Investigations into Uncertain Control Co-Design Implementations for Stochastic in Expectation and Worst-Case Robust
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Saeed Azad\textsuperscript{1}  \quad Daniel R. Herber\textsuperscript{2}

\textsuperscript{1}Postdoctoral Fellow
\texttt{saeed.azad@colostate.edu}
\textsuperscript{2}Assistant Professor
\texttt{daniel.herber@colostate.edu}

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\textsuperscript{1}Colorado State University, Department of Systems Engineering
\textsuperscript{2}Colorado State University, Department of Systems Engineering

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Introduction
Introduction

- Control co-design (CCD) refers to the integrated consideration of the physical and control system design
- Deterministic CCD has been studied\(^1\)
- Since some of the elements of CCD problem are uncertain, uncertain CCD (UCCD) strategies are needed
- Implementation challenges, implicit assumptions, and in-depth discussion of the structure of UCCD problems, method-dependent considerations, and practical insights is currently missing from the literature
- This study starts to fill these gaps by using a simple strained-actuated solar array (SASA)\(^2\) to
  - Discuss the optimal, open-loop control structure under uncertainties
  - Implement and solve a stochastic in expectation UCCD (SE-UCCD) using Monte Carlo simulation (MCS) and generalized Polynomial Chaos (gPC) expansion
  - Implement and solve a worst-case robust UCCD (WCR-UCCD) using deterministic representation of uncertainties

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\(^1\) Herber and Allison 2019; Allison and Herber 2014

\(^2\) Herber and Allison 2017; Chilan et al. 2017
A Universal UCCD Formulation

A universal UCCD formulation defined in probability space (specialized forms can be derived though the appropriate selection of the objective function and constraints\(^1\)):

- \(\tilde{\phi}\) is a time-independent uncertain variable
- \(\tilde{\phi}(t)\) is a stochastic process
- \(\tilde{\phi}(\cdot)\) is a function composition of \(\phi(\cdot)\), e.g.,
  - \(\tilde{o}(\cdot)\) is a function of the original objective function \(o(\cdot)\)
  - \(\tilde{g}(\cdot)\) is a function of the original inequality constraint vector \(g(\cdot)\)

\[
\begin{align*}
\text{minimize:} & \quad \mathbb{E} \left[ \tilde{o}(t, \tilde{u}, \tilde{\xi}, \tilde{p}, \tilde{d}) \right] \\
\text{subject to:} & \quad \mathbb{E} \left[ \tilde{g}(t, \tilde{u}, \tilde{\xi}, \tilde{p}, \tilde{d}) \right] \leq 0 \\
& \quad h(t, \tilde{u}, \tilde{\xi}, \tilde{p}, \tilde{d}) = 0 \\
& \quad \dot{\tilde{\xi}}(t) - f(t, \tilde{u}, \tilde{\xi}, \tilde{p}, \tilde{d}) = 0 \\
\text{where:} & \quad \tilde{u}(t) = \tilde{u}, \quad \tilde{\xi}(t) = \tilde{\xi}, \quad \tilde{d}(t) = \tilde{d} \\
& \quad \tilde{\phi} \in \mathcal{V}_u, \quad \tilde{\phi}(t) \in \mathcal{T}_u(t)
\end{align*}
\]

\(^1\) Azad and Herber 2022
UCCD Implementation
Open-loop Optimal Control Structures

- **Open-loop single control** (OLSC) finds a single control command that meets some criteria and is closely related to concepts from robust control theory.

- **Open-loop multiple control** (OMLC) elicits a range of optimal control responses based on the realization of uncertainties.
Uncertainty Propagation Methods
Uncertainty Propagation Methods

- A generalized Polynomial Chaos (gPC) expansion was used for uncertainty propagation and benchmarked against Monte Carlo simulation results.
- In gPC expansion, elements in an arbitrary random vector $\tilde{x}$ must have mutual independence.
- The univariate gPC basis functions of degree up to $r_i$ are denoted as $\{\phi_k(\tilde{x}_i)\}_{k=0}^{r_i}$, and satisfy the orthogonality conditions.
- The set of univariate orthogonal basis functions are obtained based on the probability distribution of $\tilde{x}$.
- A tensor product of elements in $\{\phi_k(\tilde{x}_i)\}_{k=0}^{r_i}$ is used to construct the $n_x$-variate gPC basis functions $\Phi_m(\tilde{x})$.

