Saddlepoint Approximations for Correlation Testing Among Multiple Gaussian Random Vectors

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***Abstract*—This letter considers the problem of threshold selection for a correlation test among multiple (≥ 2) random vectors. The generalized likelihood ratio test (GLRT) for this problem uses a generalized Hadamard ratio to test for block diagonality in a composite covariance matrix. As the number of realizations used to estimate the composite covariance matrix grows large, the null distribution of the likelihood ratio statistic converges to a chi-squared distribution which can be used to prescribe thresholds needed to achieve a desired false alarm rate in high sample support situations. However, this asymptotic distribution can be slow to converge, making its use dubious in many practical scenarios. To address this problem, this letter uses saddlepoint approximations for the null distribution of the generalized Hadamard ratio. Simulations are provided to demonstrate the saddlepoint approximation's ability to achieve a desired false alarm probability, even in situations with low sample support.**

***Index Terms*—Coherence, generalized coherence, generalized likelihood ratio test (GLRT), multichannel coherence, multichannel signal detection, saddlepoint approximations, Wilks' chi-squared.**

I. Introduction

DETECTING the presence of linear dependence among two or more data channels is a problem that finds its uses in many applications. For instance, in applications such as collaborative sensor networks [1], geological monitoring of seismic activity [2], radar [3], and sonar [4], one may be interested in detecting the presence of a common but unknown signal among multiple spatially distributed sensors. One solution to this problem is to construct a *generalized likelihood ratio test* (GLRT) that uses a Hadamard ratio [5], [6] to test for diagonal or block-diagonal structure in a global estimate of the channels' second-order information. Often times, however, the most difficult part of hypothesis testing lies not so much in deriving the appropriate test, but rather in finding its exact null distribution and selecting thresholds that achieve a desired false alarm rate. In many cases, Wilks' theorem [7] provides an asymptotic chi-squared distribution that can be used to identify an appropriate critical region in high sample support situations. This asymptotic distribution can, however, require overwhelming amounts of data to converge making it intractable in many practical situations. Thus, methods are needed to reliably identify an appropriate threshold under a wide range of sample support situations.

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The application of the Hadamard ratio for the detection of multiple temporally white Gaussian sequences and its distribution have been previously discussed in [5] and [8]. In [5], the geometric aspects of the Hadamard ratio and its implications for multichannel detection were discussed. Under the assumption that each channel contains white complex normal noise, the authors derived expressions for its null distribution in the two and three channel cases. These results were then extended by deriving a recursive formulation for finding the null distribution as one adds additional channels. The asymptotic distribution of the Hadamard ratio under both the null and alternative hypotheses was also discussed in [8]. Assuming that both the sample size and number of channels tend to infinity at the same rate, the Hadamard ratio is shown to be asymptotically normal with both a mean and a variance that are dependent on the true covariance matrix of the data.

The work in [5] and [8] was recently extended in [6] and [9] by considering the detection of both *spatially and temporally correlated* time series. Given multiple independent realizations of a vector-valued time series, the GLRT of [6] tests whether or not the space-time covariance matrix is block-diagonal using a *generalized* Hadamard ratio involving the sample covariance matrix. In [9], it was shown that this test statistic is stochastically equivalent to a product of independently distributed beta random variables. This stochastic representation was then used to derive an asymptotic chi-squared distribution but, as discussed above, results suggested that this asymptotic distribution is slow to converge in many cases. For this reason, this letter considers the use of saddlepoint approximations [10], [11] for accurately approximating the null distribution of the likelihood ratio statistic. The saddlepoint approximation's ability to accurately achieve a desired false alarm probability even in low-sample support situations is then demonstrated through simulation.

