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Characterizing Optimally Fault-Tolerant Manipulators Based on Relative Manipulability Indices

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Abstract— **In this article, the authors examine the problem of designing nominal manipulator Jacobians that are optimally fault tolerant to one or more joint failures. In this work, optimality is defined in terms of the worst case relative manipulability index. While this approach is applicable to both serial and parallel mechanisms, it is especially applicable to parallel mechanisms with a limited workspace. It is shown that a previously derived inequality for the worst case relative manipulability index is generally not achieved for fully spatial manipulators and that the concept of optimal fault tolerance to multiple failures is more subtle than previously indicated. Lastly, the authors identify the class of eight degree-of-freedom Gough-Stewart platforms that are optimally fault tolerant for up to two locked joint failures. Examples of optimally fault-tolerant seven and eight degree-of-freedom mechanisms are presented.**

Index Terms— **kinematic redundancy, fault tolerance, manipulability, parallel manipulators**

I. INTRODUCTION

Fault tolerant design of serial or parallel manipulators is critical for tasks requiring robots to operate in remote and hazardous environments where repair and maintenance tasks are extremely difficult [1]-[6]. In such cases, operational reliability is of prime importance. By adding kinematic redundancy to the robotic system, the robot may still be able to perform its task even if one or more joint actuators fail [7]. However, simply adding kinematic redundancy to the system does not guarantee fault tolerance [8]. One must strategically plan how the kinematic redundancy should be added to the system to ensure that fault tolerance is optimized [9].

One approach to the problem of designing fault tolerant robots is to optimize some measure of fault tolerance. While a number of measures have been proposed [10], [11], in this article we focus on the relative manipulability index, which was first introduced in [8] to quantify the fault tolerance of kinematically redundant serial manipulators. The *relative manipulability index* corresponding to locked joint failures in joints $i_1, \ldots, i_f$ is defined to be

$$\rho_{i_1,\cdots,i_f} = \frac{w({}^{i_1\cdots i_f}J)}{w(J)} \quad (1)$$

where J denotes the manipulator Jacobian, ${}^{i_1\cdots i_f}J$ denotes the manipulator Jacobian after the columns $i_1, \ldots, i_f$ corresponding to the failed joints are removed, and where $w(J) = \sqrt{\det(JJ^T)}$ is the manipulability index for J [12]. This quantity is a local measure of the amount of dexterity that is retained when a manipulator suffers one or more locked joint failures. The value of a relative manipulability index ranges from zero to one with a zero value indicating a local loss of full end-effector control and a value of one indicating that those joints in question only produce self-motion [8].

Relative manipulability indices have also been used to study the fault tolerance of redundant Gough-Stewart platforms [13]. A Gough-Stewart platform (GSP) is a parallel mechanism consisting of a base, a moving platform, and struts. For a GSP, the inverse Jacobian M maps the generalized velocity of the payload to the corresponding joint velocities of the individual struts. The matrix M has the same form as the transpose of a manipulator Jacobian J. In other words, the first three components of each row forms a unit vector that is orthogonal to the vector given by the last three components of that row. If M^TM is a diagonal matrix, then one says that the mechanism is an orthogonal Gough-Stewart platform (OGSP) [14]-[15]. OGSPs are a special class of GSPs that are particularly well-suited to various precision applications owing to the local kinematic and dynamic decoupling of the Cartesian directions they provide [16]. In [13], a class of OGSPs was identified that possess optimal fault tolerant manipulability for single joint failures based on maximizing the minimum relative manipulability index about an operating point.

