epts was the ability
ectly in performing
ons of the instanta-
o balls rolling along
e similar to the ve-
demonstration de-
the demonstrations
reinstructional per-
e preparatory phys-
nd calculus-based

sics course showed
n both the velocity
velocity compari-
d be made directly
he tasks required a
of speed and posi-
o the use of a posi-
n the speeds of the
cused on the posi-
enerally identified
ier as a time when
views, 90% of the
irse were able to
efore instruction
the students en-
no postcourse im-
s rate.
the students were
two balls rolling
e inclined at the
he two balls start-
at the end of the
al speed. By com-
e time interval or,
ding to the same
ecide which ball
performance on
ocity, change of
guished from one
comparison, the
al explicitly and
ration as the ra-
irse, none of the
ng the precourse
about 40% re-
ares favorably to
dents in a tradi-
mproved from a
% afterwards.
ind acceleration
ntributed to the
lents in the pre-
comparison task
is unlikely that
norization. Al-
demonstration
ave been able to
t distinguishing
id acceleration.
n the interview
portant kind of
ply kinematical
y small change
calculus-based

physics shows, this ability often does not develop spontaneously during a typical introductory course. For many students, the traditional emphasis on algebraic formalism does not lead to conceptual understanding.

Through special instruction that emphasized the application of the kinematical concepts to actual motions, students in the preparatory physics course were able to achieve a level of understanding that matched that of students who had a stronger background in physics but who received a more traditional type of instruction. It seems reasonable to assume that many of the better prepared students who participated in our study would also have benefitted from some of the same instructional strategies that proved effective with the less well-prepared students.

The students in the preparatory physics course initially had a great deal of difficulty in translating back and forth between actual motions and their graphical representations.[2] By the end of the kinematics portion of the course, their performance on examination questions indicated considerable improvement. About three-fourths of the class could obtain data from a demonstrated motion and plot a correct v vs t graph and about the same fraction could design an arrangement of tracks to produce a motion depicted on a v vs t graph. The written responses of the students showed that they had carefully considered the instants when the velocity was zero, the periods of positive and negative velocity, and the time intervals during which the velocity was changing.

The examples used to illustrate the instructional approach described in this paper are taken from a curriculum that emphasizes the development of a sound qualitative understanding of the kinematical concepts.[4] Central to such an understanding is the concept of velocity as a continuously varying quantity, of instantaneous velocity as a limit, and of uniform acceleration as the ratio of the change in instantaneous velocity to the elapsed time. The concepts of position, velocity, change of velocity, and acceleration should be clearly distinguished from one another. A functional understanding entails the ability to make connections among the various kinematical concepts, their representations, and the motions of real objects. To be properly addressed, each of these instructional objectives requires an intensive effort over an extended period of time. Not all concepts, of course, can be treated in an introductory physics course in the detail outlined in this paper. However, the kinematical concepts are of sufficient importance to warrant special attention. In addition to providing a basis for the study of dynamics, the concepts of velocity and acceleration are often used to introduce various instantaneous rates, not only in physics but in other disciplines.

ACKNOWLEDGMENTS

The authors would like to thank Emily H. van Zee for her substantive contributions to the work described in this paper. The research and curriculum development projects on which the paper is based have been partially supported by the National Science Foundation under grants SED81-12924 and DPE84-70081.

[a] Present address: Rocky Mountain Division, Shell Oil Company, P. O. Box 831, Houston, TX 77001.

[1] D. E. Trowbridge and L. C. McDermott, "Investigation of Student Understanding of the Concept of Velocity in One Dimension," Am. J. Phys. **48**, 1029 (1980); "Investigation of Student Understanding of the Concept of Acceleration in One Dimension," Am. J. Phys. **49**, 242 (1981).

[2] L. C. McDermott, M. L. Rosenquist, and E. H. van Zee, "Student Difficulties in Connecting Graphs and Physics," Am. J. Phys. (to be published).

[3] M. L. Rosenquist, Ph.D. Dissertation, University of Washington, 1982 (unpublished).

[4] M. L. Rosenquist and L. C. McDermott, *Kinematics* (ASUW Publishing, University of Washington, Seattle, 1985).

[5] A. H. Halloun and D. Hestenes, "The Initial Knowledge State of College Physics Students," Am. J. Phys. **53**, 1043 (1985); "Common Sense Concepts About Motion," Am. J. Phys. **53**, 1056 (1985).

