4.2 THE HOMOGENEOUS LINEAR EQUATION WITH CONSTANT COEFFICIENTS

A. Introduction

In this section we consider the special case of the nth-order homogeneous linear differential equation in which all of the coefficients are real constants. That is, we shall be concerned with the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{4.23}$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are real constants. We shall show that the general solution of this equation can be found explicitly.

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{4.24}$$

This equation is called the *auxiliary equation* or the *characteristic equation* of the given differential equation (4.23). If $y = e^{mx}$ is a solution of (4.23) then we see that the constant m must satisfy (4.24). Hence, to solve (4.23), we write the auxiliary equation (4.24) and solve it for m. Observe that (4.24) is formally obtained from (4.23) by merely replacing the kth derivative in (4.23) by m^k $(k = 0, 1, 2, \ldots, n)$. Three cases arise, according as the roots of (4.24) are real and distinct, real and repeated, or complex.

THEOREM 4.11

Consider the nth-order homogeneous linear differential equation (4.23) with constant coefficients. If the auxiliary equation (4.24) has the n distinct real roots $m_1, m_2, \ldots, m_n$, then the general solution of (4.23) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

THEOREM 4.12

1. Consider the nth-order homogeneous linear differential equation (4.23) with constant coefficients. If the auxiliary equation (4.24) has the real root m occurring k times, then the part of the general solution of (4.23) corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{mx}.$$

2. If, further, the remaining roots of the auxiliary equation (4.24) are the distinct real numbers $m_{k+1}, \ldots, m_n$, then the general solution of (4.23) is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{mx} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}.$$

3. If, however, any of the remaining roots are also repeated, then the parts of the general solution of (4.23) corresponding to each of these other repeated roots are expressions similar to that corresponding to m in part 1.

THEOREM 4.13

1. Consider the nth-order homogeneous linear differential equation (4.23) with constant coefficients. If the auxiliary equation (4.24) has the conjugate complex roots $a + bi$ and $a - bi$, neither repeated, then the corresponding part of the general solution of (4.23) may be written

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx).$$

2. If, however, $a + bi$ and $a - bi$ are each k-fold roots of the auxiliary equation (4.24), then the corresponding part of the general solution of (4.23) may be written

$$y = e^{ax}[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) \sin bx + (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1}) \cos bx].$$