

Convergence of Sequences of Random Variables

Preliminary Concepts and Notation

If $\{x_n\} = \{x_1, x_2, \dots\}$ is a sequence of real numbers, we say that $\{x_n\}$ converges to a number x^* if for all $\varepsilon > 0$, there exists N such that $|x_n - x^*| < \varepsilon$ for all $n \geq N$. In this case, we write $x_n \rightarrow x^*$ or $\lim_{n \rightarrow \infty} x_n = x^*$.

In the following, let $\{X_n\} = \{X_1, X_2, \dots\}$ be a sequence of random variables, and X^* some random variable.

Convergence Almost Surely

We say that $\{X_n\}$ converges to X^* almost surely (or a.s.) if

$$P\{X_n \rightarrow X^*\} = 1.$$

In this case, we write $X_n \rightarrow X^*$ a.s., or $\lim_{n \rightarrow \infty} X_n = X^*$ a.s., or $X_n \xrightarrow{\text{a.s.}} X^*$.

Convergence In Probability

We say that $\{X_n\}$ converges to X^* in probability if for all $\varepsilon > 0$,

$$P\{|X_n - X^*| \geq \varepsilon\} \rightarrow 0.$$

In this case, we write $X_n \rightarrow X^*$ in probability, or $\lim_{n \rightarrow \infty} X_n = X^*$ in probability, or $X_n \xrightarrow{P} X^*$.

Convergence In Mean Square

We say that $\{X_n\}$ converges to X^* in mean square (or m.s.) if

$$E[(X_n - X^*)^2] \rightarrow 0.$$

In this case, we write $X_n \rightarrow X^*$ m.s., or $\lim_{n \rightarrow \infty} X_n = X^*$ m.s., or $X_n \xrightarrow{\text{m.s.}} X^*$.

Convergence In Distribution

Let F_n be the distribution function of X_n , and F^* the distribution function of X^* . We say that $\{X_n\}$ converges to X^* in distribution if

$$F_n(x) \rightarrow F^*(x)$$

for all x such that F^* is continuous at x . In this case, we write $X_n \rightarrow X^*$ in distribution, or $\lim_{n \rightarrow \infty} X_n = X^*$ in distribution, or $X_n \xrightarrow{D} X^*$.