[6] P. W. Hewson, "Diagnosis and Remediation of an Alternative Conception of Velocity Using a Microcomputer Program," Am. J. Phys. **53**, 684 (1985).

[7] J. Lochhead, Department of Physics and Astronomy, University of Massachusetts, Amherst, MA (private communication).

[8] A. B. Arons, "Student Patterns of Thinking and Reasoning, Part Two," Phys. Teach. **22**, 21 (1984).

Rigid-body dynamics of a football

Peter J. Brancazio
Department of Physics, Brooklyn College, City University of New York, Brooklyn, New York 11210

(Received 24 June 1985; accepted for publication 4 June 1986)

The motion of a spinning football provides interesting illustrations of some of the principles of rigid-body motion. The familiar "wobbly spiral" is an example of torque-free precession; it is shown that the "wobble-to-spin" ratio depends only on the principal moments of inertia of the football. The angular momentum of a spinning football is not always conserved in flight. The response of a football to aerodynamic forces can best be understood by comparing its motion to that of a spinning top or gyroscope.

I. INTRODUCTION

The subject of rigid-body motion, as taught in the traditional mechanics course for physics majors, often presents difficulties for students and teachers alike. The concepts of rigid-body motion are often difficult to explain qualitatively and are not easily related to familiar, every-day phenomena. Consider, for example, the torque-free precession of a rotating body, which is typically presented as a solution of Euler's equations of motion. The standard illustration of this effect is the Chandler wobble of the Earth[1-3]—yet in the ensuing discussion, one learns that the predicted period of the Earth's free precession differs rather substantially from the observed value!

One attractive way to illustrate torque-free precession (as well as other concepts of rigid-body motion) is to examine the flight of a football—a motion that is very familiar to most North American students. (I am referring, of course, to the American football rather than the European version, which, being spherical, does not have interesting rigid-body dynamics.) A properly thrown forward pass must be "spiraled"—that is, the football must be given a substantial spin about its long axis. In fact, the familiar "wobbly spiral"—a poorly thrown pass in which the nose of the football circles about a line passing through its center of mass—is a fine example of torque-free precession.[4] As we shall see, the application of the methods of rigid-body dynamics to the motion of a football reveals the cause of the wobbly spiral pass (a useful bit of information for the weekend quarterback) and also yields an unexpected but readily verifiable prediction about the rate of precession. Moreover, an interesting analogy can be drawn between the motion of a spinning top and the response of a spiraling football to aerodynamic drag.

II. THEORY OF TORQUE-FREE PRECESSION

The theory of torque-free precession may be found in any of the standard classical mechanics texts. In the absence of external torques, Euler's equations become

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= 0,\\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= 0,\\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= 0, \end{aligned} \tag{1}$$

where I_x, I_y, and I_z are the principal moments of inertia with respect to the axes of symmetry, and ω_x, ω_y, and ω_z are the components of the angular velocity $\boldsymbol{\omega}$ of the body with respect to its own axes.

Let us assume that the z axis coincides with the long axis of the football (the xyz axes represent the "body" coordinate system, whose origin is fixed to the center of mass of the football). To a good aproximation, a football is a cylindrically symmetric rigid body. If we ignore the small perturbing effects of the laces and inflation valve, it is readily apparent that $I_x = I_y \equiv I_{xy}$, and $I_{xy} > I_z$. Under these conditions, the most general solution of Euler's equations may be expressed as follows:

$$\begin{aligned} \omega_x &= C\sin(\Omega t + \delta),\\ \omega_y &= C\cos(\Omega t + \delta),\\ \omega_z &= \omega_{z0} = \text{constant}, \end{aligned} \tag{2}$$

where

$$\Omega = [(I_{xy} - I_z)/I_{xy}]\omega_{zo}$$

and C and δ are constants of integration. They may be evaluated from the initial conditions: if $\omega_x = \omega_{x0}$ and $\omega_y = \omega_{y0}$ at $t = 0$, then $C = (\omega_{x0}^2 + \omega_{y0}^2)^{1/2}$ and $\delta = \tan^{-1}(\omega_{xo}/\omega_{yo})$. It follows immediately that $\boldsymbol{\omega}$ must have a constant magnitude. Given the time dependence of ω_x and ω_y, $\boldsymbol{\omega}$ must precess about the x axis with an angular frequency Ω. Moreover, the direction of precession is *opposite* to the spin direction—that is, if ω_z is aligned with the

