Investigations into Uncertain Control Co-Design Implementations for Stochastic in Expectation and Worst-Case Robust

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Introduction

Motivations and Objective

- Control co-design (CCD) refers to the **integrated** consideration of the **physical** and **control** system design through optimization
- Deterministic CCD has been studied in the literature[1]
- However, since some of the elements of CCD problem are **uncertain**,methods from uncertain CCD (UCCD) are needed
- Implementation challenges, implicit assumptions, and in-depth discussion of the structure of UCCD problems, method-dependent considerations, and practical insights are currently missing from the literature
- This study fills some of these gaps by using a simple strained-actuated solar array (SASA)[2] to
  - Introduce two optimal, open-loop control structures under uncertainties
  - Implement and solve a stochastic in expectation UCCD (SE-UCCD) using Monte Carlo simulation (MCS) and generalized Polynomial Chaos (gPC) expansion
  - Implement and solve a worst-case robust UCCD (WCR-UCCD) using bounded representation of uncertainties

[1] Herber and Allison 2019; Allison and Herber 2014 [2] Herber and Allison 2017; Chilan et al. 2017

A Universal UCCD Formulation

A universal UCCD formulation defined in probability space (specialized forms can be derived though the appropriate selection of the objective function and constraints[1]):

- $\tilde{\bullet}$ is a time-independent uncertain variable
- $\tilde{\bullet}(t)$ is a stochastic process
- $\bar{\bullet}(\cdot)$ is a function composition of $\bullet(\cdot)$, e.g.,
  - $\bar{o}(\cdot)$ is a function of the original objective function $o(\cdot)$
  - $\bar{\boldsymbol{g}}(\cdot)$ is a function of the original inequality constraint vector $\boldsymbol{g}(\cdot)$

A Universal UCCD Formulation

$$
\begin{aligned}
\underset{\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}}}{\text{minimize:}} \quad & \mathbb{E}\left[\bar{o}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] \\
\text{subject to:} \quad & \mathbb{E}\left[\bar{\boldsymbol{g}}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}})\right] \leq \boldsymbol{0} \\
& \boldsymbol{h}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \\
& \dot{\tilde{\boldsymbol{\xi}}}(t) - \boldsymbol{f}(t,\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{p}},\tilde{\boldsymbol{d}}) = \boldsymbol{0} \\
\text{where:} \quad & \tilde{\boldsymbol{u}}(t) = \tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{\xi}}(t) = \tilde{\boldsymbol{\xi}},\ \tilde{\boldsymbol{d}}(t) = \tilde{\boldsymbol{d}} \\
& \tilde{\bullet} \in \mathcal{V}_u,\ \tilde{\bullet}(t) \in \mathcal{T}_u(t)
\end{aligned}
$$

[1] Azad and Herber 2022

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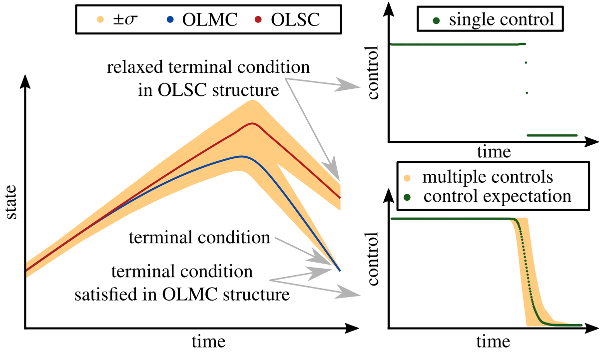
UCCD Implementation

→ Open-Loop Optimal Control Structures

- **Open-loop single control** (OLSC) finds a single control command that meets some criteria and is closely related to concepts from robust control theory.
- **Open-loop multiple control** (OLMC) elicits a range of optimal control responses based on the realization of uncertainties

Open-loop optimal control structures



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Uncertainty Propagation Methods

→ Uncertainty Propagation Methods

- A generalized Polynomial Chaos (gPC) expansion was used for uncertainty propagation and benchmarked against Monte Carlo simulation results
- In gPC expansion, elements in an arbitrary random vector $\tilde{\boldsymbol{x}}$ must have mutual independence [1]
- The univariate gPC basis functions of degree up to r_i are denoted as $\{\phi_k(\tilde{x}_i)\}_{k=0}^{r_i}$, and satisfy the orthogonality conditions
- The set of univariate orthogonal basis functions are obtained based on the probability distribution of $\tilde{\boldsymbol{x}}$ [2]
- A tensor product of elements in $\{\phi_k(\tilde{x}_i)\}_{k=0}^{r_i}$ is used to construct the n_x-variate gPC basis functions $\Phi_m(\tilde{\boldsymbol{x}})$

