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CLOSED FORM SOLUTION OF THE GENERAL THREE-DIMENSIONAL RADIATION CONFIGURATION FACTOR PROBLEM WITH MICROCOMPUTER SOLUTION

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A closed form based solution to the general problem of finding the radiation configuration factor between a differential element and a three dimensional solid object was developed. The method applies to any 3-D object which can be represented as a series of connected flat-faceted surfaces. The approach was based on insight from the Nusselt Unit Sphere Method. The developed algorithm was implemented in a microcomputer program which solves the problem and provides graphical output.

INTRODUCTION

Solving for the radiation configuration factor between a differential element and a finite area is essential in solving more general radiation problems. Various approaches have been taken in the past to solve this problem. Many authors have approached the problem by directly applying the integral expressions for the configuration factor to generate closed form solutions. The most notable authors taking this approach are Moon (1936), de Bastos (1961), and Sparrow (1963) for their application of Stokes' Theorem involving contour integration. Howell (1982) has also made a tremendous contribution in compiling many specific closed form solutions. The only obvious drawback with this traditional closed form approach is that it is limited to problems with well-defined geometry. The integral expressions become extremely cumbersome even for simple geometry.

A second approach to the problem is to apply the Unit Sphere Method developed by Nusselt (1928). Nusselt introduced solid angle geometry to simplify the problem and to provide a graphical interpretation. This graphical interpretation can be applied to determine the configuration factors experimentally. Many mechanical and photographic apparatus have been designed to implement this geometric solution. Most notable are Hamilton's Mechanical Linkage (Hamilton and Morgan, 1952), Eckert's shadow casting photographic method (Eckert and Drake, 1959), and Farrel's Scintilascope (Farrel, 1976). The advantage of these methods is that they can be applied to any arbitrarily complex object provided a model can be built. The obvious disadvantage in these methods is that they require special equipment and physical models.

Nusselt's Unit Sphere Method has also been applied in the form of direct graphical procedures. Hooper and Juhasz (1952) were probably the first to present a useful graphical procedure. Alciatore et al. (1988) have also presented an engineering graphics solution. These graphical methods are valuable in that they are intuitive, straightforward, and require very little equipment or skill. The major drawback is that they are very time-consuming and require that drawings be constructed.

A third approach is a microcomputer solution. Alciatore et al. (1988) have presented one such application which solves for the configuration factor between an elemental area and a defined 3-D contour. The computer solution is advantageous due to its speed and accuracy, but the contour is often difficult to define for certain geometries, such as arbitrary solid objects. The goals of this paper are to extend the computer solution to general 3-D objects and to develop a closed form solution which finds the configuration factor between a differential element and a 3-D contour directly.

MICROCOMPUTER IMPLEMENTATION

The first requirement of the computer algorithm was to provide a means for describing a general 3-D faceted closed surface. Two input routines were chosen for this purpose. One routine reads a standard wireframe database which contains vertex, edge, and surface information. This data can be read from a pre-existing file or entered at the keyboard. The second input routine allows the user to define the profile of a general surface of revolution. The program can take this profile and generate the complete wireframe information for the solid. This expedites entering geometry for axially symmetric bodies such as cylinders, cones, spheres, etc. Both input routines result in a faceted surface description which is illustrated in Figure 1. In this description, the one place where care must be taken is in the ordering of the facet edges. This ordering establishes the direction of the outward normal, which then determines whether or not the facet is visible

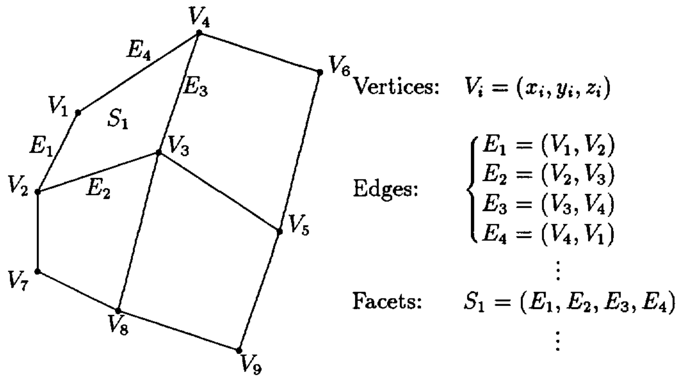

Fig. 1. Faceted Surface Description

to the differential element.