In this article, the authors determine a family of manipulators that are optimally fault tolerant to multiple failures at their nominal operating configuration. In the next section, the relationship between the relative manipulability indices and the null space of the manipulator Jacobian is established using the principal minors of the null space projection operator. Based on this formulation of fault tolerance, it is easy to establish identities and inequalities for the relative manipulability indices. Motivated by the observation that the relative manipulability indices are completely determined by the null space of the manipulator Jacobian, we then discuss some of the theoretical limitations of designing manipulator Jacobians with a prescribed null space. An optimally fault tolerant seven degree-of-freedom (DOF) manipulator is then determined in Section III. In Section IV, the authors consider the concept of equally fault tolerant configurations, i.e., configurations for which any combination of a specified number of joint failures results in the same local manipulability. It is shown through a series of results that such configurations are truly rare. In Section V the authors identify the class of 8-DOF fully spatial manipulators that have the property

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that in their nominal operating configuration the manipulators are optimally fault tolerant for up to two joint failures. Conclusions appear in Section VI.

II. FAULT TOLERANCE AND THE NULL SPACE OF THE MANIPULATOR JACOBIAN

It turns out that the amount of fault tolerance that a manipulator possesses is closely related to the null space of the manipulator Jacobian. This important fact motivates the problem of designing operating configurations for robotic mechanisms based on choosing the manipulator Jacobian to have a prescribed null space. After characterizing the relative manipulability indices in terms of the null space of the manipulator Jacobian, we will discuss the amount of freedom that a designer has in choosing the null space of a nominal manipulator Jacobian.

A. Relative Manipulability Indices and the Null Space of the Manipulator Jacobian

We begin by demonstrating that the subdeterminants of the null space projection operator of the manipulator Jacobian completely characterize the relative manipulability indices. Our analysis is applicable to serial and parallel mechanisms so throughout this work we will use M and J^T interchangeably. Let J be a full rank $m \times n$ matrix with $m < n$ and let $r = n - m$. For a manipulator, m denotes the dimension of the workspace, n denotes the number of joints, and r denotes the degree of redundancy. We will call an $n \times r$ matrix N a *null space matrix* of J if the columns of N form an orthonormal basis for the null space of J. Although the null space matrix N is not unique for a given J, any two null space matrices N and N' of J are related by an orthogonal matrix Q in the following way: $N' = NQ$.

In [8], it was shown that the relative manipulability index is related to the null space matrix by the relationship

$$\rho_{i_1,\cdots,i_f} = w(N_{i_1\cdots i_f}) = \sqrt{|N_{i_1\cdots i_f} N^T_{i_1\cdots i_f}|} \qquad (2)$$

where $N_{i_1\cdots i_f}$ is the $f \times r$ matrix consisting of rows $i_1, \ldots, i_f$ of the matrix N. We thus have the interesting observation that the relative manipulability indices are strictly a function of the null space of J. We will build on this result to address the issue of designing manipulators that are optimally fault tolerant to one or more joint failures.

The relative manipulability index squared, $\rho^2_{i_1,\cdots,i_f} = |N_{i_1\cdots i_f} N^T_{i_1\cdots i_f}|$, is perhaps best viewed as a principal minor of the null space projection operator $P_N = I - J^+ J$ where J^+ denotes the pseudoinverse of J. The $n \times n$ matrix P_N represents the orthogonal projection of the joint space onto the null space of J. Unlike a null space matrix, P_N is unique for a given J; however, given a corresponding null space matrix N, we have that $P_N = NN^T$. It then follows from (2) that the relative manipulability index squared is equal to the determinant of the matrix consisting of the $i_1, \ldots, i_f$ rows and columns of P_N.

Recall that a $k \times k$ *minor* of an $n \times n$ matrix $A = [a_{ij}]$ with $k < n$ is a subdeterminant of the form

$$A\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \triangleq \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} & \cdots & a_{i_1 j_k} \\ a_{i_2 j_1} & a_{i_2 j_2} & \cdots & a_{i_2 j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k j_1} & a_{i_k j_2} & \cdots & a_{i_k j_k} \end{vmatrix} \qquad (3)$$

where $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$. If $(j_1, \ldots, j_k) = (i_1, \ldots, i_k)$, then this quantity is called a *principal minor* of A. Hence, we have that $\rho^2_{i_1,\cdots,i_f}$ is the $(i_1, \ldots, i_f)$ principal minor of $P_N = NN^T$:

$$\rho^2_{i_1,\cdots,i_f} = P_N\begin{pmatrix} i_1 & \cdots & i_f \\ i_1 & \cdots & i_f \end{pmatrix}. \qquad (4)$$