II. Review of the Likelihood Ratio and Its Null Distribution

The problem considered here is testing for independence among L random vectors $\{\boldsymbol{x}_i\}_{i=1}^{L}$. Assuming this collection of random vectors to be zero mean, the composite vector $\boldsymbol{z} = \left[\boldsymbol{x}_1^T \cdots \boldsymbol{x}_L^T\right]^T$ has the block-structured covariance matrix

$$\mathbf{R} = E\left[\boldsymbol{z}\boldsymbol{z}^H\right] = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \cdots & \mathbf{R}_{1L} \\ \mathbf{R}_{12}^H & \mathbf{R}_{22} & \cdots & \mathbf{R}_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{R}_{1L}^H & \mathbf{R}_{2L}^H & \cdots & \mathbf{R}_{LL} \end{bmatrix} \in \mathbb{C}^{LN\times LN}$$

with $\mathbf{R}_{ik} = \mathbf{R}_{ki}^H = E\left[\boldsymbol{x}_i\boldsymbol{x}_k^H\right] \in \mathbb{C}^{N\times N}$. Note that the notation $\boldsymbol{x}^H$ denotes the Hermitian transpose of vector $\boldsymbol{x}$. The matrix $\mathbf{R}$ not only characterizes the second-order information for each channel individually but also captures the interdependence between every pair of channels. If the set of random vectors $\{\boldsymbol{x}_i\}_{i=1}^L$ is jointly proper complex normal [12], then testing for independence among all L channels simply involves testing the null hypothesis $\mathcal{H}_0$ that $\mathbf{R}$ is block-diagonal versus the alternative that it is not.

We now assume that we are given an experiment producing M *iid* realizations $\{\boldsymbol{x}_i[m]\}_{m=1}^M$ of the random vector from each channel i. All M realizations of each channel are organized into the data matrix

$$\mathbf{Z} = \begin{bmatrix} \boldsymbol{x}_1[1] & \cdots & \boldsymbol{x}_1[M] \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_L[1] & \cdots & \boldsymbol{x}_L[M] \end{bmatrix} \in \mathbb{C}^{LN\times M}. \tag{1}$$

The data matrix $\mathbf{Z}$ has the probability density function (pdf) [12]

$$f(\mathbf{Z};\mathbf{R}) = \frac{1}{\pi^{LNM}\det(\mathbf{R})^M}\exp\left\{-M\mathrm{tr}\left(\mathbf{R}^{-1}\hat{\mathbf{R}}\right)\right\}$$

where $\hat{\mathbf{R}}$ is the estimated composite covariance matrix

$$\hat{\mathbf{R}} = \frac{1}{M}\mathbf{Z}\mathbf{Z}^H = \begin{bmatrix} \hat{\mathbf{R}}_{11} & \hat{\mathbf{R}}_{12} & \cdots & \hat{\mathbf{R}}_{1L} \\ \hat{\mathbf{R}}_{12}^H & \hat{\mathbf{R}}_{22} & \cdots & \hat{\mathbf{R}}_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{R}}_{1L}^H & \hat{\mathbf{R}}_{2L}^H & \cdots & \hat{\mathbf{R}}_{LL} \end{bmatrix} \tag{2}$$

and $\hat{\mathbf{R}}_{ik}$ is an M sample estimate of the $N\times N$ cross-covariance matrix $\mathbf{R}_{ik}$. The GLRT for this problem decides to accept or reject the null hypothesis $\mathcal{H}_0$ by computing the likelihood ratio [6]

$$\Lambda = \left(\frac{\sup\limits_{\mathbf{R}\in\mathcal{R}_0} f(\mathbf{Z};\mathbf{R})}{\sup\limits_{\mathbf{R}\in\mathcal{R}} f(\mathbf{Z};\mathbf{R})}\right)^{1/M} = \frac{\det\hat{\mathbf{R}}}{\prod_{i=1}^L \det\hat{\mathbf{R}}_{ii}} \tag{3}$$

where $\mathcal{R}$ denotes the set of all positive-definite (PD) Hermitian matrices and $\mathcal{R}_0$ denotes the set of all matrices in $\mathcal{R}$ which are block-diagonal.

Under the null hypothesis that all L channels are truly independent so that $\mathbf{R} = \mathrm{blkdiag}\{\mathbf{R}_{11},\ldots,\mathbf{R}_{LL}\} \in \mathcal{R}_0$, it was shown in [9] that the likelihood ratio in (3) has the stochastic representation

$$\Lambda|\mathcal{H}_0 \stackrel{d}{=} \prod_{i=2}^{L}\prod_{n=0}^{N-1} Y_{\mathrm{in}} \tag{4}$$

where each random variable in the product is independently distributed as $Y_{\mathrm{in}} \sim \mathrm{Beta}(\alpha_{\mathrm{in}},\beta_i)$ with parameters

