2$, respectively, and it is assumed that the variances of channels are available at the destination node.

III. Performance Analysis

In this section, we will derive the analytical expression of the error performance for the system described in the preceding section. Under high SNR approximation, the symbol error rate (SER) for an $L-$ relay system using differential binary phase shift keying (DBPSK) (de-)modulation is considered. We will provide a simple and general expression for the average SER of the systems with and without a direct link. At the destination, using(6), we can evaluate the equivalent SNR for the link from the source through the kth relay node as

$$\gamma_{eq,r_k} = \frac{\gamma_{r_k,s}\gamma_{d,r_k}}{\bar{\gamma}_{r_k,s} + \gamma_{d,r_k} + 1}. \tag{10}$$

Then, the relay system can be modeled as an equivalent multi-channel system, and the SER can be therefore derived. We will first investigate the SER of the system with no direct link in detail, and then the result will be extended to the case where a direct link is also present.

Proposition 1: At high SNR, the average SER of an L-relay AF system using DBPSK signaling can be approximated as:

$$\bar{P}_{e,ND} \approx C(L)\prod_{k=1}^{L}\left[\frac{1}{\bar{\gamma}_{r_k,s}} + \frac{1}{\bar{\gamma}_{d,r_k}}\ln(\bar{\gamma}_{d,r_k})\right], \tag{11}$$

$$\bar{P}_{e,DL} \approx C(L+1)\frac{1}{\bar{\gamma}_{d,s}}\prod_{k=1}^{L}\left[\frac{1}{\bar{\gamma}_{r_k,s}} + \frac{1}{\bar{\gamma}_{d,r_k}}\ln(\bar{\gamma}_{d,r_k})\right], \tag{12}$$

where $\bar{P}_{e,ND}$ and $\bar{P}_{e,DL}$ represent the average SER for the system with no direct link and with a direct link, respectively, and

$$C(L) = \frac{1}{2^{2L-1}}\sum_{n=0}^{L-1}\binom{n+L-1}{L-1}\sum_{k=0}^{L-1-n}\binom{2L-1}{k}. \tag{13}$$

Proof: See Appendix I. ∎

The difference between (11) and (12) reveals that if the average SNR between the source and destination satisfies $\bar{\gamma}_{d,s} > C(L+1)/C(L)$, then the existence of the direct link contributes to the system performance. For $L=$1, 2, and 3, $C(L+1)/C(L)$ turns out to be $3/2$, $5/3$, and $7/4$, respectively. Therefore, as long as the SNR between the source and destination is reasonable, the system with a direct link outperforms the system without a direct link. Though a direct link would almost always benefit the communication quality, it is not always available. In addition, relays are more needed when the direct link is blocked.

Notice that, when $\bar{\gamma}_{r_k,s} = \bar{\gamma}_{d,r_k} = \bar{\gamma}_{d,s} = \tilde{\gamma}$, $\forall\, k$, and as $\tilde{\gamma} \to \infty$, using L'Hospital's rule, the SERs in (11) and (12) simplify to

$$\bar{P}_{e,ND} \approx \left(C(L)^{-1/L}\cdot\tilde{\gamma}\right)^{-L} \text{ and}$$
$$\bar{P}_{e,DL} \approx \left(C(L+1)^{-1/(L+1)}\cdot\tilde{\gamma}\right)^{-(L+1)}. \tag{14}$$

This indicates that the relay system without a direct link can achieve the coding gain $C(L)^{-1/L}$ and the diversity gain L, while the system with a direct link can achieve $C(L+1)^{-1/(L+1)}$ and $(L+1)$. The analysis shows that full diversity gain can be obtained in the AF relaying system using differential (de-)modulation. Using (13), one can evaluate the coding gain for any L. For example, $C(1)^{-1}=2$, $C(2)^{-1/2} = (4/3)^{1/2}$, $C(3)^{-1/3} = (4/5)^{1/3}$, and $C(4)^{-1/4} = (16/35)^{1/4}$.

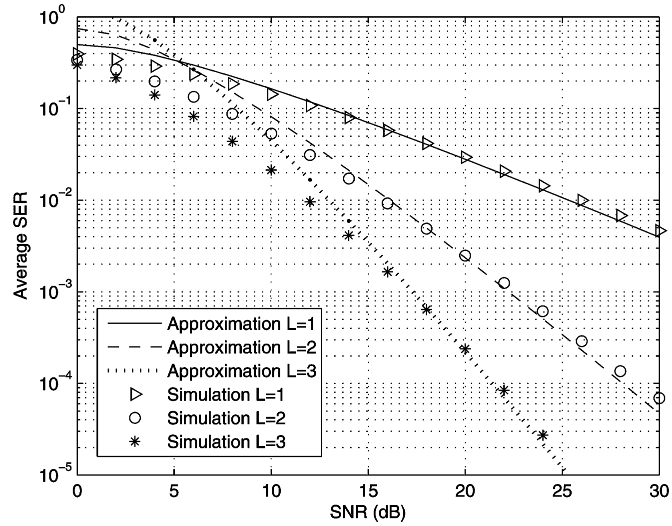

Fig. 1. SER comparison between approximation and simulation (ND, $\mathrm{SNR} = \bar{\gamma}_{r_k,s} = \bar{\gamma}_{d,r_k}$).

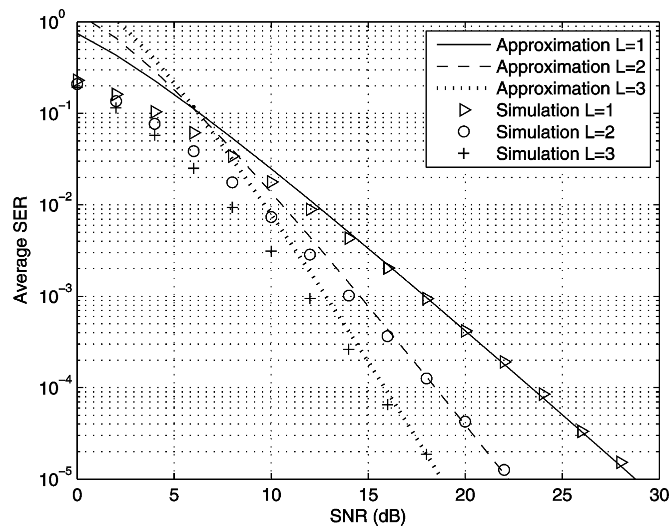

Fig. 2. SER comparison between approximation and simulation (DL, $SNR = \bar{\gamma}_{d,s} = \bar{\gamma}_{r_k,s} = \bar{\gamma}_{d,r_k}$).

We observe that the value decreases as L increases. Therefore, the system with more relays will lose some coding gain. It is also worth noting that the SER expressions in (11) and (12) coincide with the average SER of the coherent system in [18] except for the log term, which leads to a loss in terms of the coding gain in comparison with the coherent system.

In Figs. 1 and 2, we plot the approximated and simulated SERs for the systems with and without a direct link, respectively, when $L = 1$, 2, and 3. The figures show that the approximated SERs are tight to the simulations, especially at high SNR. Furthermore, these figures confirm that the diversity gain increases in direct proportion to the number of relays. Compared with Figs. 1, 2 shows that the direct transmission contributes to the diversity gain, where the DL (direct-link) system with L relays achieves the same diversity gain as the ND (no-direct-link) system with $(L+1)$ relays. However, the coding gain is slightly different because of the effect of the direct link. For example, to achieve the same SER of 10^{-4}, approximately 4.7 dB higher

SNR is required for the ND system with $L=2$, compared with the DL system with $L=1$; while 2.9 dB higher SNR is required for the ND system with $L=3$, compared with the DL system with $L=2$.

