[5] It is straightforward to generalize this problem to the case of two parallel walls having arbitrary uniform velocities.

[6] It must be emphasized that this choice does not correspond to a complete closed cycle in the phase space, as the action integrals are commonly defined in conservative Hamiltonians. For a time-dependent Hamiltonian, there are no closed cycles in phase space but open cycles. In our example, the action integral must be defined on a there and back in the configuration space, the velocity of the ball having the same direction, in order to conserve the action integral.

[7] The center of the transformation may be inside or outside the cavity. We have not considered shapes in which at least one point is singular.

[8] G. Klein and M. P. Mulholland, Proc. R. Soc. London Ser. A **361**, 447 (1978).

[9] R. Arvieu and Y. Ayant, J. Phys. A **20**, 1115 (1987) and references cited therein.

[10] In a numerical experiment where the action integrals are calculated, as in the static case between the two turning points determined by the internal caustic and the boundary, we observe a nonconservation of them if the cavity is deformed.

[11] Obviously, it may be noticed that the temperature θ is increasing when the cavity is contracting, for which the reference time τ is positive.

[12] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953); see, for example, the discussion on the moving wall, p. 1123.

An alternative approach to the teaching of rotational dynamics

James A. Lock

Physics Department, Cleveland State University, Cleveland, Ohio 44115

An alternative approach to the stability of torque-free rotations and to top nutation that emphasizes conservation laws and physical reasoning is examined. It is felt that such an approach, which is largely free of sophisticated mathematics, would be useful in teaching these topics in rotational motion at the introductory physics level.

I. INTRODUCTION

At the noncalculus physics and freshman engineering physics levels, the subject of the dynamics of tops either is not treated at all, or only a derivation of the steady-state slow precessional frequency is given.[1] It is not until the junior–senior physics major course or mechanical engineering major course in advanced mechanics that topics such as rotational stability and the nutation of tops are analyzed. Since children observe the motion of toy tops and watch the rotation of blocks thrown in the air, at a relatively early age they can develop an amazement and curiosity as to why spinning objects behave in the strange ways that they do. In light of this, it is regrettable that the discussion of these curious rotational motions is delayed so long in the scientific education of the college student.

When such topics are eventually treated in advanced mechanics textbooks, they are usually examined in terms of the solutions of the Euler equations, the set of nonlinear coupled differential equations that describe rotational motion. The mathematical analysis of these equations explains in great detail *how* the unusual motion of tops comes about. But at the end of going through the mathematics, I remain secretly uncomfortable, feeling that I still have not come any closer to understanding physically *why* the top moves as it does.

This article is an attempt to answer the question physically of why rotating objects behave as they do in two easily observable situations. These are the stability of torque-free rotations and top nutation. The analyses require no differential equations or calculus. Rather, physical reasoning is emphasized and the only mathematics used is algebra and trigonometry. It is hoped that this approach can put the analysis of advanced but interesting rotational problems within the scope of a noncalculus-based physics course.

II. THE STABILITY OF TORQUE-FREE ROTATIONS

We consider a parallelepiped block whose symmetry axes passing through the center of mass are labeled 1, 2, and 3 as in Fig. 1. The moments of inertia about these axes are I_1, I_2, and I_3, respectively, and are assumed to be unequal. When the block is spun about one of these axes and is thrown in the air, it is observed that rotations about the largest and smallest moment of inertia symmetry axis are stable but that rotations about the intermediate moment of inertia symmetry axis are unstable.[2] We wish to understand why this stability or instability occurs.

When the block is spinning in the air, gravity acts through its center of mass and produces zero torque as measured by observers whose origins are at the center of mass. As a result, such observers see the angular momentum vector as being conserved. Since there are no dissipative forces, rotational kinetic energy is conserved as well. We consider two observers who describe the rotational motion of the block. The first observer A is an inertial observer who keeps track of the block's spatial orientation. The x, y, and z axes of a coordinate system parallel to A's coordinate axes and passing through the center of mass of the block are shown in Fig. 1. The 3-symmetry axis of the block has the direction indicated by the spherical coordinate system angles θ and ϕ. Once the 3-symmetry axis points in this direction, the orientation of the cross-sectional plane of the block containing the 1- and 2-symmetry axes is given by the
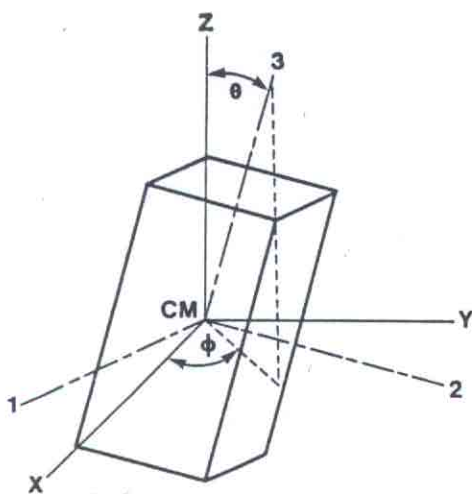
Fig. 1. A block with the symmetry axes, 1, 2, and 3 and the x, y, and z axes of an inertial observer. The 3-axis makes an angle θ with the z axis and its projection in the xy plane makes an angle ϕ with the x axis.