$$\alpha_{\mathrm{in}} = M-(i-1)N-n \tag{5}$$

$$\beta_i = (i-1)N. \tag{6}$$

That is, under the null hypothesis the likelihood ratio is distributed as a product of independent but *not* identically distributed beta random variables. Moreover, this stochastic representation is only dependent on the number of channels (L), the channel dimension (N), and the number of independent samples (M), but is invariant to the intrachannel correlation matrices $\mathbf{R}_{ii}$ for $i = 1,\ldots,L$. This stochastic representation of the likelihood ratio was then used in [9] to show that the random variable $Z = \left(\frac{1}{3}(L+1)N - 2M\right)\ln\Lambda$ is asymptotically chi-squared

$$P\left[Z\le x|\mathcal{H}_0\right] \stackrel{M\to\infty}{\longrightarrow} P\left[\chi_\nu^2 \le x\right] \tag{7}$$

with degrees of freedom

$$\nu = \dim(\mathcal{R}) - \dim(\mathcal{R}_0) = L^2N^2 - LN^2.$$

III. Approximating the Null Distribution Using the Saddlepoint Method

Saddlepoint approximations are powerful tools for obtaining accurate approximations for densities and distribution functions given a random variable's known moment generating function (MGF). Namely, given a scalar random variable X with cumulant generating function $\psi(t) = \ln E\left[e^{tX}\right]$, the saddlepoint approximation to the pdf $f(x)$ is defined as [10], [11]

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi\psi''\left(\hat{t}\right)}}\exp\left\{\psi\left(\hat{t}\right) - \hat{t}x\right\} \tag{8}$$

where, for every value of the argument x in (8), the saddlepoint $\hat{t}$ denotes the value of t such that $\psi'(t) = x$. Note that the notation ψ' and ψ'' denote the first- and second-order derivatives of the cumulant generating function, respectively. Likewise, an approximation formula for the cumulative distribution function (cdf) is given by [13]

$$\hat{F}(x) = \begin{cases} \Phi(\hat{w}) + \phi(\hat{w})\left(\frac{1}{\hat{w}} - \frac{1}{\hat{u}}\right), & \text{for } x\ne\mu \\ \frac{1}{2} + \frac{\psi'''(0)}{6\sqrt{2\pi}(\psi''(0))^{3/2}}, & \text{for } x = \mu \end{cases} \tag{9}$$

where ϕ and Φ represent the pdf and cdf of a standard normal random variable, respectively, and μ is the mean of X. The values $\hat{w}$ and $\hat{u}$ in (9) are given by

$$\hat{w} = \mathrm{sgn}(\hat{t})\sqrt{2\left(\hat{t}x - \psi(\hat{t})\right)}$$

$$\hat{u} = \hat{t}\sqrt{\psi''(\hat{t})}.$$

The first step in building a saddlepoint approximation of (3) under the null hypothesis is to derive its cumulant generating function. Given the definition of the beta function $B(x,y)$ and its relationship to the gamma function $\Gamma(\cdot)$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 z^{x-1}(1-z)^{y-1}dz$$

every random variable within the product given in (4) has Mellin transform [14]

$$\begin{aligned} E\left[Y_{\mathrm{in}}^t\right] &= \frac{1}{B(\alpha_{\mathrm{in}},\beta_i)}\int_0^1 x^{(\alpha_{\mathrm{in}}+t)-1}(1-x)^{\beta_i-1}dx \\ &= \frac{B(\alpha_{\mathrm{in}}+t,\beta_i)}{B(\alpha_{\mathrm{in}},\beta_i)} = \frac{\Gamma(\alpha_{\mathrm{in}}+t)\Gamma(\alpha_{\mathrm{in}}+\beta_i)}{\Gamma(\alpha_{\mathrm{in}})\Gamma(\alpha_{\mathrm{in}}+\beta_i+t)} \end{aligned} \tag{10}$$

with parameters α_{in} and β_i given in (5) and (6), respectively. Using (10) along with the fact that these random variables are independent, one can see that the log-likelihood ratio has the MGF

$$\begin{aligned}\phi_{\ln\Lambda}(t) &= E\left[e^{t\ln\Lambda}|\mathcal{H}_0\right] = E\left[\Lambda^t|\mathcal{H}_0\right]\\ &= \prod_{i=2}^{L}\prod_{n=0}^{N-1} E\left[Y_{\text{in}}^t\right]\\ &= \prod_{i=2}^{L}\prod_{n=0}^{N-1} \frac{\Gamma(M-n)\Gamma(M-(i-1)N-n+t)}{\Gamma(M-(i-1)N-n)\Gamma(m-n+t)}.\end{aligned}$$