Once the geometry is defined, the next step is to determine the contour which describes the border between the radiated (visible) and non-radiated (non-visible) facets. Once this contour is defined, the closed form solution developed in the next section can be applied directly. The result is the radiation configuration factor between a differential element and the defined 3-D object.

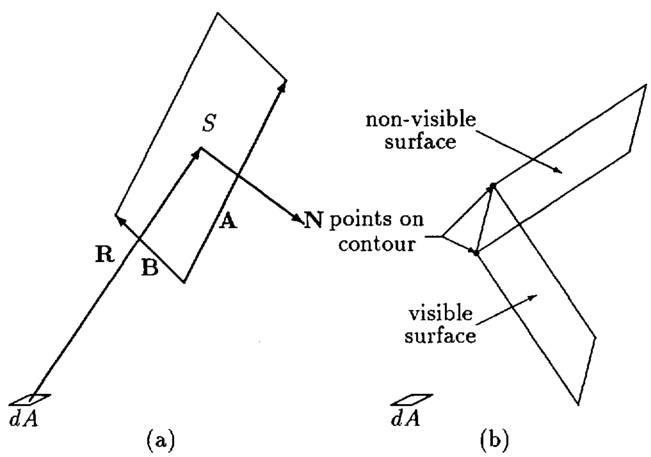

Fig. 2. Determining Surface Visibility and Object Contour

The contour is found using the common computer graphics technique called Back Plane Culling (BPC). BPC is a hidden surface removal algorithm which can be applied to wireframe data. The algorithm allows you to determine which facets are seen (radiated to) from the differential element. BPC is generally restricted to convex polyhedra although it is tolerant of low degrees of concavity. Figure 2(a) illustrates the vectors used in determining facet visibility. The two adjacent edges defined by vectors **A** and **B** define the facet outward normal **N**. Given that the line of sight from dA to facet S is defined by **R**, the following test can be used to determine visibility:

If $\mathbf{R} \cdot \mathbf{N} < 0$, then the facet is visible.

If $\mathbf{R} \cdot \mathbf{N} \geq 0$, then the facet is non-visible.

Once the visibility test has been performed for each facet, the locus of points defining the contour can be determined as illustrated in Figure 2(b). The ordering of these points is important and can be determined from the wireframe edge connectivity information. Having defined the contour for the object, the configuration factor may be calculated directly.

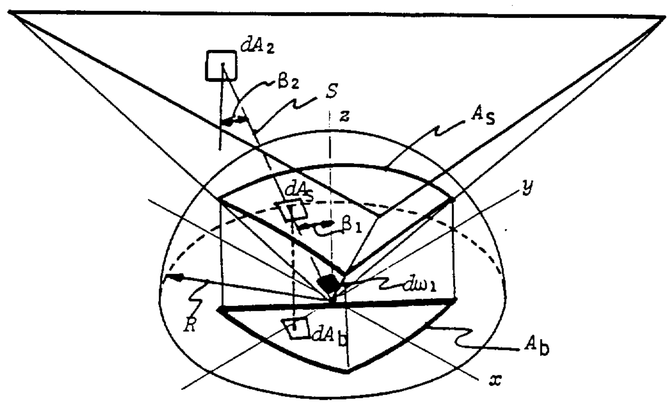

Fig. 3. Nusselt's Unit Sphere Method

CLOSED FORM SOLUTION

The definition of the configuration factor (Siegel and Howell, 1972) between a differential element dA_1 and a finite area A_2 is:

$$F_{dA_1-A_2} = \iint_{A_2} \frac{\cos\beta_1 \cos\beta_2}{\pi S^2}\, dA_2 = \frac{1}{\pi} \iint_{A_2} \cos\beta_1 \, d\omega_1 \;, \tag{1}$$

where S is the length from dA_1 to dA_2, β_1 is the angle between the normal to dA_1 and S, β_2 is the angle between the normal to dA_2 and S, and $d\omega_1$ is the solid angle defined by:

$$d\omega_1 = \cos\beta_2 \frac{dA_2}{S^2} \;.$$

(Refer to Figure 3 for an illustration of terms). This solid angle geometry was introduced by Nusselt (1928) to reinterpret the configuration factor as:

$$F_{dA_1-A_2} = \frac{1}{\pi R^2} \iint_{A_2} \cos\beta_1 \, dA_s \;,$$

where A_s is the area A_2 projected onto the hemisphere of radius R centered at dA_1. This projection onto the hemisphere is accomplished along the lines S. Thus,

$$F_{dA_1-A2} = \frac{A_b}{\pi R^2} = \frac{\textit{projection of } A_s \textit{ onto the base of the hemisphere}}{\textit{area of a circle with the radius of the hemisphere}} \;. \tag{2}$$

Determination of an analytic solution to the configuration factor between an elemental area and a closed contour using Nusselt's Unit Sphere Method is simply a matter of analytic geometry. With the N vertices of the contour given by

$$\mathbf{P}_i = (X_i, Y_i, Z_i) \qquad i = 1, \ldots, N \qquad \mathbf{P}_{N+1} = \mathbf{P}_1 \;, \tag{3}$$

the following vectors may be defined:

$$\mathbf{P}_{i(i+1)} = (X_{i+1} - X_i, Y_{i+1} - Y_i, Z_{i+1} - Z_i) = \mathbf{P}_{i+1} - \mathbf{P}_i \;. \tag{4}$$

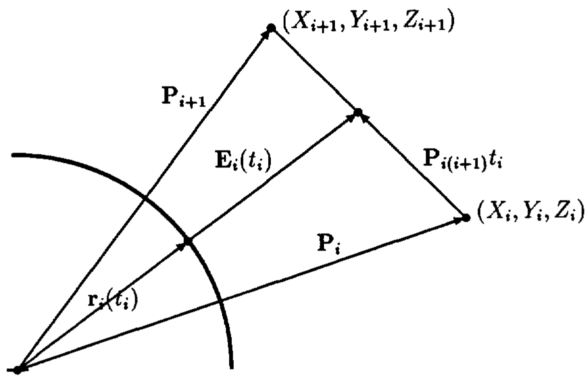

Fig. 4. Vectors Used in Edge Projection

These vectors describe the edges of the contour. To project these vectors onto the sphere of radius R, the direction cosines from the origin to the locus of points defining the edges of the contour must be found. The direction cosines are just the coordinates of the point divided by the distance from the origin. The edges of the contour may be parameterized as:

$$\mathbf{E}_i(t_i) = \mathbf{P}_i + \mathbf{P}_{i(i+1)}t_i \qquad 0 \le t_i \le 1 \ .$$

Thus the distance from the origin to points along the edges of the contour may be written as $\|\mathbf{E}_i(t_i)\|$, where $\|\cdot\|$ is the scalar magnitude of the vector quantity. This magnitude is:

$$\begin{aligned}\|\mathbf{E}_i(t_i)\| &= \left(\mathbf{E}_i(t_i)\cdot\mathbf{E}_i(t_i)\right)^{\frac{1}{2}} \\ &= \left[\mathbf{P}_i\cdot\mathbf{P}_i + 2\mathbf{P}_i\cdot\mathbf{P}_{i(i+1)}t_i + \mathbf{P}_{i(i+1)}\cdot\mathbf{P}_{i(i+1)}t_i^2\right]^{\frac{1}{2}} \ .\end{aligned}$$

The projection of the ith segment onto the sphere is defined by:

$$\mathbf{r}_i(t_i) = \frac{R}{\|\mathbf{E}_i(t_i)\|}\mathbf{E}_i(t_i) = (x_i(t_i), y_i(t_i), z_i(t_i)) \ ,$$

where $\mathbf{r}_i(t_i)$ describe the sphere point coordinates. Projecting these coordinates onto the $X-Y$ plane just means eliminating the z_i coordinate. The vector geometry is illustrated in Figure 4.