Fig. 3. An equivalent configuration of the block that has the same angular momentum vector and rotational kinetic energy. Its orientation and its spinning about the 3-axis have been reversed.

angle ψ. The second observer P keeps track of the block's spinning. P rides along on the block with his rectangular coordinate system axes nailed on top of the block's symmetry axes. He describes the spinning of the block in terms of the rotational velocities ω_1, ω_2, and ω_3 about the symmetry axes.

The initial conditions of the block's motion consist of specifying both its initial orientation and its initial spin. The block is taken to be initially spinning with the angular velocity $\omega_3(0)$ about the 3-symmetry axis, which is oriented along the z direction of observer A as in Fig. 2. Therefore, observer A sees the angular momentum vector as being

$$L = I_3\omega_3(0)\hat{u}_z \tag{1}$$

and the rotational kinetic energy as being

$$KE = \tfrac{1}{2}I_3\omega_3^2(0). \tag{2}$$

We call this the $[\theta = 0°, \omega_3(0)]$ configuration of the block. There is another configuration of the block, which possesses exactly the same angular momentum vector and the
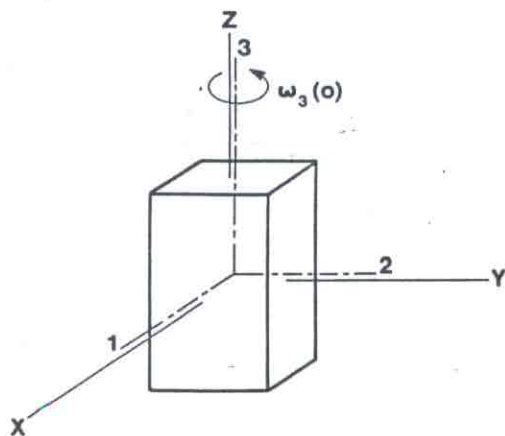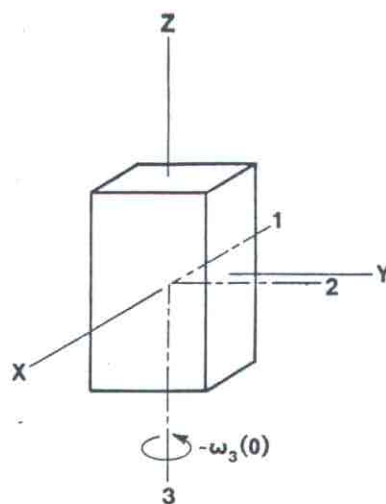

Fig. 2. The initial conditions of the block. It spins with the angular velocity $\omega_3(0)$ about the 3-axis, which is in the z direction.

same rotational kinetic energy. This is when the block's orientation is reversed and its spinning with respect to the 3-symmetry axis is reversed as in Fig. 3. We call this the $[\theta = 180°, -\omega_3(0)]$ configuration. Since both configurations are totally equivalent, when the block is spun and thrown in the air, are the initial conditions enough to specify which one of them will be dominant in the resulting motion?

This question is answered in terms of the orientation paths available to the block. A path is analogous to the frames of a movie that starts with the block in the $[\theta = 0°, \omega_3(0)]$ configuration and shows it turning until it is finally in the $[\theta = 180°, -\omega_3(0)]$ configuration. If a path between these two configurations can be found that conserves the angular momentum vector and the rotational kinetic energy at every step along the way, then the initial conditions are not enough to specify which of the equivalent configurations dominates the motion. Rather, the block traverses the path of orientations back and forth, oscillating between the two. This represents an unstable rotation. On the other hand, if there is no path between the two that conserves the angular momentum vector and the rotational kinetic energy at every step along the way, the initial conditions are compatible with only one of the configurations. The turning of the block to the other configuration is forbidden. This represents a stable rotation.