It is well known that the coefficients of the characteristic polynomial $p_A(\lambda) = |\lambda I - A|$ of A are given in terms of the sums of the principal minors of A. To be more specific, for $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$, we have that

$$a_{n-k} = (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} A\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix}. \qquad (5)$$

Since P_N is a projection, it is idempotent, i.e., $P_N^2 = P_N$, so its only possible distinct eigenvalues are 0 and 1. Furthermore, because $\mathrm{rank}(P_N) = r < n$ where $r = n - m$, it follows that the characteristic polynomial of P_N is

$$p(\lambda) = \lambda^m (\lambda - 1)^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \lambda^{n-k}. \qquad (6)$$

Equations (4), (5), and (6) then imply that

$$\sum_{1 \le i_1 < \cdots < i_f \le n} \rho^2_{i_1,\cdots,i_f} = \binom{r}{f}. \qquad (7)$$

This result, written as a slightly different but equivalent expression, was also proven in [13]; however, the proof provided there was based on repeated application of the Binet-Cauchy theorem and was less direct than applying principal minors. It is important to note, however, that the approach just given is not merely a different proof of the result in [13]. More importantly, it provides us with an approach that will be used in Section IV to address multiple joint failures.

As noted in [13], equation (7) can be used to obtain an upper bound for the worst case relative manipulability index by noting that the minimum value of any set of numbers must be less than or equal to the average so that

$$\min_{1 \le i_1 < \cdots < i_f \le n} \rho_{i_1,\cdots,i_f} \le \sqrt{\frac{\binom{r}{f}}{\binom{n}{f}}}. \qquad (8)$$

B. Designing Nominal Fully Spatial Manipulator Jacobians with a Prescribed Null Space

Based on the inequality in (8), Ukidve, et al., [13] convincingly argue the importance of designing for fault tolerance. This is especially true when there may be multiple faults. One approach to ensuring local fault tolerance is to design the manipulator based on null space properties. This is particularly applicable when the required workspace is very small as is the case in [13]. However, there are limitations to how much redundancy can be used when designing nominal manipulator Jacobians with a prescribed null space.

These limitations follow from the fact that the manipulator Jacobian for a fully spatial manipulator must satisfy certain constraints on its columns. In particular, the vector given by the first three components of a column must have unit length and must be orthogonal to the vector given by the last three components of that column. For a manipulator with n joints, this results in $2n$ constraints. If the manipulator Jacobian is required to have a prescribed null space matrix, then each of its six rows must be orthogonal to the r rows of N^T where $r = n - 6$ is the number of degrees of redundancy of the manipulator. Consequently, the manipulator Jacobian must satisfy $6r$ null space constraints. Since the manipulator Jacobian has $6n$ parameters, it follows that one has $6n - 2n - 6r = 4(6 + r) - 6r = 24 - 2r$ degrees of freedom to satisfy the design constraints. Hence, one cannot expect to arbitrarily find a manipulator with $r > 12$ degrees of redundancy that has a configuration where the manipulator Jacobian has a prescribed null space matrix.

If the mechanism is required to be an orthogonal Gough-Stewart platform (OGSP), then there is a further reduction in the degrees of freedom that one has in choosing a manipulator Jacobian with a prescribed null space. If JJ^T is required to be a diagonal matrix, there would be 15 additional constraints, decreasing the degrees of freedom to $9 - 2r$. In this case, one should not expect to be able to arbitrarily specify the null space of a manipulator with $r > 4$ degrees of redundancy. These dimension arguments will be exploited in Section IV to study the likely utility of a newly proposed fault tolerance concept.