IV. Optimum Resource Allocation

In this section, we will investigate the effects of resource allocation on the SER performance. We will show that an optimum allocation of the limited resource is possible, and it achieves the minimum system error. The resource allocation which minimizes the average SER will be addressed from three perspectives: 1) optimum energy allocation; 2) relay location optimization; and 3) joint energy and location optimization.

To perform the optimization in the ensuing subsections, we will make use of the relationship between the average power of channel fading coefficients $\sigma_{i,j}^2$ and the internode distance $D_{j,i}$ as follows:

$$\sigma_{i,j}^2 = G \cdot D_{j,i}^{-\nu}, \quad i,j \in \{s, r_k, d\} \tag{15}$$

where ν is the path loss exponent of the wireless channel and G is a constant which we henceforth set to 1 without loss of generality.[1] We also define the total SNR, $\rho := \mathcal{E}/\mathcal{N}_0$, the transmit SNR at the source node $\rho_s := \mathcal{E}_s/\mathcal{N}_0$ and at the relay nodes $\rho_{r_k} := \mathcal{E}_{r_k}/\mathcal{N}_0$ for notational convenience.

Then, the average received SNR at the relay and destination can be expressed in terms of the transmit SNRs as

$$\begin{aligned} \bar{\gamma}_{d,s} &= \rho_s \sigma_{d,s}^2 = \rho_s D_{s,d}^{-\nu}, \\ \bar{\gamma}_{r_k,s} &= \rho_s \sigma_{r_k,s}^2 = \rho_s D_{s,r_k}^{-\nu}, \\ \text{and } \bar{\gamma}_{d,r_k} &= \rho_{r_k} \sigma_{d,r_k}^2 = \rho_{r_k} D_{r_k,d}^{-\nu}. \end{aligned} \tag{16}$$

With (16), we can rewrite (11) and (12) as

$$\bar{P}_{e,ND} \approx C(L) \prod_{k=1}^{L} \left[\frac{1}{\rho_s D_{s,r_k}^{-\nu}} + \frac{1}{\rho_{r_k} D_{r_k,d}^{-\nu}} \ln\left(\rho_{r_k} D_{r_k,d}^{-\nu}\right) \right] \tag{17}$$

$$\bar{P}_{e,DL} \approx C(L+1) \frac{1}{\rho_s D_{s,d}^{-\nu}} \times \prod_{k=1}^{L} \left[\frac{1}{\rho_s D_{s,r_k}^{-\nu}} + \frac{1}{\rho_{r_k} D_{r_k,d}^{-\nu}} \ln\left(\rho_{r_k} D_{r_k,d}^{-\nu}\right) \right]. \tag{18}$$

It is clear that $\bar{P}_{e,ND}$ and $\bar{P}_{e,DL}$ are functions of ρ_s, ρ_r, D_{s,r_k} and $D_{r_k,d}$, thus the optimum energy allocation and relay location can be determined to achieve the best system error performance. Generally, the optimum solution can be derived from Lagrange multiplier. However, as L increases, the number of variables increases linearly, and the mathematical complexity renders the optimization problem analytically untractable. In this paper, we will focus on idealized systems, where all the relays have the same distance to the source and destination; i.e., $D_{s,r_k} = D_{s,r}$ and $D_{r_k,d} = D_{r,d}$. The idealized systems have an important property as stated below, which guarantees the operability of our two-dimensional optimization technique.

[1]One could incorporate the shadowing effect by adding a factor $10^{-X/10}$ to G, where X is a zero-mean Gaussian random variable. Thus our SER expressions can also be modified to incorporate the shadowing effect.

Proposition 2: In idealized relay systems with path loss exponent ν, number of relays L and source-destination distance $D_{s,d}$, the system SERs $\bar{P}_{e,ND}$ and $\bar{P}_{e,DL}$ are convex functions of ρ_s, $D_{s,r}$, ρ_r, and $D_{r,d}$ at reasonably high SNR.

Proof: See Appendix II. ■

This proposition reveals that there is a unique global optimum system setup that achieves the minimum SER. Thus, we can safely perform numerical search to find our system optimum.

In this paper, we consider resource optimization on two network topologies. One is the ellipse case and the other is the line case. For the ellipse case, $D_{s,r} + D_{r,d} = D \geq D_{s,d}$. The line case can be regarded as a special case of the ellipse case, where $D = D_{s,d}$. In the systems with the general ellipse topology, we can solve the optimum resource allocation problem at any point on a two-dimensional plane, by changing the value of D. Therefore, the energy optimization results for the idealized topologies can provide useful insights for understanding the effect of energy allocation in cooperative networks. Throughout the paper, the source-destination distance is normalized to $D_{s,d} = 1$.

A. Energy Optimization

Problem Statement 1: In a relay system with path loss exponent ν and source-destination distance $D_{s,d}$, for any given relay locations ($D_{s,r}$ and $D_{r,d}$), and the total energy per symbol ρ, we will determine the optimum energy allocation ρ_s and ρ_r, which

$$\begin{aligned} \text{minimizes} \quad & \bar{P}_{e,ND} \text{ or } \bar{P}_{e,DL} \\ \text{subject to} \quad & \rho_s + L\rho_r = \rho. \end{aligned} \tag{19}$$

By applying Lagrange multiplier, we found the optimum energy allocation for different relay locations in the following proposition. The proof is outlined in Appendix III.

Proposition 3: For an idealized L-relay system, at given $s-r$ and $r-d$ distances $D_{s,r}$ and $D_{r,d}$, and under the total energy constraint in(19), the optimum energy allocation ρ_s^o and ρ_r^o should satisfy

$$\sigma_{r,s}^2 \left[\ln\left(\rho_r^o \sigma_{d,r}^2\right) - 1\right] \rho_s^{o^2} + \rho_r^o \sigma_{d,r}^2 \rho_s^o - \rho \rho_r^o \sigma_{d,r}^2 = 0 \tag{20}$$

for the system with no direct link, and

$$\sigma_{r,s}^2 \left[\ln\left(\rho_r^o \sigma_{d,r}^2\right) - \frac{L}{L+1}\right] \rho_s^{o^2} + \left[\rho_r^o \sigma_{d,r}^2 - \frac{\rho \sigma_{r,s}^2}{L+1} \ln\left(\rho_r^o \sigma_{d,r}^2\right)\right] \rho_s^o - \rho \rho_r^o \sigma_{d,r}^2 = 0 \tag{21}$$

for the system with a direct link.

Notice that the log terms in (20) and(21) render a closed-form solution incalculable. However, thanks to the convexity property in Proposition 2, there exists a unique optimum energy allocation which minimizes the average SER, and the optimum solution of ρ_s and ρ_r can be found by numerical search. In addition, the $\rho\sigma_{r,s}^2 \ln(\rho_r^o \sigma_{d,r}^2)/(L+1)$ term in (21) leads to a distinct energy allocation for the system with a direct link, in comparison with the one with no direct link. This effect is dominant especially when the relays are located close to the source.