[1] Loeve 1978; Rosenblatt 1952 [2] Xiu 2010

→ Generalized Polynomial Chaos

- The resulting polynomials span the space of

$$\left\{\Phi_m(\tilde{\boldsymbol{x}})\right\}_{m=0}^{M-1} = \bigotimes_{|\boldsymbol{k}|\leq PC} \left\{\prod_{i=1}^{n_x} \phi_k(\tilde{x}_i)\right\}$$

- PC is either the highest polynomial order in each direction, or alternatively, is the total degree of a subset of basis elements
- Any second-order variable or process $\tilde{y}(t,\tilde{\boldsymbol{x}})$ can be expressed by polynomial chaos of PC degree as:

$$\tilde{y}(t,\tilde{\boldsymbol{x}}) \approx y_{PC}(t,\tilde{\boldsymbol{x}}) = \sum_{m=0}^{M-1} \hat{y}_m(t)\Phi_m(\tilde{\boldsymbol{x}})$$

- The unknown coefficients $\hat{y}_i(t)$ are estimated through a Galerkin projection [1] or a collocation formulation [2]

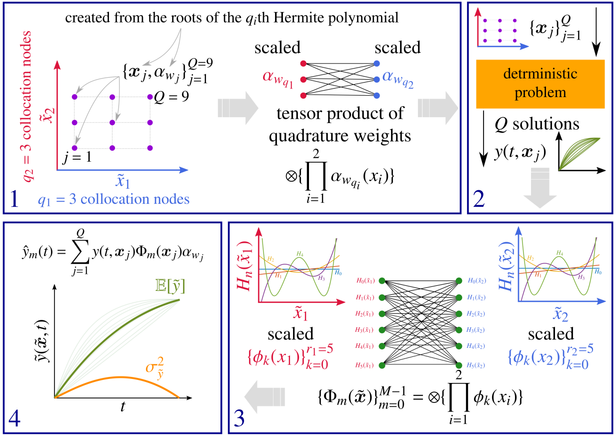
[1] Xiu 2010; Wang et al. 2019 [2] Cottrill 2012

Generalized Polynomial Chaos

The unknown coefficients are obtained from through a quadrature rule (and thus collocation points) from

$$\hat{y}_m(t) = \mathbb{E}\left[y(t,\boldsymbol{x})\Phi_j(\tilde{\boldsymbol{x}})\right] = \int_\Gamma y(t,\boldsymbol{x})\Phi_j(\boldsymbol{x})dF_{\tilde{x}}(\boldsymbol{x}) \approx \sum_{j=1}^{Q} y(t,\boldsymbol{x}_j)\Phi_m(\boldsymbol{x}_j)\alpha_{w_j}$$

Steps involved in gPC



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Simple SASA UCCD Formulations

→ Deterministic Simple SASA CCD

- Simplified strain-actuated solar array (SASA) system for spacecraft pointing control and jitter reduction [1]

$$\underset{u,\boldsymbol{\xi},k}{\text{minimize:}} \quad -\xi_1(t_f)$$

$$\text{subject to:} \quad u - u_{max} \leq 0$$

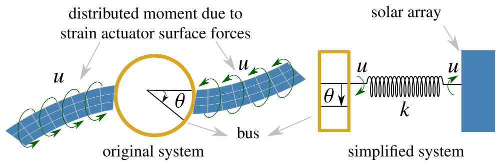
$$u_{min} - u \leq 0$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u$$

$$\boldsymbol{\xi}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \xi_2(t_f) = 0$$

$$\text{where:} \quad u(t) = u, \ \boldsymbol{\xi}(t) = \boldsymbol{\xi}$$

Original and Simplified SASA system



- Plant: stiffness of the solar array k
- Control: strain actuation $u(t)$
- State: relative displacement & velocity $\boldsymbol{\xi}(t)$
- Problem data: inertia ratio J

[1] Herber and Allison 2017

→ Stochastic in Expectation UCCD (SE-UCCD)