III. DESIGNING OPTIMALLY FAULT TOLERANT 7-DOF SPATIAL MANIPULATOR JACOBIANS

According to equation (8), the maximum worst case relative manipulability index for a 7-DOF manipulator is $1/\sqrt{7}$. This optimal value is achieved if and only if the null vector of the manipulator Jacobian has components of equal magnitude, i.e., $|\hat{n}_i| = 1/\sqrt{7}$ where $\hat{n}_i$ is the i-th component of the unit length null vector $\hat{\mathbf{n}}_J$. Hence, we can specify the null vector to obtain an optimally fault tolerant manipulator configuration. Based on the dimension arguments in Section II-B, we have 22 degrees of freedom in choosing a 7-DOF manipulator Jacobian with a prescribed null vector. If we further require that JJ^T be diagonal, the number of degrees of freedom in choosing J with a prescribed null vector reduces to seven. An example of a nominal manipulator Jacobian that is optimally fault tolerant to a single failure is given by

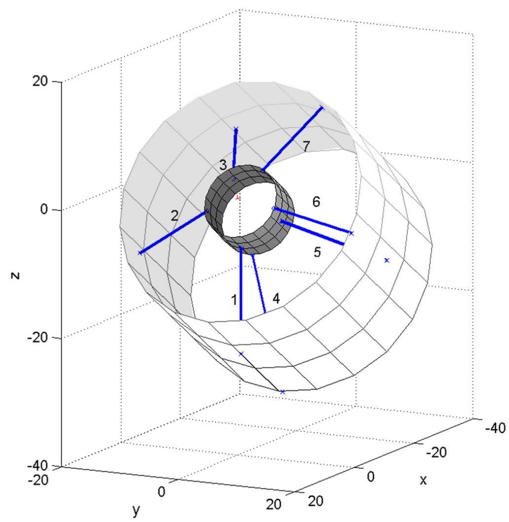

Fig. 1. An example of a cylindrical geometry for an OGSP corresponding to a realization of the optimally fault tolerant 6×7 manipulator Jacobian given in (9). The labels on the struts correspond to the respective columns of J (rows of M). Similar parallel mechanisms have been proposed for mounting in aerospace vehicles [14].

$$J^T = \begin{bmatrix} 0.000 & 0.000 & 1.000 & 0.113 & 1.065 & 0.000 \\ -0.175 & -0.827 & -0.536 & 0.870 & 0.023 & -0.314 \\ 0.877 & 0.418 & -0.239 & 0.297 & -0.159 & 0.814 \\ -0.408 & -0.004 & -0.913 & -0.696 & 0.581 & 0.308 \\ 0.473 & -0.802 & 0.364 & -0.689 & -0.553 & -0.323 \\ 0.065 & 0.983 & -0.174 & 0.020 & -0.177 & -0.993 \\ -0.836 & 0.233 & 0.497 & 0.085 & -0.781 & 0.508 \end{bmatrix}. \tag{9}$$

This manipulator Jacobian corresponds to a 7-DOF manipulator, and its null vector components are all equal. Consequently, all seven relative manipulability indices corresponding to (9) are equal to $1/\sqrt{7}$. In this case, JJ^T is diagonal so (9) corresponds to an OGSP.

There are a number of different possible manipulator realizations that can be generated from the Jacobian in (9). Clearly, the desired failure tolerance properties are not affected by multiplying one or more of the columns of J by -1. A parallel manipulator generated from this Jacobian is shown in Fig. 1.

IV. EQUALLY FAULT TOLERANT CONFIGURATIONS

Equation (8) served as a motivation in [13] for defining a manipulator operating about a single point in the workspace to be optimally fault tolerant to $f \leq r$ failures if all of its relative manipulability indices $\rho_{i_1,\cdots,i_f}$ are equal, i.e.,

$$\rho_{i_1,\cdots,i_f} = \sqrt{\frac{\binom{r}{f}}{\binom{n}{f}}} \tag{10}$$

for $1 \leq i_1 < \cdots < i_f \leq n$. In this article, we will prefer to say that a manipulator is *equally fault tolerant to $f \leq r$ failures* at an operating configuration if (10) holds for $1 \leq i_1 < \cdots < i_f \leq n$ at that configuration. Note that equal fault tolerance is a local property since it would apply to specific configurations and would be most applicable for manipulators operating in a small workspace. If a manipulator is equally fault tolerant to $f \leq r$ failures, then by (8) it is optimally fault tolerant in a worst case relative manipulability index sense to $f \leq r$ failures. However, while it is clear that an optimal value exists, it is possible that a manipulator may not have a configuration that is equally fault tolerant to f failures. In this case, the optimal value is smaller than the bound given in (8). It is the goal of this section to show that this is typically the case.