The optimum energy allocations obtained from numerical search for systems with and without a direct link are plotted in Fig. 3. We consider the total SNR value of $\rho = 30$ dB and a

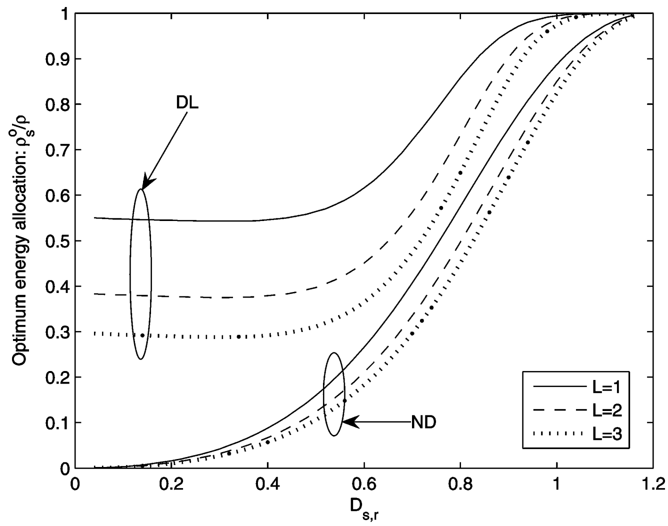

Fig. 3. Optimum energy allocation (ND and DL, $D = 1.2, D_{s,d} = 1, \rho = 30$ dB, $\nu = 4$).

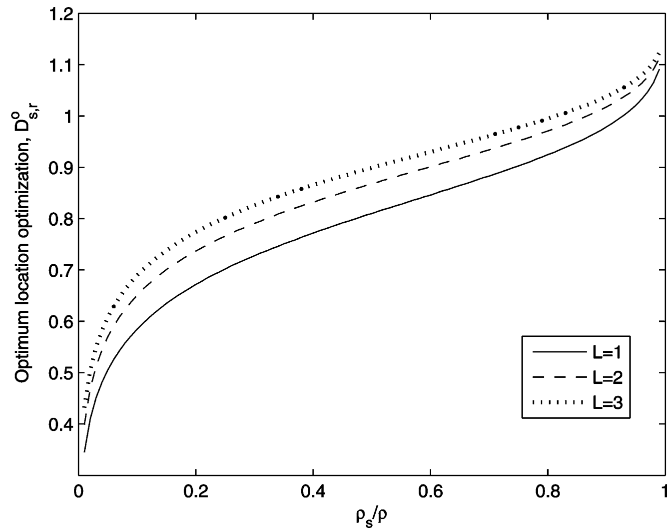

Fig. 4. Optimum relay location (ND and DL, $D = 1.2, D_{s,d} = 1, \rho = 30$ dB, $\nu = 4$).

path loss exponent of $\nu = 4$ with various L values. The ellipse setup is considered with $D = D_{s,r} + D_{r,d} = 1.2$ and $D_{s,d} = 1$. When the relays are close to the destination, both systems with and without a direct link behave similarly: most of the energy is assigned to the source to ensure that the transmitted signal can reach the relays. However, when the relays start moving towards the source, the two systems begin to behave very differently. For the no-direct-link (ND) system, the optimum energy allocation at the source ρ_s/ρ drops monotonically even when the relays are very close to the source. For the direct-link (DL) system, ρ_s/ρ drops at the beginning, but then remains almost constant when $D_{s,r}/D \leq 0.5$. This is reasonable because, intuitively, when the relays are close to the source, the DL system is reminiscent of a system with $(L+1)$ colocated antennas, which renders the uniform energy allocation across the $(L+1)$ nodes optimum, when the SNR is reasonably high.

B. Location Optimization

Consider now the optimum relay location in an idealized L-relay system. To find the optimum location of the relays given the source and the relay energies, we formulate an optimization problem as shown here.

Problem Statement 2: In a relay system with path loss exponent ν and source-destination distance $D_{s,d}$, for any given transmit energy at the source and relay nodes (ρ_s and ρ_r), and source-relay-destination distance D, we will determine the optimum relay location $D_{s,r}$, which:

$$\begin{aligned} &\text{minimizes} \quad \bar{P}_{e,ND} \text{ or } \bar{P}_{e,DL} \\ &\text{subject to} \quad D_{s,r} + D_{r,d} = D. \end{aligned} \tag{22}$$

Compared with the energy optimization, the location optimization has a different feature, which is described in the following lemma.

Lemma 1: For the given energy levels at the source and relays, the relay location optimization which minimizes the SER is independent of the presence of the direct link between the source and the destination.

Proof: This can be proved by the average SER expression in(12). The $1/\bar{\gamma}_{d,s} = D_{s,d}^{\nu}/\rho_s$ term has a fixed value given the $s-d$ distance $D_{s,d}$ and prescribed energy at the source. Therefore, a direct link does not affect the location optimization. Hence, the location optimization can be achieved without considering the direct link; i.e., the results of location optimization are identical in systems with and without a direct link. ∎

The optimum location can be found by treating the SER as a function of the distances and using the Lagrange multiplier. Similar to the proof in Appendix III, we obtain the following result.

Proposition 4: For the given source-destination distance $D_{s,d}$, source-relay-destination distance D, and the prescribed transmit energy levels ρ_s and ρ_r, there exists a unique optimum relay location which satisfies

$$\begin{aligned} &D_{s,r}^{o^{\nu-1}} \rho_r - \left(D - D_{s,r}^o\right)^{\nu-1} \\ &\qquad \rho_s \left\{ \ln\left[\rho_r \left(D - D_{s,r}^o\right)^{-\nu}\right] - 1 \right\} = 0 \end{aligned} \tag{23}$$

and accordingly, $D_{r,d}^o = D - D_{s,r}^o$.

Again, the log term and the path loss exponent ν in (23) make it difficult to find the closed-form solution, but Proposition 2 allows for numerical search of the optimum solution using (23).

Fig. 4 depicts the optimum relay location which is applicable to both systems with and without a direct link. We consider the total SNR value of $\rho = 30$ dB and a path loss exponent of $\nu = 4$ with various L values. The ellipse setup with $D = D_{s,r} + D_{r,d} = 1.2$ and $D_{s,d} = 1$ is considered. As more transmit energy is assigned to the source, the optimum relay location moves toward the destination.

C. Joint Energy and Location Optimization

So far, we have been dealing with the energy and location optimizations separately. Now let us consider the joint optimization which satisfies both the energy and location optimality conditions.

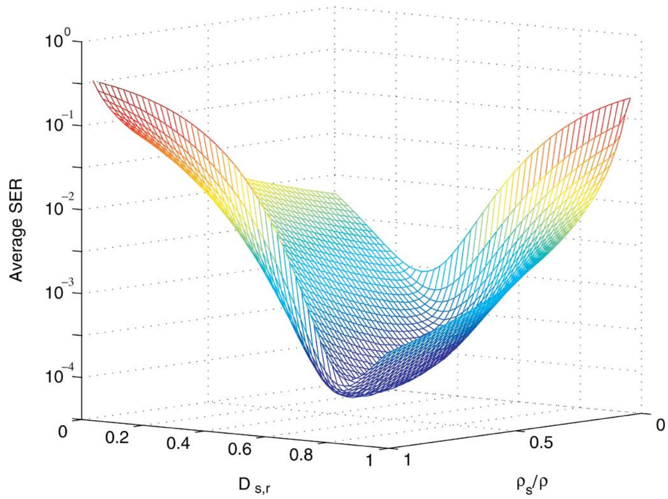

Fig. 5. Performance surface versus energy and distance (ND, $\rho = 15$ dB, $L = 3, \nu = 4$).