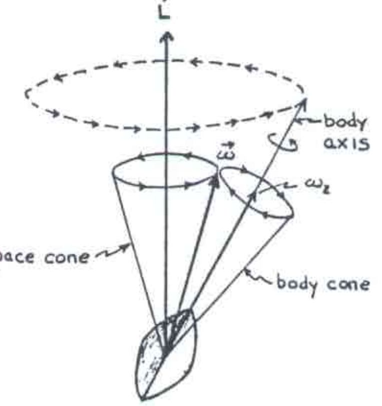

Fig. 1. Precession of ω in an inertial reference frame. In this representation, L is fixed in space and ω precesses about L. As it does so, the body cone (whose axis is fixed to the long axis of the football) rolls on the space cone. Thus the body axis precesses about L in the same direction as it is spinning.

positive z axis (corresponding to a counterclockwise spin), $\boldsymbol{\omega}$ will be seen to be turning clockwise.

An attempt at this point to compare the motion described above with the motion of a football may cause some confusion. Careful observation of the precessional motion of a real wobbly spiral reveals that the *direction of the precession is the same as the spin direction*. As seen by the receiver of the pass, the spin and precession are both counterclockwise if the football has been thrown by a righthander, or both clockwise if the passer is left-handed.

This apparent contradiction between theory and observation arises because Euler's equations describe the phenomena *as seen in a coordinate system fixed to the body*. This is the precessional motion as it would be seen by, let us say, a small bug who is firmly attached to the football (this is, in fact, the point of view taken in the explanation of the Chandler wobble of the Earth). To represent the wobbly spiral as it is seen on the field, we must view the motion from an inertial reference frame. It should be noted, therefore, that Ω does *not* correspond to the precession frequency of the wobble.

If there are no external torques acting on the football, then its angular momentum **L** must be constant with a fixed direction in inertial space, and $\boldsymbol{\omega}$ will precess about **L** at a fixed angle. In the traditional representation, $\boldsymbol{\omega}$ traces out the "space cone" as it precesses around **L** and the "body cone" as it precesses around the body axis (**L**, $\boldsymbol{\omega}$, and the body axis always lie in the same plane). Thus the body cone (whose axis coincides with the long axis of the football) rolls without slipping on the outside of the space cone (Fig. 1). It is easy to see that if $\boldsymbol{\omega}$ is turning in a clockwise sense with respect to the body axis, the body axis is turning in a counterclockwise sense about **L**. Thus as seen in an inertial reference frame, the spin and the precession of the football are in the same direction, in agreement with observations.

The frequency of the precession can be obtained by using the Eulerian angle transformations. We define the $x'y'z'$ axes as the space axes, with the angular momentum vector aligned with the z' axis. The xyz axes represent the body axes, with the z axis coinciding with the long axis of the football. In terms of the standard Eulerian angles, θ is the angle between the long axis of the football and its angula momentum vector. The rate of precession of the z axis about the z' axis is measured by $\dot{\phi}$, the rate of change of the first Eulerian angle. Thus $\dot{\phi}$ represents the observed precession frequency ω_p of the wobbly spiral.

The evaluation of $\dot{\phi}$ proceeds as follows: in terms of the Eulerian angles, **L** has the following components relative to the body (xyz) axes:

$$
\begin{aligned}
L_x &= I_{xy}\omega_x = L \sin\theta \sin\psi, \\
L_y &= I_{xy}\omega_y = L \sin\theta \cos\psi, \\
L_z &= I_z\omega_z = L \cos\theta.
\end{aligned} \tag{3}
$$

The components $\omega_x, \omega_y, \omega_z$ relative to the body axes are related to the rates of change of the Eulerian angles (the angular velocities $\dot{\theta}, \dot{\phi}$, and $\dot{\psi}$) by the following equations[5]:

$$
\begin{aligned}
\omega_x &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\
\omega_y &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi, \\
\omega_z &= \dot{\phi} \cos\theta + \dot{\psi}.
\end{aligned} \tag{4}
$$

By combining these equations, we arrive at the following simple equation for the precession frequency ω_p:

$$\omega_P \equiv \dot{\phi} = L/I_{xy}. \tag{5}$$

From Eqs. (2), (3), and (5) we obtain

$$\omega_P = \omega_{z0}\left[\left(\frac{I_z}{I_{xy}}\right)^2 + \frac{(\omega_{x0}^2 + \omega_{y0}^2)}{\omega_{z0}^2}\right]^{1/2}. \tag{6}$$