Our first result concerning equally fault tolerant configurations is the following:

Theorem 1: If a manipulator is equally fault tolerant to f failures where $1 < f \leq r$, then it is also equally fault tolerant to $f-1$ failures. Furthermore, the manipulator is equally fault tolerant to k failures for $k = 1, 2, \ldots, f$.

Proof: The proof is omitted due to space limitations.

The reason that Theorem 1 will play such an important role in this work is the fact that it forces P_N to have a particularly simple structure when the manipulator is equally fault tolerant to more than one failure. If J is equally fault tolerant to a single failure, then the diagonal elements of P_N are all equal to r/n. If J is equally fault tolerant to $f \geq 2$, then by Theorem 1 it is equally fault tolerant to single failures and to two failures. Hence, the (i,j) principal minor of the symmetric matrix P_N is

$$\begin{vmatrix} r/n & p_{ij} \\ p_{ji} & r/n \end{vmatrix} = \frac{r^2}{n^2} - p_{ij}^2 = \frac{r(r-1)}{n(n-1)} \tag{11}$$

where we have used the fact that $p_{ji} = p_{ij}$ and where the last equality follows from the assumption of equal fault tolerance to two failures. Solving for p_{ij} gives $p_{ij} = \frac{\pm 1}{n}\sqrt{\frac{r(n-r)}{n-1}}$ for all $1 \leq i < j \leq n$. Hence, when J is equally fault tolerant to $f \geq 2$ failures, the diagonal elements of P_N are all equal and the off-diagonal elements of P_N all have the same magnitude, i.e., P_N has the form

$$P_N = \begin{bmatrix} a & \pm b & \pm b & \cdots & \pm b \\ \pm b & a & \pm b & \cdots & \pm b \\ \pm b & \pm b & a & \cdots & \pm b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm b & \pm b & \pm b & \cdots & a \end{bmatrix} \tag{12}$$

where $a = \frac{r}{n}$ and $b = \frac{-1}{n}\sqrt{\frac{r(n-r)}{n-1}}$.

Consider now a manipulator with two degrees of redundancy, and suppose that the manipulator is equally fault tolerant to two failures. Since the rank of P_N would then be two, it follows that the 3×3 principal minors of P_N are zero; otherwise, the rank of P_N would be greater than or equal to three. Any 3×3 principal minor of P_N necessarily has the form

$$\begin{vmatrix} a & \pm b & \pm b \\ \pm b & a & \pm b \\ \pm b & \pm b & a \end{vmatrix} = a^3 - 3ab^2 \pm 2b^3. \tag{13}$$

Since one of these two quantities is zero, so is their product so that

$$\begin{aligned} 0 &= (a^3 - 3ab^2 + 2b^3)(a^3 - 3ab^2 - 2b^3) \\ &= (a-b)^2(a+2b)(a+b)^2(a-2b) \\ &= (a^2 - b^2)^2(a^2 - 4b^2). \end{aligned} \tag{14}$$

We thus conclude that $a^2 = b^2$ or $a^2 = 4b^2$. Substituting in the expressions for a and b yields that $n = 0$ or $n = 3$, respectively. As $n = 0$ does not make sense, we conclude that $n = 3$. Equivalently, the workspace has $m = n - r = 3 - 2 = 1$ degree of freedom so that the corresponding Jacobian is a 1×3 matrix. Equal fault tolerance then dictates that the Jacobian has the form $J = \begin{bmatrix} \pm\alpha & \pm\alpha & \pm\alpha \end{bmatrix}$ for some $\alpha > 0$.