From this expression, it is straightforward to see that, for any real-valued scalar c, the random variable $Z = c\ \ln\ \Lambda$ has the cumulant generating function

$$\begin{aligned}\psi_Z(t) &= \ln\phi_{\ln\Lambda}(ct)\\ &= \sum_{i=2}^{L}\sum_{n=0}^{N-1}\ln\frac{\Gamma(M-n)\Gamma(M-(i-1)N-n+ct)}{\Gamma(M-(i-1)N-n)\Gamma(m-n+ct)}.\end{aligned} \tag{11}$$

Note that similar expressions for the CGF in (11) can be found in [11] but for the case where the composite covariance matrix $\hat{\mathbf{R}}$ in (2) is a real-valued symmetric Wishart matrix.

As discussed above, the next step in building a saddlepoint approximation involves solving for the saddlepoint. Letting $\gamma_k(z)$ denote the polygamma function [15] of order k

$$\gamma_k(z) = \frac{d^{(k+1)}\ln\Gamma(z)}{dx^{(k+1)}}$$

the cumulant generating function given in (11) has kth-order derivative

$$\begin{aligned}\frac{d^k\psi_Z(t)}{dt^k} = c^k\sum_{i=2}^{L}\sum_{n=0}^{N-1} & \left[\gamma_{k-1}\left(M-(i-1)N-n+ct\right)\right.\\ &\left.-\gamma_{k-1}\left(M-n+ct\right)\right].\end{aligned} \tag{12}$$

For any argument x in (8), solving for the saddlepoint involves finding the value of t such that

$$\begin{aligned}\psi'_Z(t) = c\sum_{i=2}^{L}\sum_{n=0}^{N-1} & \left[\gamma_0\left(M-(i-1)N-n+ct\right)\right.\\ &\left.-\gamma_0\left(M-n+ct\right)\right] = x.\end{aligned} \tag{13}$$

Using the fact that the trigamma function $\gamma_1(t)$ is a monotonically decreasing function of t, it is straightforward to show that the function $\psi''_Z(t)$ [found by letting $k=2$ in (12)] is strictly positive. This implies that $\psi'_Z(t)$ is monotonic as well, and hence, there exists a unique solution to (13).

Unfortunately, there is no simple closed-form solution to the problem given in (13). In lieu of this fact, one must resort to numerical root-finding algorithms to find the saddlepoint. One very simple method for accomplishing this is the Newton–Raphson algorithm [16] where, for every value of x, one calculates the sequence of iterations $\left\{\hat{t}_n\right\}_{n=0}^{\infty}$ using the recursive relationship

$$\hat{t}_{n+1} = \hat{t}_n - \frac{\psi'_Z\left(\hat{t}_n\right)-x}{\psi''_Z\left(\hat{t}_n\right)}. \tag{14}$$

One of the things required by any root-finding algorithm, however, is an initial estimate of the solution to start the recursion. Although many choices exist, one possibility is to assume that the random variable $Z = c\ \ln\ \Lambda$ is normally distributed with a known mean and variance, and find the saddlepoint solution under that assumption. Given the fact that a normally distributed random variable with mean μ and variance σ^2 has the saddlepoint [10]

$$\hat{t} = \frac{x-\mu}{\sigma^2}$$

as well as the fact that the mean and variance of Z can be obtained by evaluating the first- and second-order derivatives given in (12) at $t=0$, i.e., $\psi'_Z(0) = E[Z]$ and $\psi''_Z(0) = E\left[(Z-E[Z])^2\right]$, one can match the first- and second-order moments to the saddlepoint for a normal distribution by using the initial solution

$$\hat{t}_0 = \frac{x-\psi'_Z(0)}{\psi''_Z(0)}. \tag{15}$$

The initialization in (15), although *ad hoc*, is partly justifiable given the connection that the product of beta representation in (4) has with the Wilks' lambda distribution [17] and a central limit theorem for the Wilks' lambda statistic as N and M approach infinity [18]. Thus, as the channel dimension (N) and number of samples (M) grow large, the initial saddlepoint $\hat{t}_0$ in (15) asymptotically approaches the saddlepoint $\hat{t}$ which solves (13).