The area enclosed by these parametric curves will be determined using an application of Green's theorem. With $A = \iint_A dA = \frac{1}{2}\oint_C x\,dy - y\,dx$, x_i, y_i, dx_i, and dy_i, with $i = 1, \ldots, N$, are the terms necessary to determine the configuration factor. In general,

$$d\zeta_i = \frac{d\zeta_i}{dt_i}dt_i \qquad \text{for } \zeta = x,\ y, \text{ or } z \ .$$

Hence by differentiating each parametric curve with respect to t_i and then eliminating the Z component, the integration over the projected contour becomes a matter of summing the line integral contributions from individual segments. First note that:

$$\frac{d\mathbf{E}_i(t_i)}{dt_i} = \mathbf{P}_{i(i+1)} \qquad \text{and} \qquad \frac{d}{dt_i}\frac{1}{\|\mathbf{E}_i(t_i)\|} = -\frac{\mathbf{P}_{i(i+1)}\cdot\mathbf{E}_i(t_i)}{[\mathbf{E}_i(t_i)\cdot\mathbf{E}_i(t_i)]^{\frac{3}{2}}}$$

Thus the integrand can be rewritten as:

$$x_i\,dy_i - y_i\,dx_i = \left(x_i\frac{dy_i}{dt_i} - y_i\frac{dx_i}{dt_i}\right)dt_i \ ,$$

or, equivalently, the line integrand is equal to the Z component of the vector $\mathbf{r}_i(t_i)\times\frac{d\mathbf{r}_i(t_i)}{dt_i}$ multiplied by dt_i. The cross product is given by:

$$\begin{aligned}&\mathbf{r}_i(t_i)\times\frac{d\mathbf{r}_i(t_i)}{dt_i} \\ &= \frac{R}{\|\mathbf{E}_i(t_i)\|}\mathbf{E}_i(t_i)\times\left[\frac{R}{\|\mathbf{E}_i(t_i)\|}\mathbf{P}_{i(i+1)} - \frac{R\mathbf{P}_{i(i+1)}\cdot\mathbf{E}_i(t_i)}{\|\mathbf{E}_i(t_i)\|^3}\mathbf{E}_i(t_i)\right] \\ &= \frac{R^2}{\|\mathbf{E}_i(t_i)\|^2}\mathbf{E}_i(t_i)\times\mathbf{P}_{i(i+1)} \\ &= \frac{R^2}{\|\mathbf{E}_i(t_i)\|^2}\mathbf{P}_i\times\mathbf{P}_{i(i+1)} \ .\end{aligned}$$

The cross product terms in the previous equations do not depend on t_i, so they may be taken out of the integral. All that needs be integrated is $\frac{1}{\|\mathbf{E}_i(t_i)\|^2}$. The integration is over $0 \le t_i \le 1$. Therefore, Equation 2 may be written as:

$$\begin{aligned}&\int_0^1 \frac{dt_i}{\|\mathbf{E}_i(t_i)\|^2} \\ &= \frac{1}{\|\mathbf{P}_{i(i+1)}\|^2}\int_0^1 \frac{dt_i}{t_i^2 + 2\frac{\|\mathbf{P}_i\|}{\|\mathbf{P}_{i(i+1)}\|}\cos\beta_i t_i + \left(\frac{\|\mathbf{P}_i\|}{\|\mathbf{P}_{i(i+1)}\|}\right)^2} \\ &= \frac{1}{\|\mathbf{P}_{i(i+1)}\|^2}\int_0^1 \frac{dt_i}{\left(t_i + \frac{\|\mathbf{P}_i\|}{\|\mathbf{P}_{i(i+1)}\|}\cos\beta_i\right)^2 + \left(\frac{\|\mathbf{P}_i\|}{\|\mathbf{P}_{i(i+1)}\|}\sin\beta_i\right)^2} \\ &= \frac{1}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|\sin\beta_i}\int_{\frac{\pi}{2}-\beta_i}^{\tan^{-1}\left(\frac{\|\mathbf{P}_{i(i+1)}\|+\|\mathbf{P}_i\|\cos\beta_i}{\|\mathbf{P}_i\|\sin\beta_i}\right)} d\theta \\ &= \frac{1}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|\sin\beta_i}\left[\tan^{-1}\left(\frac{\|\mathbf{P}_{i(i+1)}\|+\|\mathbf{P}_i\|\cos\beta_i}{\|\mathbf{P}_i\|\sin\beta_i}\right) + \beta_i - \frac{\pi}{2}\right] \ ,\end{aligned}$$

where β_i is the angle between $\mathbf{P}_i$ and $\mathbf{P}_{i(i+1)}$.

At this point, some simplifications are in order. First,

$$\sin\beta_i = \frac{\|\mathbf{P}_i\times\mathbf{P}_{i(i+1)}\|}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|}$$
$$\cos\beta_i = \frac{\mathbf{P}_i\cdot\mathbf{P}_{i(i+1)}}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|} \ .$$

This greatly simplifies the argument of $\tan^{-1}$:

$$\begin{aligned}\frac{\|\mathbf{P}_{i(i+1)}\| + \|\mathbf{P}_i\|\cos\beta_i}{\|\mathbf{P}_i\|\sin\beta_i} &= \frac{(\mathbf{P}_{i(i+1)} + \mathbf{P}_i)\cdot\mathbf{P}_{i(i+1)}}{\|\mathbf{P}_i\times\mathbf{P}_{i(i+1)}\|} \\ &= \frac{\mathbf{P}_{i+1}\cdot\mathbf{P}_{i(i+1)}}{\|\mathbf{P}_i\times\mathbf{P}_{i+1}\|} \ .\end{aligned}$$

Also,

$$\begin{aligned}\frac{\mathbf{P}_i\times\mathbf{P}_{i(i+1)}\big|_Z}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|\sin\beta_i} &= \frac{\mathbf{P}_i\times\mathbf{P}_{i(i+1)}\big|_Z}{\|\mathbf{P}_i\times\mathbf{P}_{i(i+1)}\|} \\ &= \frac{\mathbf{P}_i\times\mathbf{P}_{i+1}\big|_Z}{\|\mathbf{P}_i\times\mathbf{P}_{i+1}\|} \ ,\end{aligned}$$

where $|_Z$ indicates the Z-component of the vector. Thus the configuration factor between the differential element dA and the closed contour C (in closed form) is:

$$\begin{aligned}F_{dA-C} = \frac{1}{2\pi}\Bigg|\sum_{i=1}^{N}&\frac{\mathbf{P}_i\times\mathbf{P}_{i+1}|_Z}{\|\mathbf{P}_i\times\mathbf{P}_{i+1}\|} \\ &\cdot\left[\tan^{-1}\left(\frac{\mathbf{P}_{i+1}\cdot\mathbf{P}_{i(i+1)}}{\|\mathbf{P}_i\times\mathbf{P}_{i+1}\|}\right) - \sin^{-1}\left(\frac{\mathbf{P}_i\cdot\mathbf{P}_{i(i+1)}}{\|\mathbf{P}_i\|\|\mathbf{P}_{i(i+1)}\|}\right)\right]\Bigg| \ ,\end{aligned} \tag{5}$$

where the vectors involved are defined by Equations 3 and 4. The absolute value is used to allow the contour to be traversed in either a positive or negative direction.