The above observations prove the following result:

Theorem 2: No 8-DOF spatial manipulator can be equally fault tolerant to two simultaneous joint failures.

An 8-DOF *optimally* fault tolerant manipulator Jacobian will be determined in the next section. Of course the worst case relative manipulability index to two failures will necessarily be smaller than $1/\sqrt{28}$, the upper bound given by (8).

We are now ready to consider the case when J is equally fault tolerant to $f \geq 3$ failures. Applying similar arguments as above, we obtain the following result:

Theorem 3: Regardless of a manipulator's geometry or the amount of kinematic redundancy present in a manipulator, no fully spatial manipulator Jacobian can be equally fault tolerant to three or more joint failures.

Proof: The proof is omitted due to space limitations.

Theorem 3 is in fact applicable to any manipulator whose workspace dimension is greater than one, e.g., no planar manipulator can be equally fault tolerant to three or more failures regardless of how many joints it may have.

We now consider the case when a fully spatial manipulator is equally fault tolerant to two failures. We have already shown that this is impossible for $r = 2$. To simplify matters, note that multiplying any of the columns of J by -1 does not affect the fault tolerance properties of J. In doing so, the corresponding rows and columns of P_N are also multiplied by -1 so that we can assume without loss of generality that P_N has the form

$$P_N = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & \pm b & \cdots & \pm b \\ b & \pm b & a & \cdots & \pm b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \pm b & \pm b & \cdots & a \end{bmatrix}. \tag{15}$$

We use the property that P_N is a projection to determine restrictions on the number of degrees of redundancy that a fully spatial manipulator can have for the equal fault tolerance property to hold. As a projection, $P_N^2 = P_N$ so that for $j > 1$,

$$b = p_{1j} = (P_N)_{1j} = (P_N^2)_{1j} = 2ab + qb^2 \tag{16}$$

where q is the integer $q = n_1 - n_2 - 1$ where n_1 denotes the number of elements in the j-th column of P_N that are equal to b and n_2 denotes the number of elements equal to $-b$. Clearly $n_1 + n_2 = n - 1$ as $(P_N)_{jj} = a$ and $(P_N)_{ij} = \pm b$ for $i \neq j$. Since $b \neq 0$, (16) yields

$$q = \frac{1-2a}{b}. \tag{17}$$

For a redundant fully spatial manipulator, $m = 6$ and $n = r+6$. Substituting the expressions for a and b into (17) gives

$$q = \frac{1 - \frac{2r}{n}}{\frac{-1}{n}\sqrt{\frac{r(n-r)}{n-1}}} = (r-6)\sqrt{\frac{r+5}{6r}}. \tag{18}$$

The requirement that (17) is an integer is a necessary condition for the existence of a manipulator having $r > 1$ degrees of redundancy with the property that it is equally fault tolerant to two failures.

Unfortunately, the requirement that q is an integer eliminates most if not all practical manipulator designs since only specific values of r are feasible. Indeed, it was shown in Section II-B that one can only expect to be able to design for a prescribed null space if $r \leq 12$. Testing $r = 2, 3, \ldots, 12$, one finds that only $r = 3$, 6, and 10 result in integer values of q in (18). Note that this further confirms that no fully spatial manipulator Jacobian corresponding to an 8-DOF manipulator can be equally fault tolerant to two failures. Consider now the case when $r = 3$. We have already noted that $n_1 - n_2 = q + 1$ and $n_1 + n_2 = n - 1 = r + 5$ so that $2n_1 = q + r + 6$, or, equivalently, $q + r = 2n_1 - 6$. Hence, $q+r$ is an even number so that q and r have the same parity, i.e., both are even or both are odd. However, for $r = 3$, we have $q = -2$ implying that $r = 3$ is not a feasible solution. Thus, if a redundant fully spatial manipulator with $r \leq 12$ degrees of redundancy is equally fault tolerant to two joint failures then $r = 6$ or 10.