There are many paths between the two configurations for which the angular momentum vector is conserved. If the block is at some moment in the (θ, ϕ, ψ) orientation and its spinning is given by

$$\omega_1 = -(I_3/I_1)\omega_3(0)\sin\theta\cos\psi,$$
$$\omega_2 = (I_3/I_2)\omega_3(0)\sin\theta\sin\psi, \tag{3}$$
$$\omega_3 = \omega_3(0)\cos\theta,$$

then the angular momentum vector seen by observer A is given by Eq. (1). However, it is difficult to make any of these paths conserve the rotational kinetic energy

$$KE = \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2 \tag{4}$$

at every step along the way as well. At the point on the path specified by Eq. (3), conservation of the rotational kinetic
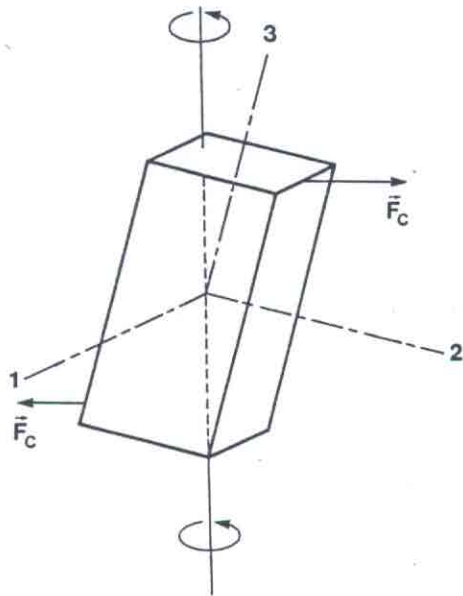
Fig. 4. The initial spinning of the block approximately about the z axis. $\vec{F}_c$ is the centrifugal force acting on the asymmetric mass concentrations.

Fig. 5. The initial conditions of the top. It spins with the angular velocity $\omega_3(0)$ about the 3-axis, which is in the xy plane of an inertial observer. The base support S is fixed and the tangential direction T is the direction of precession.

energy requires that Eq. (3) substituted into Eq. (4) must be equal to the initial kinetic energy of Eq. (2). When this is done, the θ dependence cancels out and one is left with

$$\sin^2 \psi = I_2(I_1 - I_3)/I_3(I_1 - I_2). \tag{5}$$

The condition that $0 \leqslant \sin^2 \psi \leqslant 1$ further requires that either $I_1 > I_3 > I_2$ or $I_2 > I_3 > I_1$. Thus only when I_3 is the intermediate moment of inertia can one find a path of orientations between the two equivalent configurations that conserves both energy and angular momentum. When I_3 is the intermediate moment of inertia, what gives the block the initial push to start it along the path between the two equivalent configurations? The answer is found in the fact that when one throws the block in the air, it is difficult to release it so that it spins *exactly* about the 3-axis.[2] It is more likely that the released block will end up spinning *approximately* about the 3-axis as shown in Fig. 4. Since the block is slightly asymmetric in its initial spinning, the extra concentrations of mass with respect to the spin axis in the upper right and lower left of Fig. 4 will be pulled perpendicular to the spin axis by centrifugal force. This force is denoted by $\mathbf{F}_c$ in Fig. 4 and it causes an additional rotation of the block about the 1-axis. The combination of the rotation of the block about the spin axis and the rotation about the 1-axis produced by centrifugal force starts the block moving along the path of allowed orientations oscillating between the two equivalent configurations.