We can also obtain an expression for the angle θ between the long axis and the direction of **L**:

$$\cos\theta = \frac{L_z}{L} = \left[1 + \left(\frac{I_{xy}}{I_z}\right)^2 \frac{(\omega_{x0}^2 + \omega_{y0}^2)}{\omega_{z0}^2}\right]^{-1/2}. \tag{7}$$

III. THE DYNAMICS OF THE WOBBLY SPIRAL

We have identified the wobble of a poorly thrown forward pass as a torque-free precession of the football whose frequency and amplitude are given by Eqs. (6) and (7). But is the flight of a football truly torque-free? The motion of a football in flight is affected by gravity and aerodynamic forces. While gravity cannot produce a torque on an unsupported football, aerodynamic forces can and do produce torques. The magnitude and direction of these torques depend upon how the aerodynamic forces are distributed over the surface of the football. At the very least, we would expect air resistance to slow the rate of spin of the ball. However, if the ball is traveling a short distance at a relatively slow speed and the long axis is not too steeply inclined to its trajectory, the torques due to aerodynamic forces will have a negligible effect on the trajectory and spin of the football. In these circumstances, the angular momentum vector maintains a constant magnitude and direction in space. The precessional motion is thereby torque-free and independent of the trajectory of the football. In Sec. V we shall consider what happens when the motion is not torque-free.

What, then, causes the football to wobble in flight? Examining the equations of torque-free precession, we see that *there is no precession if* $\omega_{x0} = \omega_{y0} = 0$. With these initial conditions, $\omega_x = 0$ and $\omega_y = 0$ for all t. From Eq. (7) we find that $\cos\theta = 1$; consequently both **L** and **ω** are aligned with the long axis of the football. This corresponds to the so-called "bullet pass" that is the hallmark of a good quarterback. A precession will arise if the ball is given even a small rotational motion about a second axis. Thus the wobble is a consequence of the rotational motion imparted when the ball is launched.

To throw a perfect (nonwobbling) spiral, the ball must be launched with a spin about its long axis only. In terms of throwing technique, the wrist must remain locked as the ball spins off the fingers; any slight rotation of the wrist at the moment of release will give the football some angular velocity about a transverse axis. A typical wobbly spiral is released with a substantial spin about its long axis and a very small spin about one or both of its transverse axes. That is, $\omega_{z0} \gg \omega_{x0}$ and ω_{y0}. To a good approximation, Eq. (6) reduces to

$$\omega_p/\omega_{z0} = I_z/I_{xy}. \tag{8}$$

That is, the "wobble-to-spin ratio"—the ratio of the rate of precession to the rate of spin about the long axis—is a fixed quantity that depends only on the principal moments of inertia.

A. Principal moments of inertia of a football

Numerical values for I_z and I_{xy} for a football can be obtained by calculation or by direct measurement, and it is of interest to see how closely the results compare. The principal moments of inertia can be calculated if we make the reasonable approximation that a football is an ellipsoid of revolution about its long axis (i.e., a prolate spheroid). For a solid, homogeneous ellipsoid of mass density ρ, the principal moments are

$$
\begin{aligned}
I_z &= 8\pi\rho a b^4/15, \\
I_{xy} &= 4\pi\rho a b^2(a^2 + b^2)/15,
\end{aligned} \tag{9}
$$

where a and b are the radii measured along the major and minor axes, respectively. For a hollow ellipsoid of shell thickness t, the principal moments may be computed by evaluating $I = I(a+t, b+t) - I(a,b)$. If the ellipsoid is thin-walled (i.e., $a,b \gg t$) then the principal moments of inertia are, to first order in t,

$$
\begin{aligned}
I_z &= 8\pi\rho b^3 t(4a + b)/15, \\
I_{xy} &= 4\pi\rho b t\,[(a+b)^3 + a(a^2 + b^2)]/15.
\end{aligned} \tag{10}
$$

Consequently the ratio of the two principal moments of a thin-shelled, hollow ellipsoid is

$$\frac{I_z}{I_{xy}} = \frac{2(1+4R)}{(1+R)^3 + R(1+R^2)}, \quad R = \frac{a}{b}. \tag{11}$$

An offical American football is 11–11¼ in. long and 6.73–6.85 in. in diameter. These dimensions yield average values of $a = 14.1$ cm, $b = 8.6$ cm. Thus $R = 1.64$ and $I_z/I_{xy} = 0.618$.