- Uncertainties are

$$\tilde{k} \sim \mathcal{N}(\mu_k, \sigma_k)$$
$$\tilde{J} \sim \mathcal{N}(\mu_J, \sigma_J)$$
$$\tilde{\xi}_{2,t_0} \sim \mathcal{N}(\mu_{\xi_{2,t_0}}, \sigma_{\xi_{2,t_0}})$$

- k_s is the constraint shift index
- Dynamics are satisfied $a.s.$ or almost surely
- Terminal b.c. applied only when using a OLMC structure
- Risk-neutral formulation

SE-UCCD

$$\begin{aligned}
\underset{u,\tilde{\boldsymbol{\xi}},\mu_k}{\text{minimize:}} \quad & -\mathbb{E}[\tilde{\xi}_1(t_f)] \\
\text{subject to:} \quad & u - u_{max} \leq 0 \\
& u_{min} - u \leq 0 \\
& k_s\sigma_k - \mu_k \leq 0 \\
& \begin{bmatrix} \dot{\tilde{\xi}}_1 \\ \dot{\tilde{\xi}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\tilde{k}}{\tilde{J}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tilde{J}} \end{bmatrix} u \quad \text{(a.s.)} \\
& \tilde{\boldsymbol{\xi}}(t_0) = \begin{bmatrix} 0 \\ \mathcal{N}(\mu_{\xi_{2,t_0}}, \sigma_{\xi_{2,t_0}}) \end{bmatrix} \\
& \tilde{\xi}_2(t_f) = 0 \quad \text{(if OLMC)} \\
\text{where:} \quad & \tilde{k} = \mathcal{N}(\mu_k, \sigma_k), \quad \tilde{J} = \mathcal{N}(\mu_J, \sigma_J) \\
& u(t) = u, \quad \tilde{\boldsymbol{\xi}}(t) = \tilde{\boldsymbol{\xi}}
\end{aligned}$$

Worst-Case Robust UCCD (WCR-UCCD)

A risk-averse formulation described in epigraph form with bounded uncertainties

WCR-UCCD

Outer loop problem

$$\begin{aligned}
\underset{u,\mu_k,v}{\text{minimize:}} \quad & -v \\
\text{subject to:} \quad & v - \Phi\left(t, u, \tilde{\xi}, \tilde{k}(\mu_k), \tilde{J}, \tilde{\xi}_{2,t_0}\right) \leq 0 \\
& \hat{u} - u_{max} \leq 0 \\
& u_{min} - \hat{u} \leq 0 \\
& k_s \sigma_k - \mu_k \leq 0 \\
\text{where:} \quad & u(t) = u
\end{aligned}$$

Inner loop problem

$$\begin{aligned}
\underset{\boldsymbol{\xi},k,J,\xi_{2,t_0}}{\text{minimize:}} \quad & o_{in} = \xi_1(t_f) \\
\text{subject to:} \quad & \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u \\
& \boldsymbol{\xi}(t_0) = \begin{bmatrix} 0 \\ \xi_{2,t_0} \end{bmatrix} \\
& -0.3 \leq \xi_2(t_f) \leq 0.3 \\
\text{where:} \quad & k(\mu_k) \in \mathcal{R}_k, \quad J(\mu_J) \in \mathcal{R}_J \\
& \xi_{2,t_0}(\mu_{\xi_{2,t_0}}) \in \mathcal{R}_{\xi_{2,t_0}}, \quad \boldsymbol{\xi}(t) = \boldsymbol{\xi}
\end{aligned}$$

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Results and Discussion

→ Results

- A nested coordination strategy used for OLMC-SE-UCCD
- Inner-loop optimal control subproblem solved using direct transcription (DT)
- All DT implementations done in MATLAB-based *DTQP* toolbox [1]
- A direct single shooting (DSS) used for the outer-loop WCR-UCCD
- Implementations available on Github [2]

Table: Settings for UCCD implementations.