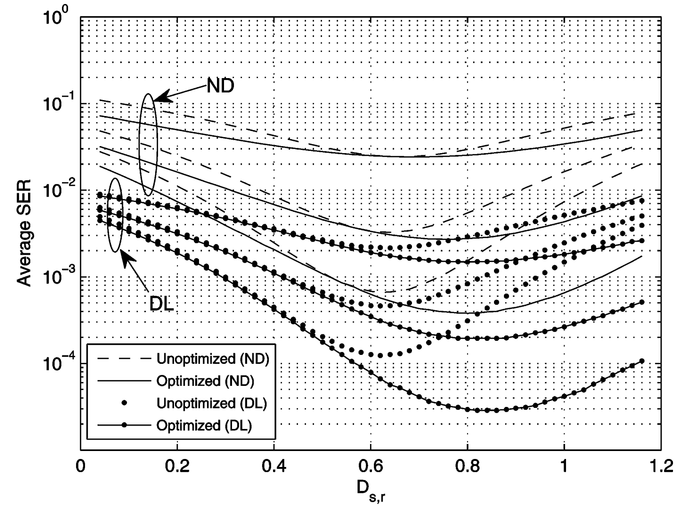

Fig. 6. SER comparison between the system with and without optimum energy allocation ($D = 1.2, \rho = 15$ dB, $\nu = 4$).

Problem Statement 3: In a relay system with path loss exponent ν and source-destination distance $D_{s,d}$, for any given total transmit energy ρ, and source-relay-destination distance D, we will determine the optimum energy allocation ρ_s and the optimum relay location $D_{s,r}$, which

$$\begin{aligned} \text{minimizes} \quad & \bar{P}_{e,ND} \text{ or } \bar{P}_{e,DL} \\ \text{subject to} \quad & \rho_s + L\rho_r = \rho, \\ & D_{s,r} + D_{r,d} = D. \end{aligned} \tag{24}$$

The analytical solution can be obtained by solving the equation set (20) and (23) for the system with no direct link, or (21) and (23) for systems with a direct link. The solution achieves the global minimum SER. However, the explicit solutions cannot be easily obtained even for the idealized case as we have seen in the preceding section. Fortunately, the convexity proved in Proposition 2 guarantees one unique global optimum resource allocation scheme, thus the joint optimization can be obtained by carrying out a two-dimensional numerical search iteratively. The steps are as follows:

Step 1. *Step 1. Initialization*) Set the uniform energy allocation as the optimum, i.e., $\rho_s^o = \rho/(L+1)$.

Step 2. Step 2. (*Location Optimization*) For a given energy allocation, find the optimum relay location, $D_{s,r,\text{new}}^o$, which is SER-minimizing. If the optimum location differs from the original location, set the optimum location to the new one, $D_{s,r}^o = D_{s,r,\text{new}}^o$ and continue to Step 3; otherwise, stop.

Step 3. Step 3. (*Energy Optimization*) For a given relay location, find the optimum energy allocation, $\rho_{s,\text{new}}^o$. If the optimum energy allocation differs from the original, set $\rho_s^o = \rho_{s,\text{new}}^o$, and go back to Step 2; stop otherwise.

Note that Step 2 and 3 are interchangeable. Fig. 5 shows the SER surface when $L = 3$ with $\rho = 15$ dB, $D_{s,d} = D = 1$, and $\nu = 4$. Using the above steps, the global minimum value can be obtained. This value provides the joint energy and location optimization. More examples and comparisons are given in Section V.

V. Simulations and Discussions

In this section, we will discuss the performance of the AF-based relay system combined with differential demodulation and the optimum resource allocation. We will compare the performance of the system with and without optimization.

A. Benefits of Energy and Location Optimizations

To verify the advantages of the optimum energy allocation and relay location, the SERs of the relay systems with and without optimization are depicted in Figs. 6–8. The following system parameters will be used: $\rho = 15$ dB, $D = 1.2$, $\nu = 4$, and $L =$ 1, 2, 3. In the system without energy optimization, a uniform energy allocation is employed, that is, $\rho_s = \rho_r = \rho/(L+1)$ at any $D_{s,r}$. In the system without location optimization, the relays are placed at a point that is equally spaced from the source and the destination; that is, $D_{s,r} = D_{r,d} = D/2$.

Fig. 6 depicts the benefits of energy optimization for systems with no direct link and with a direct link. In both ND and DL systems, the three pairs of unoptimized and optimized curves from top to bottom represent the systems with $L =$1, 2, and 3, respectively. From this figure, we observe that: i) the energy-optimized systems universally outperform the unoptimized systems; ii) SER minima of the optimized systems and unoptimized ones are quite different except for the single-relay $(L = 1)$ system with no direct link; iii) the unoptimized systems have the minimum SER almost at the midpoint, which agrees with the results in [16], [18], and [21]; while the optimized systems achieve the minimum SER by having the relays closer to the destination, except for the single-relay system without a direct link; and iv) in the system with a direct link and when the relays are located close to the source, the SERs of the optimized and unoptimized systems are almost identical, this is because the uniform energy allocation is nearly optimum. As expected, this feature is not shared by the system with no direct link. These observations confirm our analysis in the preceding section.

The benefits of location optimization are illustrated in Figs. 7 and 8: *i)* the location-optimized systems universally outperform

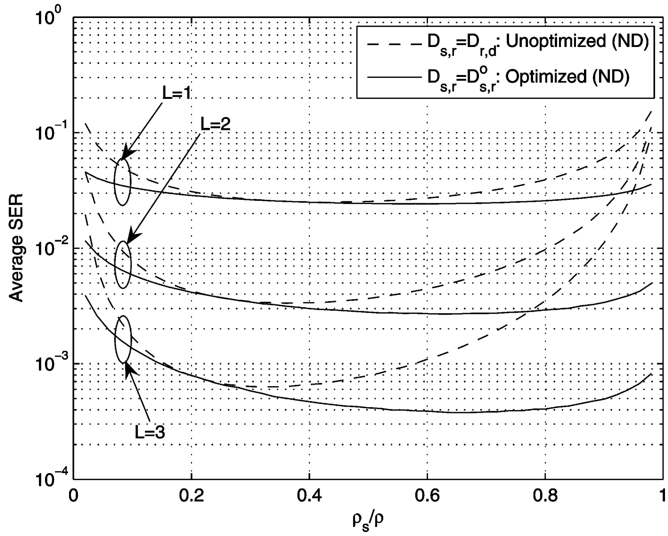

Fig. 7. SER comparison between the system with and without relay location optimization (ND, $D = 1.2, \rho = 15$ dB, $\nu = 4$).

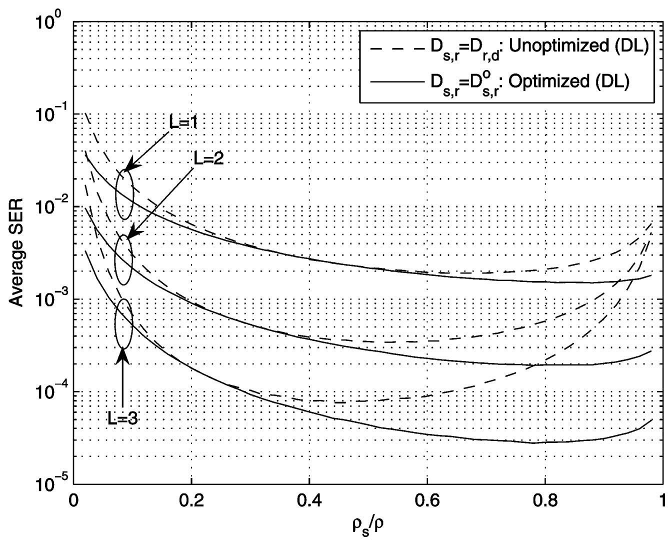

Fig. 8. SER comparison between the system with and without relay location optimization (DL, $D_{s,d} = 1, D = 1.2, \rho = 15$ dB, $\nu = 4$).