III. TOP NUTATION

We consider a symmetric top with the base point fixed and whose center of mass is a distance h from the base point. The moments of inertia with respect to the 1-, 2-, and 3-axes of Fig. 5 are $I_1 = I_2 = I_{12}$ and I_3. The top is spun with the speed $\omega_3(0)$ about the 3-symmetry axis. If the top is given a small tangential push as it is released, it is observed that it does not fall due to gravity. Instead, it precesses uniformly. If the top is released without the small tangential push, it is observed that for a short time its cen-
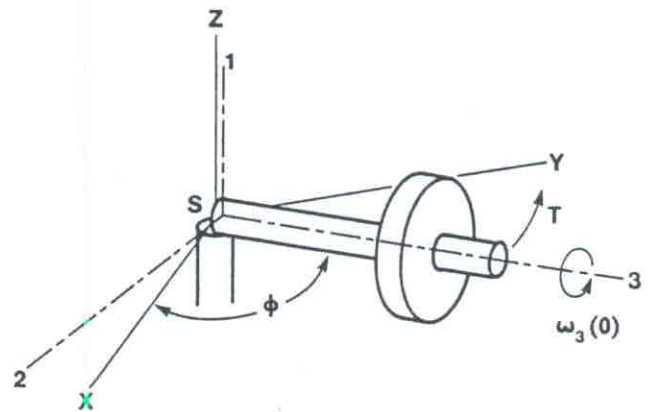
ter of mass does fall due to gravity. But then the center of mass reaches a low point and the top bounces back upwards as if it were bouncing off of something that we cannot see. This vertical bouncing of the center of mass is called nutation. We wish to understand why the nutation bouncing occurs and what the top seems to be bouncing off.

Initially the top is oriented with its 3-symmetry axis in the xy plane of observer A whose origin is now at the base of the top as in Fig. 5. Gravity again acts through the top's center of mass and produces a torque in the xy plane perpendicular to the 3-axis. This perpendicular or tangential direction T is shown in Fig. 5. Since the gravitational torque points in the T direction, the angular momentum in that direction changes and gives rise to the precession of the top.[1] The reaction forces at S produce no torque on the top with respect to the coordinate system whose origin is at S. Thus the gravitational torque is the only torque in the problem and the portion of the angular momentum vector perpendicular to T is conserved. In this example, this perpendicular portion is the plane containing the z axis and the 3-symmetry axis. These conserved components of the an-
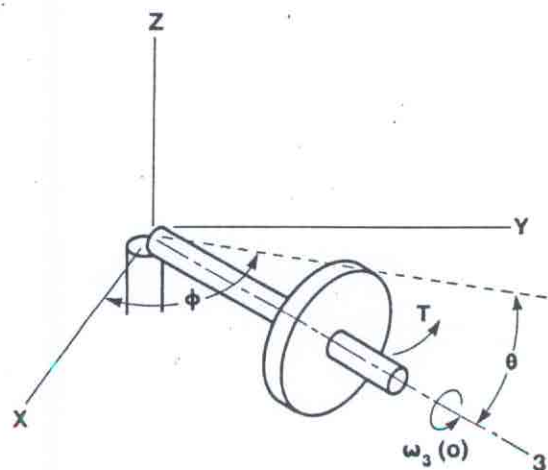

Fig. 6. The fallen configuration of the top. The 3-axis is an angle θ below the xy plane and the top precesses in the tangential direction T with the angular velocity ω_{ds}.
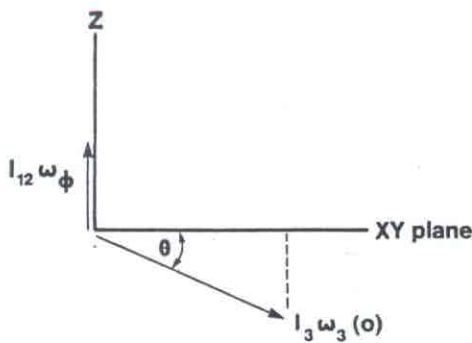
Fig. 7. The angular momenta of spinning and precession when the top is in the fallen configuration. Conservation of angular momentum requires that $L_z = 0$.

gular momentum are then

$$L_3 = I_3\omega_3(0),$$
$$L_z = 0,$$
(6)

as seen by observer A. Since gravity initially causes the top's center of mass to fall, a path of orientations between the initial configuration of Fig. 5 and the fallen configuration of Fig. 6 must exist, which conserves L_3, L_z, and energy at every step along the way. It should be noted in Fig. 6 that as a matter of convenience the angle θ is now taken to be the angle of the 3-symmetry axis below the xy plane.