1 Loeve 1978; Rosenblatt 1952
2 Xiu 2010
Generalized Polynomial Chaos

- The resulting polynomials span the space of

\[
\left\{ \Phi_m(\tilde{x}) \right\}_{m=0}^{M-1} = \bigotimes_{|k| \leq PC} \prod_{i=1}^{n_x} \phi_k(\tilde{x}_i)
\]

- \(PC\) is either the highest polynomial order in each direction, or alternatively, is the total degree of a subset of basis elements

- Any second-order variable or process \(\tilde{y}(t, \tilde{x})\) can be expressed by polynomial chaos of \(PC\) degree as:

\[
\tilde{y}(t, \tilde{x}) \approx y_{PC}(t, \tilde{x}) = \sum_{m=0}^{M-1} \hat{y}_m(t) \Phi_m(\tilde{x})
\]

- The unknown coefficients \(\hat{y}_i(t)\) are estimated through a Galerkin projection \(^1\) or a collocation formulation \(^2\)

\(^1\) Xiu 2010; Wang et al. 2019  \(^2\) Cottrill 2012
Generalized Polynomial Chaos

The unknown coefficients are obtained from through a quadrature rule (and thus collocation points) from

\[
\hat{y}_m(t) = \mathbb{E}[y(t, x) \Phi_j(\tilde{x})] = \int_{\Gamma} y(t, x) \Phi_j(x) dF_\tilde{x}(x) \approx \sum_{j=1}^{Q} y(t, x_j) \Phi_m(x_j) \alpha_{wj}
\]

Steps involved in gPC:
1. \( q_2 = 3 \) collocation nodes
2. \( q_1 = 3 \) collocation nodes
3. \( \hat{y}(\tilde{x}, t) \)
4. \( \mathbb{E}[\hat{y}] \)
5. \( \alpha_{wj} \)
6. \( \{\phi_k(x_1)\}_{k=0}^{r_1=5} \)
7. \( \{\phi_k(x_2)\}_{k=0}^{r_2=5} \)
8. \( \{\Phi_m(\tilde{x})\}_{m=0}^{M-1} = \otimes_{i=1}^{2} \{\phi_k(x_i)\} \)
9. \( \{x_j\}_{j=1}^{Q} \)
10. \( \alpha_{wq1} \)
11. \( \alpha_{wq2} \)
12. \( \otimes_{i=1}^{2} \alpha_{wq_i}(x_i) \)
Simple SASA UCCD Formulations
Deterministic Simple SASA CCD

- Simplified strain-actuated solar array (SASA) system for spacecraft pointing control and jitter reduction

\[
\text{minimize:} \quad -\xi_1(t_f) \\
\text{subject to:} \quad u - u_{max} \leq 0 \\
\quad \quad \quad \quad u_{min} - u \leq 0 \\
\quad \quad \quad \quad \frac{\dot{\xi}_1}{\xi_2} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u \\
\xi(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \xi_2(t_f) = 0 \\
\text{where:} \quad u(t) = u, \quad \xi(t) = \xi
\]

- Plant: stiffness of the solar array \( k \)
- Control: strain actuation \( u(t) \)
- State: relative displacement & velocity \( \xi(t) \)
- Problem data: inertia ratio \( J \)

\(^1\) Herber and Allison 2017
Stochastic in Expectation UCCD (SE-UCCD)

- Uncertainties are
  \[ \tilde{k} \sim \mathcal{N}(\mu_k, \sigma_k) \]
  \[ \tilde{J} \sim \mathcal{N}(\mu_J, \sigma_J) \]
  \[ \tilde{\xi}_{2,t_0} \sim \mathcal{N}(\mu_{\xi_{2,t_0}}, \sigma_{\xi_{2,t_0}}) \]

- \( k_s \) is the constraint shift index
- Dynamics are satisfied \( a.s. \) or almost surely
- Terminal b.c. applied only when using a OLMC structure
- Risk-neutral formulation

SE-UCCD

\[
\begin{align*}
\text{minimize:} & \quad - \mathbb{E}[\tilde{\xi}_1(t_f)] \\
\text{subject to:} & \quad u - u_{\text{max}} \leq 0 \\
& \quad u_{\text{min}} - u \leq 0 \\
& \quad k_s\sigma_k - \mu_k \leq 0 \\
& \quad \dot{\tilde{\xi}}_1 = \begin{bmatrix} 0 & 1 \\ -\tilde{k} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (a.s.) \\
& \quad \tilde{\xi}(t_0) = \mathcal{N}(\mu_{\xi_{2,t_0}}, \sigma_{\xi_{2,t_0}}) \\
& \quad \tilde{\xi}_2(t_f) = 0 \quad \text{(if OLMC)} \\
\end{align*}
\]

where:
\[ \tilde{k} = \mathcal{N}(\mu_k, \sigma_k), \quad \tilde{J} = \mathcal{N}(\mu_J, \sigma_J) \]
\[ u(t) = u, \quad \tilde{\xi}(t) = \tilde{\xi} \]
Worst-Case Robust UCCD (WCR-UCCD)