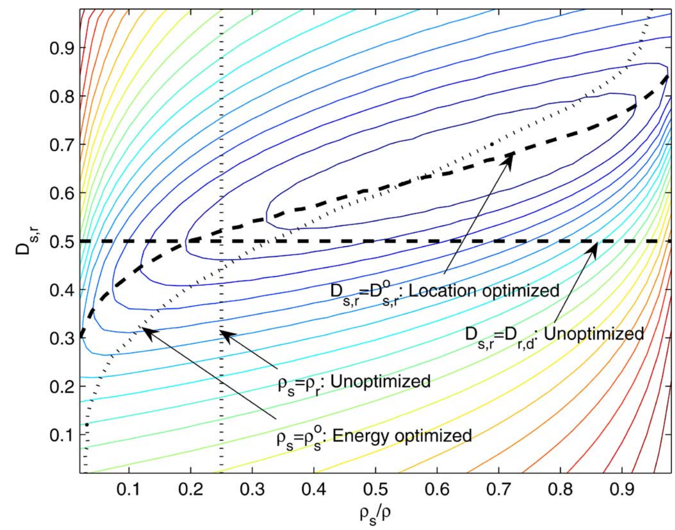

Fig. 9. The SER contour with AF (ND, $D_{s,d} = D = 1, \rho = 15$ dB, $L = 3$).

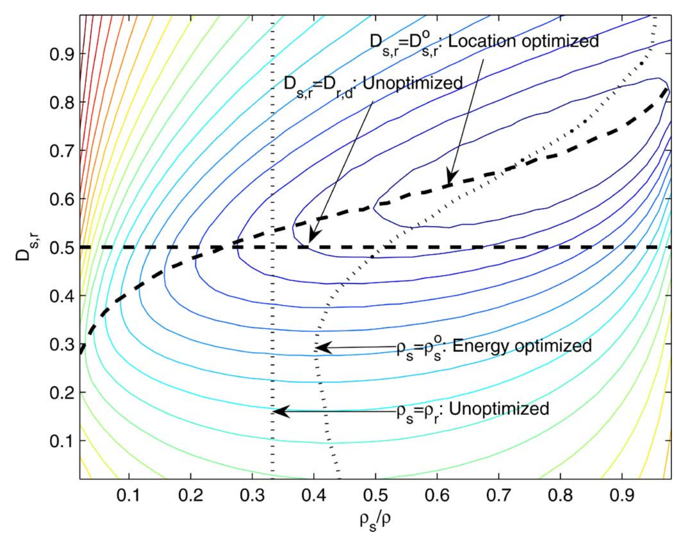

Fig. 10. The SER contour with AF (DL, $D_{s,d} = D = 1, \rho = 15$ dB, $L = 2$).

the unoptimized systems; *ii)* the minima of the location-optimized SER curves are very different from those of the unoptimized ones; and *iii)* the optimum SER can be achieved by assigning more energy to the source except for the single-relay system with no direct link. Furthermore, the curves in Figs. 7 and 8 show more flatness compared with the energy optimized systems in Fig. 6. This implies that the average SER is more sensitive to the relay location than to the energy allocation.

In brief, the advantages of the resource optimization are evident from Fig. 6–8, and the optimum SER can be achieved by locating the relays closer to the destination with more transmit energy at the source.

B. Benefits of Joint Optimization

Next, we consider the joint optimization. Figs. 9 and 10 depict the SER contour of the relay system without and with a direct link, respectively. When $\rho = 15$ dB and $D_{s,d} = D = 1$, systems with $L = 3$ and $L = 2$ are considered, respectively. In both figures, the vertical line and horizontal line represent the SER of the system with a uniform energy allocation and mid-distance relay location, respectively. We also plot the curves for the energy optimization and location optimization. Notice that these two curves intersect at the optimum point, which corresponds to the joint energy and location optimization. From these two figures, it is clear that the SER of the unoptimized systems is much worse than the jointly optimized ones. This indicates that we cannot obtain the minimum SER with a uniform energy allocation or mid-distance relay location. Therefore, to achieve the minimum SER, one has to adapt a system via the joint optimization.

To sum up, the minimum SER can be achieved by locating the relays closer to the destination and assigning more energy to the source than the relays. This is due to the fact that the SER decreases by allowing the source to transmit signals with more energy while reducing the effect of noise on the amplification factor, and enhancing the error performance of relay-destination links.

VI. CONCLUSION

In this paper, we investigated the optimum resource allocation of the relay systems with a differential AF protocol. Based on the average SER analysis, a two-dimensional optimization was studied. Our analysis reveals that the differential AF-based system can achieve full diversity gain, and the optimum energy allocation depends on the existence of a direct link, while the optimum relay location does not. We showed that, under the constraints of the total transmit energy, the source-destination distance, and the source-relay-destination distance: i) given relay locations, the average SER can be minimized by appropriately distributing the prescribed total energy per symbol across the source and the relays; ii) given the source and relay energy levels, there is an optimum relay location which minimizes the average SER; and iii) the global minimum SER can be achieved by the joint energy-location optimization. Our numerical examples confirm that both the energy and location optimizations provide considerable SER advantages. We also observed that the minimum SER can be achieved by locating the relays closer to the destination with a large source energy for the AF-based systems.

APPENDIX I
PROOF OF PROPOSITION 1

In an L-relay AF system with no direct link between the source and the destination, the received SNR can be obtained as

$$\gamma_{ND} = \sum_{k=1}^{L} \gamma_{eq,r_k}. \tag{25}$$

Given γ_{ND}, the conditional SER for multiple independent channels using DBPSK signaling can be calculated as in [17, Ch. 12]

$$P_{e|\gamma_{ND}} = \frac{1}{2^{2L-1}} e^{-\gamma_{ND}} \sum_{n=0}^{L-1} c_n \gamma_{ND}^n \tag{26}$$

where $c_n = 1/n! \sum_{k=0}^{L-1-n} \binom{2L-1}{k}$. Using (25), γ_{ND}^n can be explicitly expanded as shown in (27) at the bottom of the page, where $m_L = n - \sum_{i=1}^{L-1} m_i$. Then, (26) can be reorganized as

$$P_{e|\gamma_{ND}} = \sum_{n,m}^{L} \prod_{k=0}^{L} e^{-\gamma_{eq,r_k}} \gamma_{eq,r_k}^{m_k} \tag{28}$$

where [see (29) at the bottom of the page]. With the conditional SER, we can obtain the average SER by averaging it with respect to the probability density function (pdf) of γ_{eq,r_k}, $p(\gamma_{eq,r_k})$ as follows:

$$\bar{P}_{e,ND} = \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{L} P_{e|\gamma_{ND}} \times \prod_{k=1}^{L} p(\gamma_{eq,r_k}) d\gamma_{eq,r_1} d\gamma_{eq,r_2} \cdots d\gamma_{eq,r_L}. \tag{30}$$