To obtain absolute values of I_z and I_{xy}, we must estimate the density ρ of the football shell. Since the shell thickness t is relatively small, $\rho = m/St$, where m and S are the mass and surface area of the football. An official football weighs 14.5 ounces, equivalent to 0.411 kg. The surface area of an ellipsoid of revolution is given by the equation

$$S = 2\pi b(be + a\sin^{-1} e)/e, \quad e = (a^2 - b^2)^{1/2}/a. \tag{12}$$

Thus the substitution $\rho t = m/S$ can be made in Eqs. (10), and values for I_z and I_{xy} can be calculated. The results are

$$
\begin{aligned}
I_z &= 0.00212 \text{ kg m}^2, \\
I_{xy} &= 0.00343 \text{ kg m}^2.
\end{aligned}
$$

The principal moments can be measured directly by suspending the football on a torsional pendulum and measuring its period of oscillation. With an apparatus that he previously used to determine the moments of inertia of tennis rackets,[6] Brody has measured I_z and I_{xy} for a standard professional football. He obtained the following values[7]:

$$I_z = 0.00194 \text{ kg m}^2,$$
$$I_{xy} = 0.00321 \text{ kg m}^2,$$
$$I_z/I_{xy} = 0.604.$$

The calculated values are high, but within 10% of the measured values. This difference may be attributed in part to the fact that a football has a more tapered shape (and pointier ends) than an ellipsoid. Nevertheless, the calculated and measured ratios differ by only 2%. Considering the assumptions made in the calculations, the agreement is surprisingly good.

B. The "wobble-to-spin" ratio

According to Eq. (8), the wobble-to-spin ratio of a football is determined only by the size and shape of the football, provided that the amplitude of the wobble is not too large. Substituting numerical values for I_z and I_{xy}, we obtain $\omega_p/\omega_{z0} \approx 0.6$. That is, to a good approximation, *there should be three wobbles to every five spins, independent of the rate of spin, linear speed, or trajectory of the football.* [According to Eqs. (6) and (7), if ω_{z0} exceeds ω_{x0} or ω_{y0} by a factor of five or more, then $0.6 < \omega_p/\omega_{z0} < 0.63$, and $0^0 < \theta < 18^0$.] The demonstration of this phenomenon does not require a passer, receiver, and a football field; it can be done in a classroom simply by tossing a football lightly in a vertical direction. With a little ingenuity, it is not too difficult to set up a procedure to measure the wobble-to-spin ratio. (I have been able to verify the 3:5 ratio for a few forward passes that were videotaped in slow motion replays.) However one chooses to demonstrate this effect, the example of the wobbly spiral provides a very instructive and off-beat illustration of the phenomenon of torque-free precession.

IV. THE "END-OVER-END" KICK

As a corollary to the discussion of torque-free motion, it should be noted that precession can occur only if the football is given an initial spin about its long axis. Because of the cylindrical symmetry ($I_x = I_y$), the solution of Euler's equations when $\omega_{z0} = 0$ is simply

$$\omega_x = \omega_{x0}, \quad \omega_y = \omega_{y0}, \quad \omega_z = 0.$$

Thus both $\boldsymbol{\omega}$ and $\mathbf{L}$ are constant; i.e., the football will not precess. This solution is illustrated by making the football spin "end-over-end" about a transverse axis. During a game this motion is normally seen on a kickoff. Here the football is kicked off a tee, and the kicker deliberately aims his foot at a point below the center of mass so as to set the football spinning about a transverse axis. Unless the football is affected by strong torques, its flight will be especially stable; its spin axis will maintain a constant direction in space without any precessional motion.

V. THE FOOTBALL AS GYROSCOPE: RESPONSE TO AERODYNAMIC FORCES

If there are no torques acting on a football during its flight, then its angular momentum vector should maintain a constant direction with respect to the ground. However, careful observation of football passes and kicks reveals that $\mathbf{L}$ does not always maintain a constant direction over the entire trajectory. In general, the motion of a spinning football will fall into one of three categories (see Fig. 2):

Type I: The angular momentum vector of the football maintains a fixed orientation over the entire trajectory.

Type II: The angular momentum vector maintains a fixed orientation until shortly after passing the peak of the trajectory; then the ball suddenly turns over and descends nose down, often wobbling strongly.

Type III: The angular momentum vector pivots continuously, remaining tangent to the trajectory throughout the flight.