Again, it is easy to find a path that conserves L_z. In Fig. 6, in order to balance the downward component of the angular momentum due to spinning, $I_3\omega_3(0)\sin\theta$, the top must precess in the tangential direction with the angular speed ω_ϕ generating an upward angular momentum, $I_{12}\omega_\phi$. This is shown in Fig. 7. Conservation of L_z then requires that

$$\omega_\phi = (I_3/I_{12})\omega_3(0)\sin\theta.$$
(7)

In order to conserve L_3 as well, the top must increase slightly its spinning speed. However, this increase is proportional to $\sin^2\theta$, which is small and can be ignored since we assume that θ remains near $0°$. If ω_θ is the angular velocity of the falling of the top in the direction of increasing θ, then the initial energy is

$$E = \tfrac{1}{2}I_3\omega_3^2(0)$$
(8)

and the energy in the fallen configuration is

$$E = \tfrac{1}{2}I_3\omega_3^2(0) + \tfrac{1}{2}I_{12}\omega_\phi^2 + \tfrac{1}{2}I_{12}\omega_\theta^2 - mgh\sin\theta.$$
(9)

Simultaneous conservation of energy and L_z for the path requires that

$$\tfrac{1}{2}I_{12}\omega_\theta^2 = mgh\sin\theta - \tfrac{1}{2}(I_3^2/I_{12})\omega_3^2(0)\sin^2\theta.$$
(10)

Equations (9) and (10) have two interesting physical interpretations. The first one deals with why falling motion of the top ceases. As the top falls, its potential energy is converted into kinetic energy. The kinetic energy is of two types; the kinetic energy of falling, $\tfrac{1}{2}I_{12}\omega_\theta^2$, and the kinetic energy of precession, $\tfrac{1}{2}I_{12}\omega_\phi^2$. The decrease in potential energy goes as $\sin\theta$ but the increase in precessional kinetic energy goes as $\sin^2\theta$ in order to conserve L_z. For small θ, $\sin^2\theta$ increases much slower than $\sin\theta$ and so the difference between the potential energy loss and the precessional kinetic energy gain goes into the kinetic energy of falling. However, for somewhat larger θ, $\sin^2\theta$ increases faster

than $\sin\theta$, the precessional kinetic energy claims all the gravitational potential energy that is released, and nothing is left over for the kinetic energy of falling.[3,4] As a result, when

$$\sin\theta = 2mghI_{12}/I_3^2\omega_3^2(0),$$
(11)

the lowest point in the falling is reached. Below this angle, the kinetic energy of the falling motion cannot obtain any of the potential energy loss in order that the falling continue.

The second interpretation of Eqs. (9) and (10) deals with why the top bounces upward when the low point of Eq. (11) is reached. For small angles θ, the behavior of the top in the θ direction is as if it were subject to the potential energy[5]

$$U = -mgh\sin\theta + \tfrac{1}{2}(I_3^2\omega_3^2(0)/I_{12})\theta^2.$$
(12)

This potential energy may be interpreted as arising from the torque due to gravity and the torque produced by an effective torsional spring whose stiffness constant is

$$K = I_3^2\omega_3^2(0)/I_{12}$$
(13)

and which always tries to restore the orientation of the top to its initial condition of $\theta = 0°$. In a number of situations in mechanics, it is useful to speak of fictitious forces or pseudoforces such as centrifugal force or the Coriolis force. An inertial observer does not consider these entities as true forces. Rather, he considers them to be part of the acceleration of the object. However, to a noninertial observer riding along with the object, these entities feel like forces and are compensated for by various passive forces such as tension or reaction forces. The tendency of the top to try to maintain its initial conditions can be interpreted as another fictitious force. To an inertial observer, $\tfrac{1}{2}K\theta^2$ is really the precessional kinetic energy. But, to a noninertial observer riding along with the precession and watching only the falling motion of the top, it would appear that the top uses the spring pseudotorque to attempt to recover its initial orientation when gravity has pulled the 3-axis lower than $\theta = 0°$. The faster the top is spinning, the larger K becomes, and the larger the pseudotorque of initial conditions maintenance becomes.[6] This explains the rebound of the top from its low point. The effective spring torque is pulling the top up try-
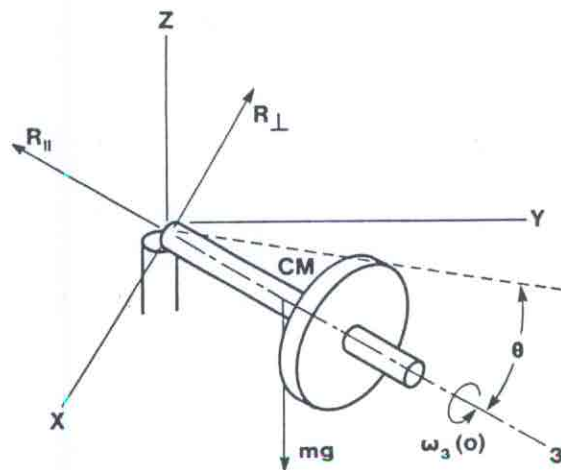

Fig. 8. The force of gravity and the reaction forces $R_\perp$ and $R_\parallel$ provided by the support when the top is in the fallen configuration.

ing to recover its initial orientation just as a mass hung vertically from a spring and released is pulled back up from its low point by the force of the extended spring.