By substituting (28) into (30), we have

$$\bar{P}_{e,ND} = \sum_{n,m}^{L} \prod_{k=1}^{L} \int_0^\infty e^{-\gamma_{eq,r_k}} \gamma_{eq,r_k}^{m_k} p(\gamma_{eq,r_k}) d\gamma_{eq,r_k} \tag{31}$$

where the pdf of γ_{eq,r_k} is given by [22]

$$p(\gamma_{eq,r_k}) = \frac{\beta^2}{2} e^{-\gamma_{eq,r_k}/\bar{\gamma}_{r_k,s}} K_0(\beta\sqrt{\gamma_{eq,r_k}}) + \frac{\beta}{\bar{\gamma}_{r_k,s}} \sqrt{\gamma_{eq,r_k}} e^{-\gamma_{eq,r_k}/\bar{\gamma}_{r_k,s}} K_1(\beta\sqrt{\gamma_{eq,r_k}}) \tag{32}$$

where $\beta = 2\sqrt{(1 + 1/\bar{\gamma}_{r_k,s})/\bar{\gamma}_{d,r_k}}$. Then the integral in (31) can be evaluated as

$$\begin{aligned} I_{n,m_k} &:= \int_0^\infty e^{-\gamma_{eq,r_k}} \gamma_{eq,r_k}^{m_k} p(\gamma_{eq,r_k}) d\gamma_{eq,r_k} \\ &= \frac{\beta^2}{2} \int_0^\infty \gamma_{eq,r_k}^{m_k} e^{-\alpha\gamma_{eq,r_k}} K_0(\beta\sqrt{\gamma_{eq,r_k}}) d\gamma_{eq,r_k} \\ &\quad + \frac{\beta}{\bar{\gamma}_{r_k,s}} \int_0^\infty \gamma_{eq,r_k}^{m_k+1/2} e^{-\alpha\gamma_{eq,r_k}} \\ &\quad \times K_1(\beta\sqrt{\gamma_{eq,r_k}}) d\gamma_{eq,r_k} \end{aligned} \tag{33}$$

$$\gamma_{ND}^n = \underbrace{\sum_{m_1=0}^{n} \binom{n}{m_1} \gamma_{eq,r_1}^{m_1} \sum_{m_2=0}^{n-m_1} \binom{n-m_1}{m_2} \gamma_{eq,r_2}^{m_2} \cdots \sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2} m_i} \binom{n-\sum_{i=1}^{L-2} m_i}{m_{L-1}} \gamma_{eq,r_{L-1}}^{m_{L-1}} \gamma_{eq,r_L}^{m_L}}_{L-1} \tag{27}$$

$$\sum_{n,m}^{L} = \frac{1}{2^{2L-1}} \sum_{n=0}^{L-1} c_n \underbrace{\sum_{m_1=0}^{n} \binom{n}{m_1} \sum_{m_2=0}^{n-m_1} \binom{n-m_1}{m_2} \cdots \sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2} m_i} \binom{n-\sum_{i=1}^{L-2} m_i}{m_{L-1}}}_{L-1}. \tag{29}$$

where $\alpha = 1 + 1/\bar{\gamma}_{r_k,s}$. Each integral in (33) can be computed by using the integration property of Bessel functions [10 (6.631.3)]

$$\int_0^\infty x^\theta e^{-\alpha x^2} K_\phi(\beta x)dx = \frac{1}{2}\alpha^{-1/2\theta}\beta^{-1}\Gamma\left(\frac{1+\theta+\phi}{2}\right)\Gamma\left(\frac{1-\phi+\theta}{2}\right) \times e^{\beta^2/8\alpha}W_{-\theta/2,\phi/2}\left(\frac{\beta^2}{4\alpha}\right) \quad (34)$$

where $W_{m,n}(\cdot)$ is the Whittaker function $W_{m,n}(z) = e^{-z/2}z^{n+1/2}U(1/2+n-m, 1+2n, z)$, with $U(\cdot,\cdot,\cdot)$ denoting confluent hypergeometric function of the second kind. By further using the approximation $U(a,1,1/x) \approx \ln(x)/\Gamma(a)$, $U(a,2,1/x) \approx x/\Gamma(a)$ [1(13.5.9) and (13.5.7)], and $\alpha = 1 + 1/\bar{\gamma}_{r_k,s} \approx 1$ at high SNR, (33) can be simplified to

$$I_{n,m_k} \approx m_k!\left[\frac{1}{\bar{\gamma}_{r_k,s}} + \frac{1}{\bar{\gamma}_{d,r_k}}\ln\bar{\gamma}_{d,r_k}\right]. \quad (35)$$

Plugging the above result back into (31), we get the following:

$$\bar{P}_{e,ND} = \sum_{n,m}^{L}\prod_{k=1}^{L} m_k!\left[\frac{1}{\bar{\gamma}_{r_k,s}} + \frac{1}{\bar{\gamma}_{d,r_k}}\ln\bar{\gamma}_{d,r_k}\right]. \quad (36)$$

Now, let us evaluate the coefficient part in (36), [see (37) at the bottom of the page]. By using the fact that

$$\underbrace{\binom{n}{m_1}\binom{n-m_1}{m_2}\cdots\binom{n-\sum_{i=1}^{L-2}m_i}{m_{L-1}}}_{L-1}\prod_{k=1}^{L} m_k! = n! \quad (38)$$

the coefficient, defined as $C(L)$, can be simplified to

$$\begin{aligned} C(L) &:= \sum_{n,m}^{L}\prod_{k=1}^{L} m_k! \\ &= \frac{1}{2^{2L-1}}\sum_{n=0}^{L-1} c_n \underbrace{\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\cdots\sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2}m_i}}_{L-1} n! \\ &= \frac{1}{2^{2L-1}}\sum_{n=0}^{L-1}\sum_{k=0}^{L-1-n}\binom{2L-1}{k} \\ &\quad\times \underbrace{\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\cdots\sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2}m_i}}_{L-1} 1. \end{aligned} \quad (39)$$

By using mathematical induction, we can prove that

$$\xi(L) := \underbrace{\sum_{m_1=0}^{n}\sum_{m_2=0}^{n-m_1}\cdots\sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2}m_i}}_{L-1} 1 = \binom{n+L-1}{L-1}. \quad (40)$$

We know that, for $L=1$, $\xi(1) = 1 = \binom{n+1-1}{1-1}$. Suppose for an arbitrary $L = l \geq 1$, $\xi(l) = \binom{n+l-1}{l-1}$, then, we have $\xi(l+1) = \sum_{m=0}^{n}\binom{n-m+l-1}{l-1}$ when $L = l+1$. We can also show that

$$\begin{aligned} \binom{n+l}{l} &= \frac{(n+l)!}{n!\,l!} \\ &= \frac{(n+l-1)!}{n!\,l!}(n+l) \\ &= \frac{(n+l-1)!}{n!(l-1)!} + \frac{(n+l-1)!}{(n-1)!(l-1)!} \\ &\vdots \\ &= \frac{(n+l-1)!}{n!(l-1)!} + \frac{(n-1+l-1)!}{(n-1)!(l-1)!} \\ &\quad + \cdots + \frac{(l-1)!}{0!(l-1)!} \\ &= \sum_{m=0}^{n}\binom{n-m+l-1}{l-1} = \xi(l+1). \end{aligned} \quad (41)$$

Therefore, (40) holds for any $L \geq 1$. Using (40), we can simplify (39) to (13). Finally, the average SER yields(11).