For all three types, the football may also be wobbling as it travels. Thus a football sometimes exhibits both torque-free precession and gyroscopic precession (i.e., a precession caused by external torques) in flight. Since these two effects have different causes and occur independently, let us assume for simplicity that the ball is launched without a wobble, so that the angular momentum vector coincides with the long axis.

In general, the type of trajectory followed by the football depends upon such factors as the launching speed and angle, the rate of spin, and the orientation of the long axis of the football with respect to its trajectory. The aerodynamic force that acts on the football arises from the uneven distribution of air pressure over the surface of the ball. This pressure distribution can be resolved into a single resultant force that acts at a point known as the center of pressure. The magnitude of this force will depend on the air speed of the football (more specifically, on the square of the speed) as well as on the orientation of the long axis of the football to the airflow. In fact, for the same speed, the aerodynamic

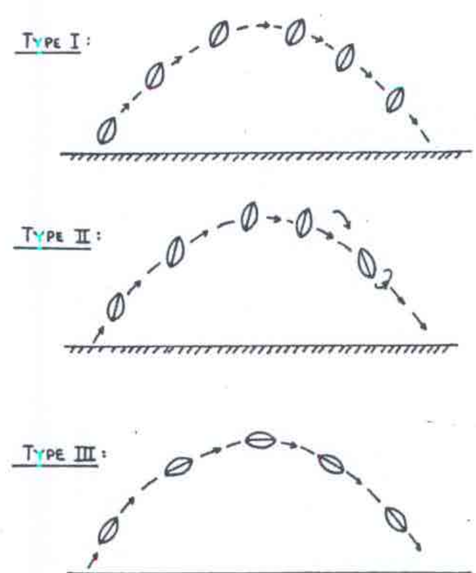

Fig. 2. Classification of football trajectories. The angular momentum vector maintains a constant direction on a type I trajectory, but not on type II or type III.

ootball during its
r should maintain
ground. However,
kicks reveals that
direction over the
of a spinning foot-
(see Fig. 2):
or of the football
tire trajectory.
ctor maintains a
g the peak of the
ver and descends

or pivots continu-
y throughout the

o be wobbling as
bits both torque-
n (i.e., a preces-
. Since these two
dependently, let
unched without a
vector coincides

ed by the football
ng speed and an-
f the long axis of
The aerodynamic
he uneven distri-
of the ball. This
a single resultant
nter of pressure.
n the air speed of
are of the speed)
is of the football
the aerodynamic

lar momentum vec-
y, but not on type II

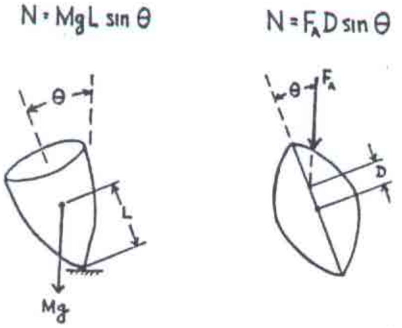

Fig. 3. Comparison of torques on a spinning top and a football. The line of action of the aerodynamic torque F_A intersects the long axis of the football a distance D from the center of the football, producing a torque $F_A D \sin\theta$ about the center.

force will be about 10 times greater when the football is traveling broadside (long axis perpendicular to its trajectory) than when it is traveling "nose first"—i.e., with its long axis parallel to its trajectory.[8]

In the most general case, the line of action of the aerodynamic force will intersect the long axis at a distance D from the center of mass. Thus the aerodynamic force produces a torque $N = F_A D \sin\theta$ about the center of mass, where θ is the angle between the line of action of $\mathbf{F}_A$ and the long axis (see Fig. 3). However, the magnitude and direction of $\mathbf{F}_A$ and the location of the center of pressure (and hence the value of D) both vary over the course of flight in response to changes in the air speed and orientation of the football. For certain orientations the line of action of $\mathbf{F}_A$ will pass through the center of mass, resulting in no aerodynamic torque. This will occur, for example, when the football is traveling in the nose-first orientation.

When a football is launched without spin, it tends to tumble slowly in flight, indicating that a small aerodynamic torque is acting. A spinning football, on the other hand, becomes a gyroscope and tends to undergo precession rather than tumbling. Indeed, as Fig. 3 shows, the aerodynamic torque on a football about its center takes the same mathematical form as the torque due to gravity on a spinning top about its point of support. Consequently, the classical analysis of the spinning top can be used to give some insight into the motion of the spiraling football.