EXAMPLE

Figure 5 contains the microcomputer solution's graphic output for various orientations of a tetrahedron. The figure illustrates the ability of the program to determine which facets contribute to the configuration factor. This ability allows for the desired outer contour to be defined, thus allowing application of the closed form solution (Equation 5). To illustrate the use of the closed form solution, case iv from Figure 5 will be presented as an example.

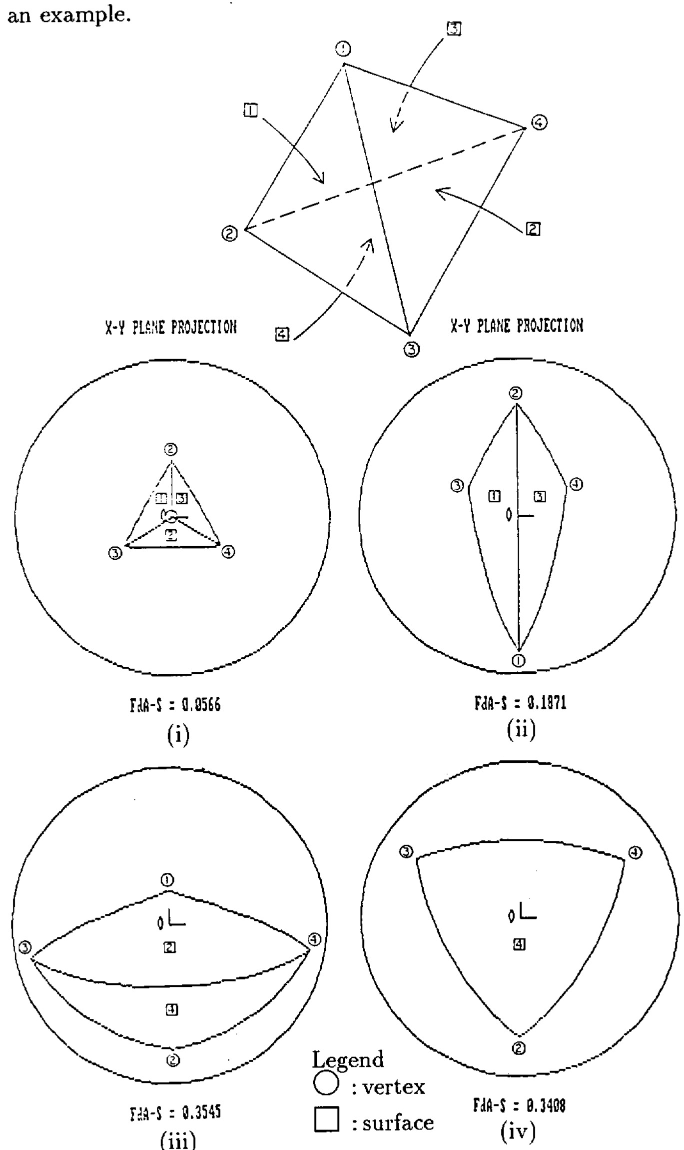

Fig. 5. Microcomputer Solution for a Tetrahedron

The contour vertices for case iv are defined by:

$$\mathbf{P}_1 = \mathbf{P}_4 = (0, -2/\sqrt{3}, 1) \quad \mathbf{P}_2 = (-1, 1/\sqrt{3}, 1) \quad \mathbf{P}_3 = (1, 1/\sqrt{3}, 1)$$

$$\|\mathbf{P}_1\| = \|\mathbf{P}_2\| = \|\mathbf{P}_3\| = \|\mathbf{P}_4\| = \sqrt{7/3}$$