IV. Simulation Results

To demonstrate the ability of the saddlepoint approximation to model the null distribution of (3), simulations were conducted to compare the approximation in (8) to the asymptotic chi-squared distribution in (7). For $L=3$ channels, an $N=12$ dimensional observation per channel, $M=250$ samples, and using the scaling factor $c = \frac{1}{3}(L+1)N - 2M$ as in (7), Fig. 1(a) plots the saddlepoint solution to (13) found using the recursion in (14) as a function of the dependent variable x. The oblique dashed line in this plot shows the initial estimate $\hat{t}_0$ given in (15). From this plot, one can see that the initialization closely approximates the saddlepoint solution within a neighborhood of the mean (shown by a vertical dashed line in the figure) but experiences a larger deviation the farther one gets into the tails of the distribution. Likewise, the dark-colored line in Fig. 1(b) plots the pdf approximation $\hat{f}(x)$ found by substituting the saddlepoint $\hat{t}$ into the expression given in (8). This is compared to a histogram of log-likelihood values produced through a Monte Carlo simulation which is denoted by light-colored values in this plot. This histogram was formed by generating 5000 instances of the data matrix $\mathbf{Z}$ in (1) containing *iid* $\mathcal{CN}(0,1)$ random variables, forming the estimated covariance matrix $\hat{\mathbf{R}}$ in (2) for each instance, and computing the scaled logarithm of the likelihood ratio given in (3). Finally, the dark-colored dashed curve in Fig. 1(b) plots the pdf corresponding to the asymptotic chi-squared distribution given in (7). From this plot, it is clear that, for the specific parameters L, N, and M chosen for this example, the saddlepoint

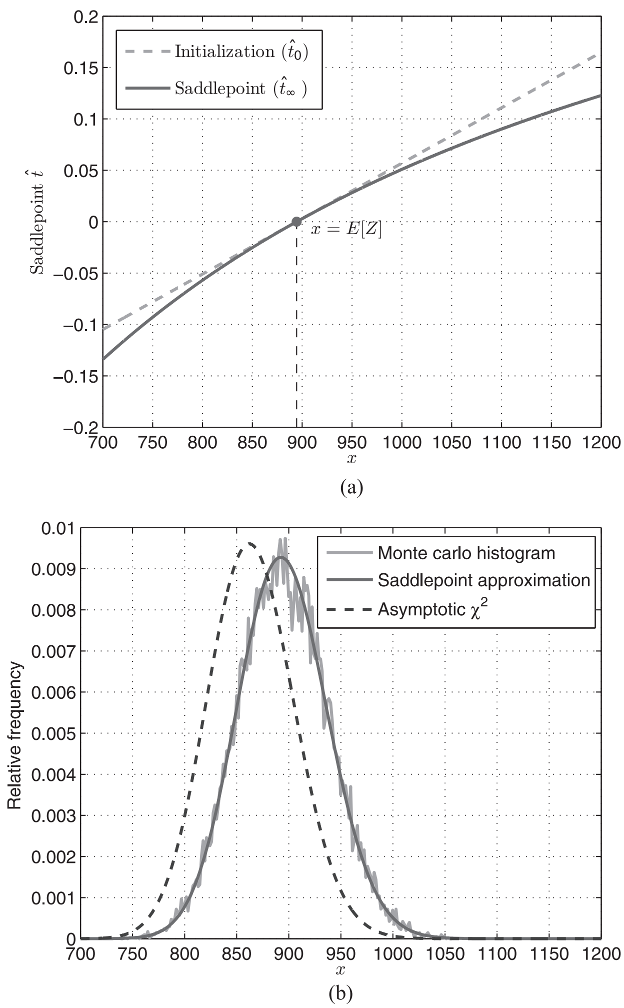

Fig. 1. Comparison of the saddlepoint pdf in (8) to the histogram of likelihood values obtained through simulation as well as the asymptotic chi-squared distribution in (7) for $L = 3$, $N = 12$, and $M = 250$. (a) Initialization and saddlepoint solution. (b) Saddlepoint pdf and asymptotic χ^2.