In the relay system with a direct link between the source and the destination, the approximated SER can be evaluated through the same procedure as above. The received SNR of the DL system can be calculated by adding the direct link

$$\gamma_{DL} = \gamma_{ND} + \gamma_{d,s} = \sum_{k=1}^{L}\gamma_{eq,r_k} + \gamma_{d,s}. \quad (42)$$

Following the same steps as the no-direct-link case, we have

$$\bar{P}_{e,DL} = \sum_{n,m}^{L+1}\prod_{k=1}^{L}\int_0^\infty e^{-\gamma_{eq,r_k}}\gamma_{eq,r_k}^{m_k}p(\gamma_{eq,r_k})d\gamma_{eq,r_k} \int_0^\infty e^{-\gamma_{d,s}}\gamma_{eq,r_k}^{m_{L+1}}p(\gamma_{d,s})d\gamma_{d,s}. \quad (43)$$

The integration for the direct link SNR can be calculated as (33)

$$\begin{aligned} I(d,s) &:= \int_0^\infty e^{-\gamma_{d,s}}\gamma_{d,s}^{m_{L+1}}p(\gamma_{d,s})d\gamma_{d,s} \\ &= \int_0^\infty e^{-\gamma_{d,s}}\gamma_{d,s}^{m_{L+1}}\frac{1}{\bar{\gamma}_{d,s}}e^{-\gamma_{d,s}/\bar{\gamma}_{d,s}}d\gamma_{d,s} \\ &= \frac{1}{\bar{\gamma}_{d,s}}(m_{L+1})! \end{aligned} \quad (44)$$

$$\sum_{n,m}^{L}\prod_{k=1}^{L} m_k! = \frac{1}{2^{2L-1}}\sum_{n=0}^{L-1} c_n \underbrace{\sum_{m_1=0}^{n}\binom{n}{m_1}\sum_{m_2=0}^{n-m_1}\binom{n-m_1}{m_2}\cdots\sum_{m_{L-1}=0}^{n-\sum_{i=1}^{L-2}m_i}\binom{n-\sum_{i=1}^{L-2}m_i}{m_{L-1}}}_{L-1}\prod_{k=1}^{L} m_k!. \quad (37)$$

where $p(\gamma_{d,s})$ is the pdf of the direct link. By combining the above result with (35), we can reach the result for $\bar{P}_{e,DL}$ in (12).

APPENDIX II
PROOF OF PROPOSITION 2

In the idealized system, all the relays have the same average SNR $\bar{\gamma}_{r,s}$ and $\bar{\gamma}_{d,r}$, thus (11) and (12) can be simplified to

$$\bar{P}_{e,ND} \approx C(L)\left[\frac{1}{\bar{\gamma}_{r,s}} + \frac{1}{\bar{\gamma}_{d,r}}\ln(\bar{\gamma}_{d,r})\right]^L, \tag{45}$$

$$\bar{P}_{e,DL} \approx C(L+1)\frac{1}{\bar{\gamma}_{d,s}}\left[\frac{1}{\bar{\gamma}_{r,s}} + \frac{1}{\bar{\gamma}_{d,r}}\ln(\bar{\gamma}_{d,r})\right]^L. \tag{46}$$

After replacing the average SNR at each node by the transmit energy and internode distance using (16), we can reexpress (45) and (46) as

$$\bar{P}_{e,ND} \approx C(L)\left[f(\rho_s, D_{s,r}, \rho_r, D_{r,d})\right]^L$$
$$\bar{P}_{e,DL} \approx C(L+1)\left[g(\rho_s, D_{s,r}, \rho_r, D_{r,d})\right]^L$$

where

$$f(\rho_s, D_{s,r}, \rho_r, D_{r,d}) = \rho_s^{-1}D_{s,r}^{\nu} + \rho_r^{-1}D_{r,d}^{\nu}\ln\left(\rho_r D_{s,r}^{-\nu}\right)$$
$$g(\rho_s, D_{s,r}, \rho_r, D_{r,d}) = \left(\rho_s^{-1}D_{s,d}^{\nu}\right)^{1/L} \times\left[\rho_s^{-1}D_{s,r}^{\nu} + \rho_r^{-1}D_{r,d}^{\nu}\ln\left(\rho_r D_{s,r}^{-\nu}\right)\right].$$

The power function $p(x) = x^L$ is a nondecreasing univariate convex function when $L \geq 1$ for $x \geq 0$. According to the property of convex functions [3, Ch. 3], if $f(\rho_s, D_{s,r}, \rho_r, D_{r,d})$ and $g(\rho_s, D_{s,r}, \rho_r, D_{r,d})$ are convex functions, then $\bar{P}_{e,ND}$ and $\bar{P}_{e,DL}$ are also convex functions. So it suffices to prove the convexity of f and g.

The Hessian matrix of f can be computed as

$$\mathbf{H}_f = \begin{bmatrix} 2\rho_s^{-2}\bar{\gamma}_{r,s}^{-1} & -\nu\rho_s^{-1}D_{s,r}^{-1}\bar{\gamma}_{r,s}^{-1} & 0 & 0 \\ -\nu\rho_s^{-1}D_{s,r}^{-1}\bar{\gamma}_{r,s}^{-1} & \nu(\nu-1)D_{s,r}^{-2}\bar{\gamma}_{r,s}^{-1} & 0 & 0 \\ 0 & 0 & H_{f22} & H_{f23} \\ 0 & 0 & H_{f32} & H_{f33} \end{bmatrix} \tag{47}$$

where

$$H_{f22} = \frac{\partial^2 f}{\partial\rho_r^2} = \rho_r^{-2}\bar{\gamma}_{d,r}^{-1}\left[2\ln(\bar{\gamma}_{d,r}) - 3\right]$$
$$H_{f23} = H_{f32} = \frac{\partial^2 f}{\partial\rho_r\partial D_{r,d}} = \nu\rho_r^{-1}D_{r,d}^{-1}\bar{\gamma}_{r,s}^{-1}\left[2 - \ln(\bar{\gamma}_{d,r})\right],$$
$$H_{f33} = \frac{\partial^2 f}{\partial D_{r,d}^2} = \nu D_{r,d}^{-2}\bar{\gamma}_{r,s}^{-1}\left[(\nu-1)\ln(\bar{\gamma}_{d,r}) - (2\nu-1)\right].$$

And the determinant of the matrix is

$$\|\mathbf{H}_f\| = \nu^2(\nu-2)\rho_s^{-4}\rho_r^{-4}D_{s,r}^{2\nu-2}D_{r,d}^{2\nu-2} \left[\ln(\bar{\gamma}_{d,r}) - 1\right]\left[(\nu-2)\ln(\bar{\gamma}_{d,r}) - (2\nu-3)\right]. \tag{48}$$

It is obvious that $\|\mathbf{H}_f\| > 0$ if $\nu > 2$, and $\ln(\bar{\gamma}_{d,r}) > 2 + 1/(\nu-2)$. Hence $f(\rho_s, D_{s,r}, \rho_r, D_{r,d})$ is strictly convex over $\rho_s, D_{s,r}, \rho_r, D_{r,d}$ under this condition.