Category	**Option**	**Value**
General	defects	trapezoidal (TR)
	mesh	equidistant
	quadrature	composite TR
	outer-loop solver	fmincon
	solver tolerance	10^{-6}
SE-UCCD	inner-loop solver	quadprog
	derivatives	symbolic
	n_t	100
	N_{mcs}	$10,000$
	Q	10^3
	r_i	8
	M	9^3
WCR-UCCD	inner-loop solver	fmincon
	derivatives	forward
	n_t	100

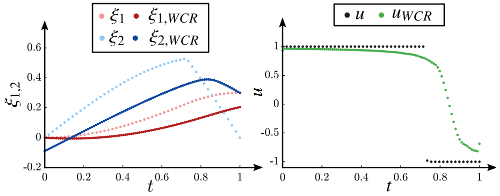
[1] Herber 2017 [2] Azad 2022

→ Results

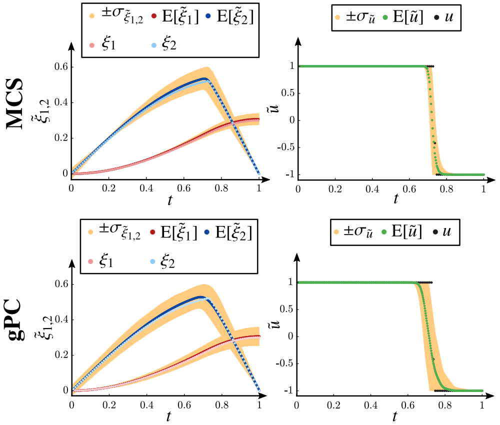
Table: UCCD solutions.

Formulation	Structure	$\tilde{o}$	μ_k	$t(s)$	t_{switch}
CCD	OLSC	-0.301	3.441	3	0.727
Stc-MCS	OLMC	-0.308	3.311	5717	0.737
Stc-gPC	OLMC	-0.306	3.185	562	0.742
$\|\Delta\|$	-	0.65%	3.81%	90%	0.68%
WcR	OLSC	0.204	0.705	2592	0.838

WCR-UCCD Solution



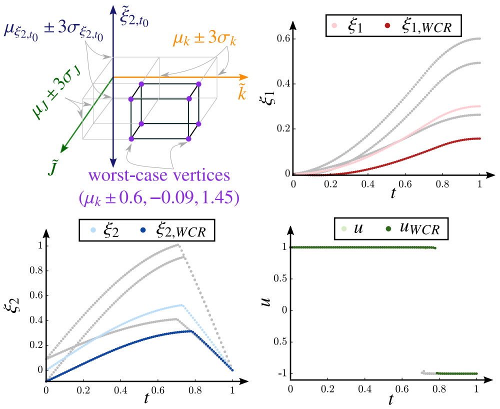
SE-UCCD Solution



→ Results - Polytopic Uncertainties

- A polytope is a bounded, closed, and convex polyhedron
- For a *linear program*, the feasible region is the convex hull of the vertices of the polytope
- Therefore, the optimal solution is achieved at a polytope vertex
- The OLMC-WCR-UCCD of simple SASA has polytopic uncertainties and is linear with nested formulation
- The number of required evaluations reduced to vertices of the polytope, i.e. 2^3 vertices

OLMC-WCR-UCCD solution with polytopic uncertainties



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Conclusion

→ Conclusions and Future Work

- Open-loop single control (OLSC) and open-loop multiple control (OLMC) structures were introduced
- Results indicate that gPC offers promising improvements in the computational time
- Uncertainty considerations impact system and design judgment
- Extension to problems with probabilistic path constraints, especially stochastic chance-constraints UCCD formulations
- Inclusion of time-dependent disturbances in the dynamic system model
- Various geometries (such as ellipsoidal, hexagonal, etc.) for WCR-UCCD
- Non-probabilistic propagation methods such as interval analysis and methods from fuzzy programming

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Questions?

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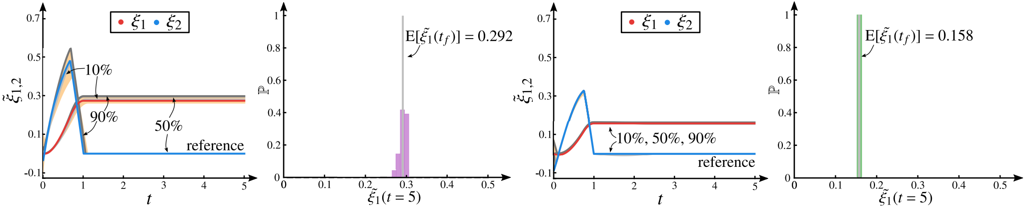
→ Closed-loop Implementations

Closed-loop investigations



(a) SE-UCCD state (b) SE-UCCD performance (c) WCR-UCCD state. (d) WCR-UCCD objective