Ten or even six degrees of redundancy would be a considerable amount of redundancy to add to a manipulator and adding that much redundancy may even make the manipulator more prone to a joint failure. So it could be argued that even if one could design a manipulator to be equally fault tolerant to two failures, it would be undesirable to do so because of the high number of degrees of redundancy required. This observation is even more significant for an orthogonal GSP. The additional requirement that JJ^T be diagonal reduces our freedom in designing a manipulator Jacobian with a prescribed null space to $9 - 2r$ degrees of freedom. For $r = 6$, this value becomes $9 - 2(6) = -3$ so that there are three more design constraints than degrees of freedom to design such a manipulator.

V. OPTIMALLY FAULT TOLERANT CONFIGURATIONS

In the last section it was shown that no 8-DOF Gough-Stewart platform could be designed to operate at a configuration that is equally fault tolerant to two simultaneous failures. However, one can still design for worst case optimal fault tolerance for up to any two simultaneous failures at a nominal configuration in a manner that will be made precise later. This is the goal of the current section.

We have already seen that, from a relative manipulability index perspective, the null space of the manipulator Jacobian completely characterizes fault tolerance. Basically, the fault tolerance of a manipulator Jacobian depends on how spread out the rows of the null space matrix N are. It is natural then to try to spread these rows out as far as possible. This intuitive notion of spreading out the rows of the null space matrix will serve as a guide to identifying a configuration that is optimally fault tolerant in terms of the worst case relative manipulability index.

The case for manipulators with multiple failures is similar. In designing a fault tolerant nominal manipulator Jacobian, we first want the manipulator to be optimally fault tolerant to a single failure, i.e., we require the null space matrix N to have rows of equal norm. Among all manipulator Jacobians meeting this requirement, we choose one that optimizes fault tolerance to a second joint failure. This process can be continued to $f \leq r$ failures. In this case, we will say that the manipulator configuration is *optimally fault tolerant in terms of the worst case relative manipulability index to up to f failures*.

For a manipulator with two degrees of redundancy, it is convenient to consider the rows of the $n \times 2$ null space matrix N to be points or vectors in a plane with each vector originating from the origin. Optimal fault tolerance to a single failure dictates that the norm of the rows be equal, i.e., that the terminal points of the vectors lie on a circle centered at the origin with radius $\sqrt{2/n}$. Now the relative manipulability index ρ_{ij} is equal to the absolute value of the determinant of the matrix consisting of rows i and j of N. Since these rows have length $\sqrt{2/n}$, we have that $\rho_{ij} = \frac{2}{n}|\sin\phi_{ij}|$ where ϕ_{ij} is the angle between the corresponding i-th and j-th vectors. Fault tolerance is then characterized by the $\binom{n}{2}$ angles between the n vectors in the plane.

Before proposing a candidate optimal solution, we note the invariance of the unordered set $\{\rho_{i_1,\cdots,i_f} \mid 1 \leq i_1 < \cdots < i_f \leq n\}$ to two simple operations on the columns of J. If J' is obtained from J by multiplying some of the columns of J by -1, by reordering some of the columns, or by a combination of these two operations, then J and J' have the same unordered set of relative manipulability indices. Now rearranging columns and/or multiplying some of them by -1 affects the rows of the null space matrix in exactly the same way. These observations help simplify our study of fault tolerance. In particular, when identifying an optimal null space matrix N for the $r = 2$ case, we can assume that when the rows of N are viewed as points in the plane, they appear in the upper half plane, for if a particular row does appear in the lower half plane, simply multiply that row by -1 and its point representation will appear in the upper half plane.