Similarly, the Hessian matrix of function g is

$$\mathbf{H}_g = \begin{bmatrix} H_{g00} & H_{g01} & H_{g02} & H_{g03} \\ H_{g10} & H_{g11} & 0 & 0 \\ H_{g20} & 0 & H_{g22} & H_{g23} \\ H_{g30} & 0 & H_{g32} & H_{g33} \end{bmatrix} \tag{49}$$

where

$$\begin{aligned} H_{g00} &= \frac{\partial^2 g}{\partial\rho_s^2} \\ &= \left(\frac{1}{L+1}\right)\rho_s^{-(1/L+2)} \\ &\quad \left[\left(\frac{1}{L+2}\right)\bar{\gamma}_{r,s}^{-1} + \frac{1}{L}\bar{\gamma}_{r,s}^{-1}\ln(\bar{\gamma}_{d,r})\right] \\ H_{g01} &= H_{g10} = \frac{\partial^2 g}{\partial\rho_s\partial D_{s,r}} \\ &= -\nu\left(\frac{1}{L+1}\right)\rho_s^{-(1/L+1)}D_{s,r}^{-1}\bar{\gamma}_{r,s} \\ H_{g02} &= H_{g20} = \frac{\partial^2 g}{\partial\rho_s\partial\rho_r} \\ &= \frac{1}{L}\rho_s^{-(1/L+1)}\rho_r^{-1}\bar{\gamma}_{d,r}^{-1}\left[\ln(\bar{\gamma}_{d,r}) - 1\right] \\ H_{g03} &= H_{g30} = \frac{\partial^2 g}{\partial\rho_s\partial D_{r,d}} \\ &= -\frac{1}{L}\nu\rho_s^{-(1/L+1)}D_{r,d}^{-1}\bar{\gamma}_{d,r}^{-1}\left[\ln(\bar{\gamma}_{d,r}) - 1\right] \\ H_{g11} &= \frac{\partial^2 g}{\partial D_{s,r}^2} = \nu(\nu-1)\rho_s^{-1/L}D_{s,r}^{-2}\bar{\gamma}_{r,s}^{-1} \\ H_{g22} &= \frac{\partial^2 g}{\partial\rho_r^2} = \rho_s^{-1/L}\rho_r^{-2}\bar{\gamma}_{d,r}^{-1}\left[2\ln(\bar{\gamma}_{d,r}) - 3\right] \\ H_{g23} &= H_{g32} = \frac{\partial^2 f}{\partial\rho_r^2} = \nu\rho_s^{-1/L}\rho_r^{-1}D_{r,d}^{-1}\left[2 - \ln(\bar{\gamma}_{d,r})\right] \\ H_{g33} &= \frac{\partial^2 g}{\partial D_{r,d}^2} \\ &= \nu\rho_s^{-1/L}D_{r,d}^{-2}\bar{\gamma}_{d,r}^{-1}\left[(\nu-1)\ln(\bar{\gamma}_{d,r}) - (2\nu-1)\right]. \end{aligned}$$

The determinant of $\mathbf{H}_g$ is

$$\begin{aligned} \|\mathbf{H}_g\| = \nu^2\rho_s^{-(4/L+2)}\rho_r^{-2}D_{s,r}^{-2}D_{r,d}^{-2}\bar{\gamma}_{r,s}^{-1}\bar{\gamma}_{d,r}^{-2}\left[\ln(\bar{\gamma}_{d,r}) - 1\right] \\ \cdot\left\{\frac{(\nu-1)}{L}\bar{\gamma}_{d,r}^{-1}G_1\left(\bar{\gamma}_{d,r}, \nu, \frac{1}{L}\right)\right. \\ \left. + \left(\frac{1}{L+1}\right)\left(\frac{\nu-1}{L-2}\right)\bar{\gamma}_{r,s}^{-1}G_2\left(\bar{\gamma}_{d,r}, \nu, \frac{1}{L}\right)\right\} \end{aligned} \tag{50}$$

where

$$\begin{aligned} G_1\left(\bar{\gamma}_{d,r}, \nu, \frac{1}{L}\right) = \left(\frac{\nu-1}{L-2}\right)\ln^2(\bar{\gamma}_{d,r}) \\ - \left(\frac{2\nu-1}{L-3}\right)\ln(\bar{\gamma}_{d,r}) - \frac{(\nu-1)}{L} \end{aligned}$$

$$G_2\left(\bar{\gamma}_{d,r},\nu,\frac{1}{L}\right)=(\nu-2)\ln(\bar{\gamma}_{d,r})-(2\nu-3).$$

If $\ln(\bar{\gamma}_{d,r})>2+1/(\nu-2)$, then $G_2(\bar{\gamma}_{d,r},\nu,1/L)>0$. If $\nu>2+1/L$, and $\ln(\bar{\gamma}_{d,r})>N(\nu,L)$, we have $G_1(\bar{\gamma}_{d,r},\nu,1/L)>0$, where

$$N(\nu,L)=\frac{\left(2\nu-\frac{1}{L-3}\right)+\sqrt{\left(1+\frac{1}{L}\right)\left[4\nu^2-4\left(3+\frac{1}{L}\right)\nu+(5\nu+9)\right]}}{2\nu-2-\frac{1}{L}} \quad (51)$$

which is a decreasing function of L and ν. For example, $N(\nu,L)=6.7$ when $L=1$ and $\nu=3.5$, and $N(\nu,L)=2.2$ when $L=2$, $\nu=4$.

Put these two conditions together, if $\nu>2+1/L$, and $\ln(\bar{\gamma}_{d,r})>\max\{2+1/(\nu-2),N(\nu,L)\}$, then $\|\mathbf{H}_g\|>0$, which proves the convexity of function g. It is worthy pointing out that these requirements on $\ln(\bar{\gamma}_{d,r})$ are sufficient conditions, while the necessary condition will have less stringent requirement on the SNR.

To sum up, $\bar{P}_{e,ND}$ is a convex function when $\nu>2$, and $\ln(\bar{\gamma}_{d,r})>2+1/(\nu-2)$; while $\bar{P}_{e,DL}$ is convex if $\nu>2+1/L$, and $\ln(\bar{\gamma}_{d,r})>\max\{2+1/(\nu-2),N(\nu,L)\}$.

Appendix III Proof of Proposition 3

For the system with no direct link, we have the following first-order conditions:

$$-\frac{L\bar{P}_{e,ND}}{\rho_s}\frac{\rho_r\sigma_{d,r}^2}{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\ln\left(\rho_r\sigma_{d,r}^2\right)}-\lambda=0, \quad (52)$$

$$-\frac{L\bar{P}_{e,WD}}{\rho_r}\frac{\rho_s\sigma_{r,s}^2\left[\ln\left(\rho_r\sigma_{d,r}^2\right)-1\right]}{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\ln\left(\rho_r\sigma_{d,r}^2\right)}-L\lambda=0. \quad (53)$$

With these two equations, we have the total energy constraint as in (19). Then, by substituting ρ_s and ρ_r into (52) and (53) for the total energy constraint, we have

$$-\frac{L\bar{P}_{e,ND}}{\lambda}=\frac{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\ln\left(\rho_r\sigma_{d,r}^2\right)}{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\left[\ln\left(\rho_r\sigma_{d,r}^2\right)-1\right]}\rho. \quad (54)$$

With (52) and(54), we arrive at(20). Similarly, the first-order conditions of the system with a direct link are given by

$$-\frac{\bar{P}_{e,DL}}{\rho_s}\left(1+L\frac{\rho_r\sigma_{d,r}^2}{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\ln\left(\rho_r\sigma_{d,r}^2\right)}\right)-\lambda=0, \quad (55)$$

$$-\frac{L\bar{P}_{e,DL}}{\rho_r}\frac{\rho_s\sigma_{r,s}^2\left[\ln\left(\rho_r\sigma_{d,r}^2\right)-1\right]}{\rho_r\sigma_{d,r}^2+\rho_s\sigma_{r,s}^2\ln\left(\rho_r\sigma_{d,r}^2\right)}-L\lambda=0. \quad (56)$$

By using the same steps in the above, we have (21).

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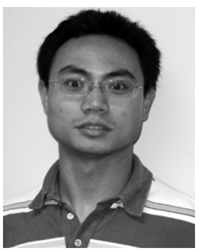

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