The theory of the symmetrical top is too lengthy to reproduce here; it can be found in any classical mechanics textbook. Accordingly, we will simply summarize the results that are relevant to this discussion:

(1) The motion of a top depends upon the stability factor

$$k_T = I_z^2\omega_z^2/4MgLI_{xy}, \tag{13}$$

where I_z and I_{xy} are the moments of inertia about the long axis and transverse axis, respectively, and ω_z is the spin rate.

(2) If a spinning top is set at an angle θ_0 to the vertical and then released, it will undergo stable precession, provided that $k_T \geqslant 1$. In general, the axis of the top will also exhibit nutation; that is, the angle θ will oscillate between two values θ_0 and θ_1 as the top precesses. During nutation the precession is not steady, but the average precession frequency is

$$\dot{\phi} = MgL/I_z\omega_z. \tag{14}$$

(3) If a top is launched with its axis vertical and with $k_T \geqslant 1$, the top will not precess and its axis will remain vertical (this is referred to as a "sleeping" top). Moreover, if the axis of the top is disturbed slightly, it will move back toward the vertical.

The spiraling football is analogous, but not identical, to a spinning top. From the football's point of view, the "vertical" is the direction of the aerodynamic force $\mathbf{F}_A$, as shown in Fig. 3. The analogous stability factor for a spiraling football is

$$k_F = I_z^2\omega_z^2/4F_A D I_{xy} \tag{15}$$

and the average precession rate is

$$\dot{\phi} = F_A D/I_z\omega_z. \tag{16}$$

These equations predict that if the football is spinning rapidly and the aerodynamic torque is relatively small, then k_F will be large and $\dot{\phi}$ small, indicating that the precession of the football should be stable but slow. Here, however, the identity with the simple top ends, because $\mathbf{F}_A$ and D are not constant. Thus the gyroscopic motion of the football is considerably more complicated than that of a spinning top, and an analytic solution is beyond reach. Nevertheless, we can still draw upon the analogy between the two to arrive at some qualitative conclusions.

On a type I trajectory, the axis of the football maintains a constant direction in space. Typically, these trajectories are characterized by fairly low launching speeds and relatively short times of flight. In this case the aerodynamic torque is rather small, so the period of precession will be long compared to the flight time. Thus there is no perceptible precession of the spin axis; this is essentially torque-free flight.

The type II trajectory is typical of punts. They are generally launched at fairly steep angles (50° to 60° to the horizontal) with the nose of the ball tipped upwards relative to its trajectory. The flight times are typically about 4 to 5 s. The football appears to maintain a constant inclination past the peak of its trajectory, whereupon it suddenly "turns over" and undergoes precession. We can understand this behavior in the following way: as the ball rises on its steep trajectory it loses speed rapidly, so the aerodynamic torque is relatively small on the ascent. By the descending part of the trajectory, the angle θ has grown rather large; the football is now almost broadside to the air flow and is gaining speed as it falls. This creates a sudden increase in the aerodynamic torque and initiates a precession of the football about its trajectory.

The type III trajectory represents the "perfect spiral" in which the long axis is always aligned with its trajectory. The football exhibits a remarkable self-correcting motion; there is a continuous change in the direction of the angular momentum vector so as to maintain a nose-first orientation over the entire trajectory. Whether the football is thrown or kicked for long or short distances, on a flat or a steep trajectory, the key feature is that the football is launched with its long axis parallel to its trajectory. Thus the line of action of $\mathbf{F}_A$ passes through the center of the ball and there is no aerodynamic torque.

In this configuration, the football is similar to a sleeping top. The curvature of the trajectory shifts the line of action away from the center to create a small torque. This is equivalent to a small disturbance in which the top is tipped slightly from the vertical. Just as the top responds by returning to the vertical, the football responds by turning its spin axis into the direction of its trajectory, thereby reducing the aerodynamic torque to zero. The only requirement is that $k_F > 1$; since the football experiences its minimum aerodynamic force in this orientation, k_F should be rather large if the ball is thrown with a lot of spin.

VI. EPILOGUE

A substantial fraction of physicists and physics students are football fans. The presentation of the subject of rigid body dynamics can be enlivened considerably by using the motion of a football to illustrate key principles and applications of the theory. Students can be encouraged to test these out first-hand by throwing a football around or while watching football games on television. Last but not least, the identification of a football as a sometime gyroscope definitely adds a new dimension to one's enjoyment of the game. The enlightened observer will come to look at forward passes, kickoffs, and punts with a whole new perspective.