A natural candidate for an optimal N can now be obtained by spreading the rows of N out as points on the upper half circle of radius $\sqrt{2/n}$. In other words, we choose the n rows of N to appear as points at the ordered angles

$$\phi_k = \frac{\pi k}{n} \quad k = 0, 1, \ldots, n-1 \tag{19}$$

on the circle with radius $\sqrt{2/n}$ and centered at the origin. The fact that (19) is an optimal null space matrix follows directly from the following result:

Theorem 4: Let N be the $n \times 2$ matrix whose i-th row is given by

$$N_i = \left[\sqrt{\tfrac{2}{n}}\cos(\tfrac{\pi(i-1)}{n}) \quad \sqrt{\tfrac{2}{n}}\sin(\tfrac{\pi(i-1)}{n})\right], \quad i = 1, 2, \ldots, n. \tag{20}$$

Then N is the null space matrix for an $(n-2) \times n$ matrix J that is optimally fault tolerant for up to two failures.

Proof: The proof is omitted due to space limitations.

Equation (19) is not the only optimal null space matrix. As mentioned earlier, post-multiplication of a null space matrix by an arbitrary $r \times r$ orthogonal matrix will result in a null space matrix corresponding to the same null space of J. Also note that permuting or multiplying rows of N by -1 will not change the set of values of the relative manipulability indices. In particular, any null space matrix N obtained from (19) where one or more rows may possibly be multiplied by -1 and/or where the rows may possibly be permuted will still result in an optimally fault tolerant albeit different manipulator Jacobian. These operations on the rows result in precisely the same operations on the columns of J. Note that while permuting the columns of J does not affect the overall geometry of a parallel mechanism, it has a significant impact on the geometry of a serial mechanism.

Using (19) directly gives a null space matrix

$$N = \begin{bmatrix} 0.500 & 0.000 \\ 0.462 & 0.191 \\ 0.354 & 0.354 \\ 0.191 & 0.462 \\ 0.000 & 0.500 \\ -0.191 & 0.462 \\ -0.354 & 0.354 \\ -0.462 & 0.191 \end{bmatrix} \tag{21}$$

with the property that any corresponding manipulator Jacobian is optimally fault tolerant for up to two joint failures. A specific manipulator Jacobian corresponding to (21) is

$$J^T = \begin{bmatrix} 0.000 & 0.000 & 1.000 & -0.639 & 0.765 & 0.000 \\ -0.835 & 0.503 & -0.225 & -0.395 & -0.845 & -0.423 \\ -0.017 & 0.148 & -0.989 & 0.251 & 0.955 & 0.139 \\ 0.719 & -0.688 & -0.101 & -0.498 & -0.618 & 0.663 \\ 0.714 & 0.533 & 0.454 & 0.614 & -0.123 & -0.822 \\ -0.831 & 0.054 & 0.554 & 0.572 & 0.046 & 0.853 \\ -0.482 & -0.875 & -0.052 & -0.474 & 0.314 & -0.879 \\ 0.163 & 0.978 & -0.131 & -0.975 & 0.199 & 0.277 \end{bmatrix}. \tag{22}$$

In this case, the rows of the manipulator Jacobian are mutually orthogonal and (22) corresponds to an OGSP that is optimally fault tolerant for up to two joint failures with a worst case relative manipulability index $\sqrt{2/8} = 0.5$ for single failures and $\frac{2\sin(\pi/8)}{8} = 0.0957$ for two simultaneous failures. Note that the second quantity compares reasonably well to the upper bound $\frac{1}{\sqrt{28}} = 0.1890$ given in (8).

VI. CONCLUSIONS

In this article, the authors examined the issue of designing kinematically redundant manipulators that are optimally fault tolerant to multiple joint failures. The authors provided an alternative proof of the recent result that the sum of the squares of the relative manipulability indices corresponding to f failures is equal to $\binom{r}{f}$. This result provides an upper bound for the worst case relative manipulability index of a manipulator with one or more failed joints. Previously, this upper bound was used to characterize optimal fault tolerance to multiple failures. However, in this article, it was shown that this upper bound is typically not achieved and is therefore not suitable for judging optimal fault tolerance. This clearly indicates the need for further consideration when designing robotic systems that are tolerant to multiple joint failures. By identifying the required properties of the null space of the manipulator Jacobian, the authors presented a general method for finding a family of 8-DOF Gough-Stewart platforms with optimal worst case fault tolerance for up to two failures.

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