As mentioned earlier, the effects of pseudoforces are observed in the reaction forces necessary to balance them. This compensation also occurs for the springlike pseudotorque of Eq. (12). Figure 8 shows the top in its fallen orientation and indicates the reaction forces $R_\perp$ and $R_\parallel$, perpendicular and parallel to the 3-symmetry axis, provided by the base support. If α_θ is the angular acceleration of the falling motion, then the motion of the center of mass is given by

$$mh\alpha_\theta = mg \cos \theta - R_\perp. \tag{14}$$

However, gravity and the effective torsional spring exert the torques

$$\tau = I_{12}\alpha_\theta = mgh \cos \theta - [I_3^2\omega_3^2(0)/I_{12}]\theta. \tag{15}$$

Combining Eqs. (14) and (15), we obtain

$$R_\perp = mg \cos \theta - \frac{m^2gh^2}{I_{12}} \cos \theta + \frac{mh}{I_{12}}\left(\frac{I_3^2\omega_3^2(0)}{I_{12}}\theta\right). \tag{16}$$

The first two terms of Eq. (16) correspond to the value that the reaction force would have if the top were not spinning and were merely falling over. $R_\perp$ would balance a component of gravity except for the fact that the reaction is reduced by the downward acceleration of the falling top. The last term in Eq. (16) describes the increase in $R_\perp$ due to the

support having to balance also the pressure exerted on it by the restoring pseudotorque when the top has been perturbed from its initial configuration. The effective lever arm of the increased reaction force is I_{12}/mh.

IV. CONCLUSION

Again, it should be emphasized that in each of these two situations no mathematics beyond trigonometry and the definition of spherical coordinates was employed. The analyses instead were based on physical intuition and the requirements of conservation laws. It is hoped that with such a point of view, many topics in advanced mechanics that deal with easily observable phenomena may be introduced to physics students before they reach the junior–senior-level course in advanced mechanics. If this were the case, the advanced course could then focus on supplying the mathematical framework for phenomena that the students were already familiar with from a physical point of view.

[1]J. Higbie, Phys. Teach. **18**, 210 (1980).
[2]J. Walker, Phys. Teach. **19**, 57 (1981).
[3]W. Case, Am. J. Phys. **45**, 1107 (1977).
[4]P. L. Tea, Phys. Teach. **26**, 22 (1988).
[5]A. P. French, *Newtonian Mechanics* (Norton, New York, 1971), pp. 691–694.
[6]Another interpretation of the restoring torque is given in E. F. Barker, Am. J. Phys. **28**, 808 (1960) and in P. M. Rinard, Am. J. Phys. **34**, 365 (1966).

Conserved vectors and orbit equations for autonomous systems governed by the equation of motion r̈ + f ṙ + gr = 0

V. M. Gorringe and P. G. L. Leach
Centre for Nonlinear Studies and Department of Computational Applied Mathematics, The University of the Witwatersrand, 2050 Wits Post Office, Republic of South Africa[a]

The existence of conserved vectors that are analogous to the conserved angular momentum and Laplace–Runge–Lenz vectors of the classical Kepler problem makes it possible to determine the orbit equations for a wide variety of autonomous systems governed by the equation of motion of the form $\ddot{r} + f\dot{r} + gr = 0$.

I. INTRODUCTION

Kaplan[1] has recently pointed out that the Laplace–Runge–Lenz vector is in a sense an "inessential constant" of the motion for the classical Kepler problem since the existence of the conserved energy and angular momentum is sufficient to derive the orbit equation. In that article, Kaplan derived the Laplace–Runge–Lenz vector for the classical Kepler problem by assuming the existence of a first integral that is quadratic in the velocities. We shall see that the existence of such a vector is rather more wide-

spread and need not be quadratic in the velocities. The procedure we have adopted demonstrates that constants of the motion for the classical Kepler problem can in fact be calculated in a systematic way without any *a priori* knowledge of the functional form of these constants. We describe how these ideas can be extended to deal with more general problems and derive analogs of the classical angular momentum, Hamilton's vector, and the Laplace–Runge–Lenz vector of the classical Kepler problem.

It is well known that the orbit equations for the two great classical intergrable systems, the Kepler problem with