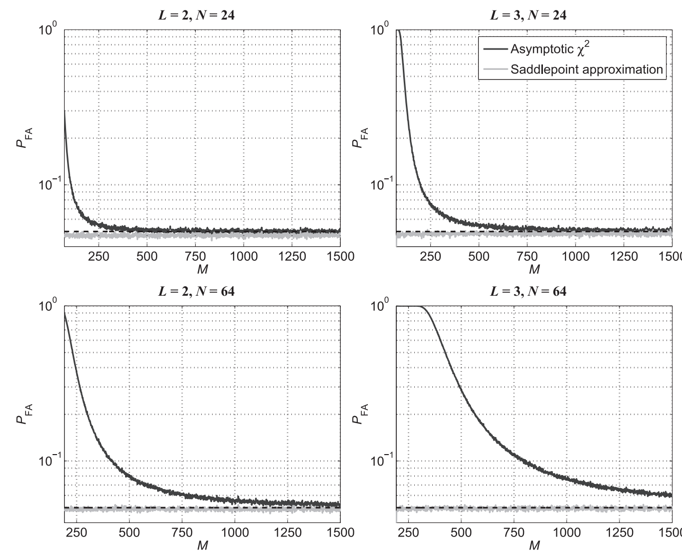

Fig. 2. Empirical false alarm probabilities versus M for several choices in L and N.

approximation gives one a much better description of how the log-likelihood ratio is distributed compared to the asymptotic chi-squared distribution.

To further demonstrate the usefulness of the saddlepoint approximation in modeling the null distribution of the log-likelihood ratio, its ability to achieve a desired false alarm rate was tested through simulation. Using the statistic $Z = c \ln \Lambda$ with scaling factor $c = \frac{1}{3}(L+1)N - 2M$, two thresholds were defined to achieve a desired false alarm probability of $P_{\mathrm{FA}} = 0.05$. The first threshold was based on the asymptotic chi-squared distribution in (7), i.e., the threshold η_{χ^2} was set such that $P\left[\chi^2_\nu \le \eta_{\chi^2}\right] = 0.95$. Using the recursions in (14), the second threshold η_{sp} was determined such that $\hat{F}\left(\eta_{\mathrm{sp}}\right) = 0.95$ using the cdf approximation in (9).

Using these two thresholds, Fig. 2 plots the empirical false alarm probabilities achieved by both methods versus the number of samples M for several values of L and N. Simulating the random variable Z using the same procedure as in Fig. 1, the dark-colored lines in Fig. 2 plot the percentage of repetitions in each Monte Carlo simulation that produce a log-likelihood ratio which exceeds the asymptotic threshold η_{χ^2}, while the light-colored lines plot the percentage which exceed the saddlepoint threshold η_{sp}. The desired false alarm rate of 0.05 is denoted in each plot using a horizontal dashed line. One can see that defining thresholds according to the asymptotic result in (7) will yield the appropriate false alarm rate only for high sample support. The number of samples needed to achieve convergence can be rather large and grows with increasing L and N. On the other hand, the saddlepoint approximation yields a false alarm performance which is constant over M and is generally far better at achieving the desired value compared to that obtained using the asymptotic chi-squared distribution.

V. Conclusion

This letter considers the problem of detecting the presence of common characteristics among two or more data channels using a generalized Hadamard ratio. Under the null hypothesis that the covariance matrix is truly block-diagonal, the generalized Hadamard ratio is statistically equivalent to a product of independent beta random variables, the logarithm of which is asymptotically chi-squared. However, in situations where the number of channels or channel dimension is large, the asymptotic distribution can be slow to converge.

For this reason, we considered the use of the saddlepoint method for building more practical approximations of the log-likelihood ratio's null distribution. This method requires finding the saddlepoint using the log-likelihood ratio's cumulant generating function which, in this case, is easily derived given the fact that the null distribution is distributed as a product of independent beta random variables. Simulation results demonstrate that the saddlepoint approximation does indeed provide a practical alternative to the asymptotic distribution and is capable of accurately characterizing the null distribution, even in low-sample support scenarios.

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