- Deterministic uncertainties
- Epigraph form
- Risk-averse

\[ \text{Outer loop problem} \]

\[
\begin{align*}
\text{minimize:} & \quad -v \\
\text{subject to:} & \quad v = \Phi(t, u, \tilde{\xi}, \tilde{k}(\mu_k), \tilde{J}, \tilde{\xi}_{2,t_0}) \\ 
& \quad \dot{u} - u_{\max} \leq 0 \\
& \quad u_{\min} - \dot{u} \leq 0 \\
& \quad k \sigma_k - \mu_k \leq 0 \\
\text{where:} & \quad u(t) = u
\end{align*}
\]

\[ \text{Inner loop problem} \]

\[
\begin{align*}
\text{minimize:} & \quad \alpha_{in} = \xi_1(t_f) \\
\text{subject to:} & \quad \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \tilde{f}_j & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \frac{1}{j} u \\
& \quad \xi(t_0) = \begin{bmatrix} 0 \\ \xi_{2,t_0} \end{bmatrix} \\
& \quad -0.3 \leq \xi_2(t_f) \leq 0.3 \\
\text{where:} & \quad k(\mu_k) \in \mathcal{R}_k, \quad J(\mu_j) \in \mathcal{R}_J \\
& \quad \tilde{\xi}_{2,t_0}(\mu_{\xi_{2,t_0}}) \in \mathcal{R}_{\tilde{\xi}_{2,t_0}}, \quad \xi(t) = \xi
\end{align*}
\]

\[ ^1 \text{Azad and Herber 2022} \]
Results and Discussion
Results

- A nested coordination strategy used for OLMC-SE-UCCD
- Inner-loop optimal control subproblem solved using direct transcription (DT)
- All DT implementations done in MATLAB-based DTQP toolbox \(^1\)
- A direct single shooting (DSS) used for the outer-loop WCR-UCCD
- Implementations available on Github \(^2\)

Table: Settings for UCCD implementations.

<table>
<thead>
<tr>
<th>Category</th>
<th>Option</th>
<th>Value</th>
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<tbody>
<tr>
<td>General</td>
<td>defects</td>
<td>trapezoidal (TR)</td>
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<td></td>
<td>mesh</td>
<td>equidistant composite TR</td>
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<td></td>
<td>quadrature</td>
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<td></td>
<td>outer-loop solver</td>
<td>fmincon</td>
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<td></td>
<td>solver tolerance</td>
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<td>(M)</td>
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<td></td>
<td>derivatives</td>
<td>forward</td>
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<tr>
<td></td>
<td>(n_t)</td>
<td>100</td>
</tr>
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</table>

\(^1\) Herber 2017  \(^2\) Azad 2022
Results

Table: UCCD solutions.

<table>
<thead>
<tr>
<th>Formulation</th>
<th>Structure</th>
<th>$\phi$</th>
<th>$\mu_k$</th>
<th>$t(s)$</th>
<th>$t_{\text{switch}}$</th>
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</thead>
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<td>OLSC</td>
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<td>0.727</td>
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<tr>
<td>Stc-MCS</td>
<td>OLMC</td>
<td>−0.308</td>
<td>3.311</td>
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<td>0.737</td>
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<tr>
<td>Stc-gPC</td>
<td>OLMC</td>
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<td>3.185</td>
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<tr>
<td>$</td>
<td>\Delta</td>
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<td></td>
<td>0.65%</td>
<td>3.81%</td>
</tr>
<tr>
<td>WcR</td>
<td>OLSC</td>
<td>0.204</td>
<td>0.705</td>
<td>2592</td>
<td>0.838</td>
</tr>
</tbody>
</table>

SE-UCCD Solution

WCR-UCCD Solution
Results - Polytopic Uncertainties

- A polytope is a bounded, closed, and convex polyhedron
- For a linear program, the feasible region is the convex hull of the vertices of the polytope
- Therefore, the optimal solution is achieved at a polytope vertex
- The OLMC-WCR-UCCD of simple SASA has polytopic uncertainties and is linear with nested formulation
- The number of required evaluations reduced to vertices of the polytope, i.e. \(2^3\) vertices

OLMC-WCR-UCCD solution with polytopic uncertainties

\[\mu_{\xi_{2,t_0}} \pm 3\sigma_{\xi_{2,t_0}}\]
\[\mu_k \pm 3\sigma_k\]

worst-case vertices
\((\mu_k \pm 0.6, -0.09, 1.45)\)
Conclusion
Conclusions and Future Work

- Open-loop single control (OLSC) and open-loop multiple control (OLMC) structures were introduced.
- Results indicate that gPC offers promising improvements in the computational time.
- Uncertainty considerations impact system and design judgment.
- Extension to problems with probabilistic path constraints, especially stochastic chance-constraints UCCD formulations.
- Inclusion of time-dependent disturbances in the dynamic system model.
- Various geometries (such as ellipsoidal, hexagonal, etc.) for WCR-UCCD.
- Non-probabilistic propagation methods such as interval analysis and methods from fuzzy programming.
References

References (continued)

Questions?

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