The terms necessary for the use of Equation 5 are:

$$\mathbf{P}_1 \times \mathbf{P}_2 = (-\sqrt{3}, -1, -2/\sqrt{3}) \quad \mathbf{P}_2 \times \mathbf{P}_3 = (0, 2, -2/\sqrt{3})$$

$$\mathbf{P}_3 \times \mathbf{P}_4 = (\sqrt{3}, -1, -2/\sqrt{3})$$

$$\|\mathbf{P}_1 \times \mathbf{P}_2\| = \|\mathbf{P}_2 \times \mathbf{P}_3\| = \|\mathbf{P}_3 \times \mathbf{P}_4\| = 4/\sqrt{3}$$

$$\mathbf{P}_{12} = (-1, \sqrt{3}, 0) \qquad \mathbf{P}_{23} = (2, 0, 0) \qquad \mathbf{P}_{34} = (-1, -\sqrt{3}, 0)$$

$$\|\mathbf{P}_{12}\| = \|\mathbf{P}_{23}\| = \|\mathbf{P}_{34}\| = 2$$

$$\mathbf{P}_1 \cdot \mathbf{P}_{12} = \mathbf{P}_2 \cdot \mathbf{P}_{23} = \mathbf{P}_3 \cdot \mathbf{P}_{34} = -2$$

$$\mathbf{P}_2 \cdot \mathbf{P}_{12} = \mathbf{P}_3 \cdot \mathbf{P}_{23} = \mathbf{P}_4 \cdot \mathbf{P}_{34} = 2$$

For this example, all terms in the summation are identical. Calculating the i=1 term, we find:

$$\frac{\mathbf{P}_1 \times \mathbf{P}_2|_Z}{\|\mathbf{P}_1 \times \mathbf{P}_2\|}\left[\tan^{-1}\left(\frac{\mathbf{P}_2 \cdot \mathbf{P}_{12}}{\|\mathbf{P}_1 \times \mathbf{P}_2\|}\right) - \sin^{-1}\left(\frac{\mathbf{P}_1 \cdot \mathbf{P}_{12}}{\|\mathbf{P}_1\|\|\mathbf{P}_{12}\|}\right)\right]$$

$$= \frac{-2/\sqrt{3}}{4/\sqrt{3}}\left[\tan^{-1}\left(\frac{2}{4/\sqrt{3}}\right) - \sin^{-1}\left(\frac{-2}{\sqrt{7/3}\,2}\right)\right]$$

$$= -\frac{1}{2}\left[.7137 - (-.7137)\right] = -.7137 \ .$$

Therefore, the desired configuration factor is:

$$F_{dA-C} = \frac{1}{2\pi}\left|\sum\right| = \frac{3}{2\pi}(.7137) = .3408 \ .$$

This value can be verified with a closed form solution for a parallel polygon centered above a differential element (Howell, 1982, p. 31):

$$F_{dA-C} = \frac{nH}{\pi\sqrt{1+H^2}} \tan^{-1}\sqrt{\frac{R^2 - H^2}{1 + H^2}}, \qquad H = \frac{h}{\ell},\ R = \frac{r}{\ell}$$

where n is the number of vertices, r is the radius from the center to a vertex, h is the perpendicular distance between the center and a side, and ℓ is the height above the differential element. For the triangle above, $n = 3$, $r = \frac{2}{\sqrt{3}}$, $h = \frac{1}{\sqrt{3}}$, $l = 1$, and

$$F_{dA-C} = \frac{3}{2\pi}\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = .3408$$

which agrees identically with Equation 5.

CONCLUSIONS

A closed form solution was developed for the problem of finding the radiation configuration factor between an elemental area and a general 3-D contour. The contour referred to is defined as a closed, segmented curve which could be the perimeter of a surface, or the visible outer contour of a 3-D object. The method is therefore very general.

A microcomputer program was written to apply the closed form solution to the case of a general 3-D object. The algorithm requires that the solid object be represented in wireframe form (vertex, edge, and surface information). The program's output consists of the desired configuration factor, and graphical output relating to Nusselt's interpretation of the problem.

As with most fundamental work with radiation configuration factors, the results apply only to diffuse radiation traveling through a non-absorbing medium.

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