ACKNOWLEDGMENTS

The author wishes to thank Cliff Frohlich (University of Texas, Austin) and an anonymous reviewer for helpful suggestions and corrections.

[1] H. Goldstein, *Classical Mechanics*, (Addison-Wesley, Reading, MA, 1980), 2nd ed., p. 210.

[2] R. Baierlien, *Newtonian Dynamics* (McGraw-Hill, New York, 1983), p. 260.

[3] J. Marion, *Classical Dynamics* (Academic, New York, 1970), 2nd ed., p. 392.

[4] The wobbly spiral as an illustration of torque-free precession is mentioned very briefly in only one of the well-known classical mechanics textbooks (Ref. 2, p. 258).

[5] Reference 1, p. 176.

[6] H. Brody, Phys. Teach. **23**, 213 (1985).

[7] H. Brody (personal communication).

[8] P. J. Brancazio, Phys. Teach. **23**, 403 (1985).

Magnetic force due to a current-carrying wire: A paradox and its resolution

Denise C. Gabuzda

Department of Physics, Brandeis University, Waltham, Massachusetts 02254

(Received 10 February 1986; accepted for publication 2 July 1986)

A straightforward investigation at an introductory level of the interconnection between electricity and magnetism initially leads to the paradoxical result that a charge at rest with respect to a current-carrying wire feels a magnetic force due to that current. Students may benefit from a presentation of this paradox and its resolution.

I. INTRODUCTION

Many students consider concepts in electrostatics simpler and more intuitive than concepts in magnetostatics. In fact, there is a sense in which electrostatic phenomena may be considered more fundamental and less complicated than magnetostatic phenomena. Electrostatic problems concern the case of all charges at rest in the lab frame; magnetostatic problems concern the more complicated case of charges not generally at rest and in motion relative to each other in the lab frame. But, in fact, all magnetic fields and forces acting on a charge may be understood in terms of electric fields and forces felt by the charge in its own rest frame.

Presentation of this aspect of the connection between electrostatics and magnetostatics is sometimes included in introductory courses. It may prove valuable both in giving students a sense of the unity of electricity and magnetism and in helping them become more comfortable with magnetism; the problem typically considered is that of a point charge moving near a current-carrying wire. If this problem is straightforwardly approached by first stating the wire to be uncharged without current flowing, and then considering the current in the wire to be a perturbation due to electron motion at the drift velocity with the ions in the metal lattice remaining at rest in the lab frame, a paradox arises; a charge that is at rest in the lab frame is found to experience a magnetic force. Symmetries[1] or other simplifying assumptions[2] which serve partially to avoid this paradox are used in some texts. More benefit may come to the students by a presentation of both the paradox and its resolution than by avoiding it in the interest of simplicity.

Section II is concerned with the arisal of the paradox itself; Sec. III develops its resolution.

II. THE PARADOX

Consider the force felt by a point charge near a current-carrying wire. This force can be explained by modeling the wire as a superposition of positive (ion) and negative (electron) charge distributions, and looking at the apparent net charge distribution in the wire, as seen in the rest frame of the point charge.

If there is no current in the wire and the point charge is stationary in the lab frame, the wire as seen in the rest frame of the point charge is electrically neutral; this is simply a statement of the neutrality of unionized atoms. If there is no current in the wire but the point charge is moving along the wire, as in Fig. 1(b), the ion and electron distributions will both appear to be Lorentz contracted by the same

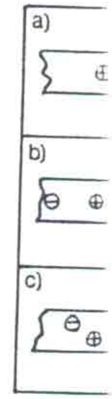

Fig. 1. Appar... The circled p... and v_- are th... (a) correspon... sponds to q ... current in the... the right with... velocity of th...

amount as ... the wire wil...

If there i... moving alo... and electro... be different... tracted by ... appear neut... 1(c), the ve... point charg... distribution... left, where v... the current... might initia... density as se...

$\lambda'_+ = \lambda_0$...

and the appa...

$\lambda'_- = -$...

where λ_0 an... electrons in ... charge densi... this case and... found a desc... moving near... by the charg...

An appare... the charge q... above framew... case: it sees t... tron distribu... and therefore... electric force... an arbitrary ... force whene... charge is zero... occur whene... tions are the... charge:

$v = (v + v$...