Ph.D. Final Examination

Continuum Limits of Markov Chains with Application to Wireless Network Modeling and Control

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October 23, 2013

Background

Stochastic networks with a large number of nodes.

- Traditional approach: Monte Carlo simulation involving individual nodes.

  For networks with a very large number of nodes — slow ...

- Continuum modeling: describe global characteristics by continuum limits – PDE solutions.

  Approximate networks by PDEs and solve by software (e.g., Matlab) — fast!

  The larger the network, the closer the PDE approximates it.

Overview

Prelim's talk:

- Theory: analyze convergence of underlying Markov chain to continuum limits (PDE).
- Application: model wireless networks by PDE.

Today's talk:

- Theory: updated results of convergence analysis.
- Application: extensions to
  - networks with more general transmission rules.
  - networks with nonuniform location of nodes.
  - control of nonuniform networks using continuum models.

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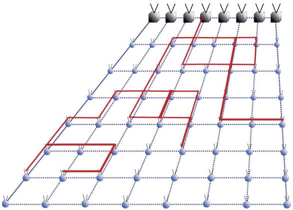
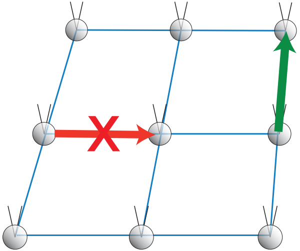
Wireless sensor network model

- For now consider uniform network: nodes located on uniform grid. (Later consider nonuniform networks.)
- Sensor nodes: generate messages and relay them to destination nodes at boundary.
- All nodes have message queues.
- Nodes only communicate with immediate neighbors (1D: left and right; 2D: north, south, east, west). (Later consider communication with further neighbors – more general transmission rules.)
- Nodes transmit or receive at each time step $k = 1, 2, \ldots$, but not both.



Channel model

- All nodes share the same wireless channel.
- A transmission from a transmitter to a neighboring receiver is successful iff none of the other neighbors of the receiver is a transmitter.

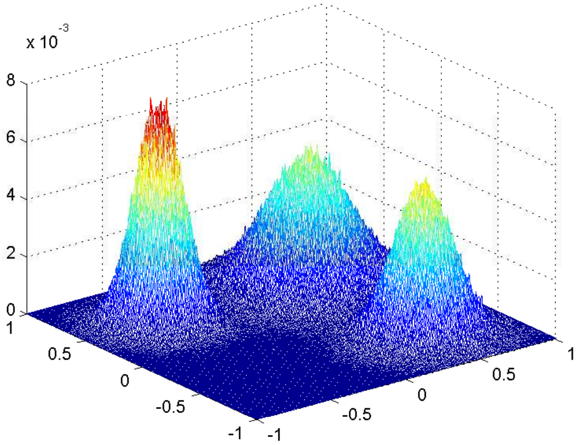


Reception at a node fails when more than one of its neighbors transmit (interference). (Later consider interference from further neighbors.)

Notation

- Interested in calculating message queue lengths at nodes.
- Consider N-node network:
  - N-point uniform grid $V_N = \{v_N(1), \dots, v_N(N)\} \subset \mathscr{D} \subset \mathbb{R}^1$ or $\mathbb{R}^2$.
  - ds_N: distance between any two neighboring grid points.
  - dt_N: time step length.
  - $X_{N,M}(k,n)$: queue length of node n (located at $v_N(n)$) at time k, normalized by an “averaging” parameter M.
  - $W(n, X_{N,M}(k,n))$: probability that node n tries to transmit at time k.
  - Set $W(n,y) = min(1,y)$: nodes transmit with probability proportional to queue length.
  - $P_n(n), P_s(n), P_e(n), P_w(n)$: probabilities of node n transmitting to north, south, east, west (sum up to 1).
    $P_e(n) = 1/4 + c_e(v_N(n))ds_N$, $P_w(n) = 1/4 + c_w(v_N(n))ds_N$
    $P_n(n) = 1/4 + c_n(v_N(n))ds_N$, $P_s(n) = 1/4 + c_s(v_N(n))ds_N$
  - $G(k,n)$: number of messages generated at node n at time k; modeled by independent Poisson random variables with mean $g(n) = Mg_p(v_N(n))dt_N$. (g_p: incoming traffic function)

Monte Carlo simulation result

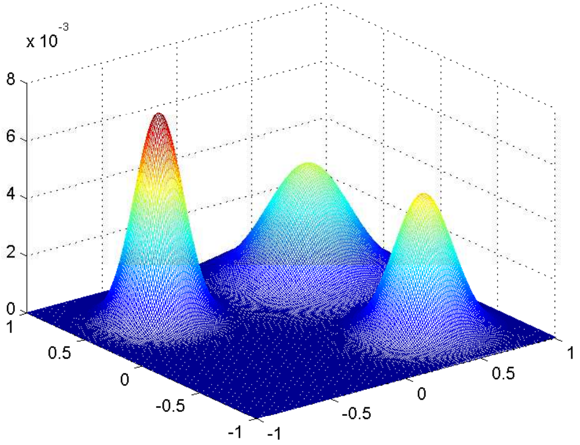


Continuum limit: PDE solution

$$
\begin{aligned}
\frac{\partial z}{\partial t}(t,x,y) =& \nabla \cdot \left(\frac{1}{4}(1-z(t,x,y))^3(1+5z(t,x,y))\nabla z(t,x,y) \right. \\
&+ \left. \begin{pmatrix} c_w(x,y)-c_e(x,y) \\ c_s(x,y)-c_n(x,y) \end{pmatrix} z(t,x,y)(1-z(t,x,y))^4 \right) + g_p(x,y)
\end{aligned}
$$

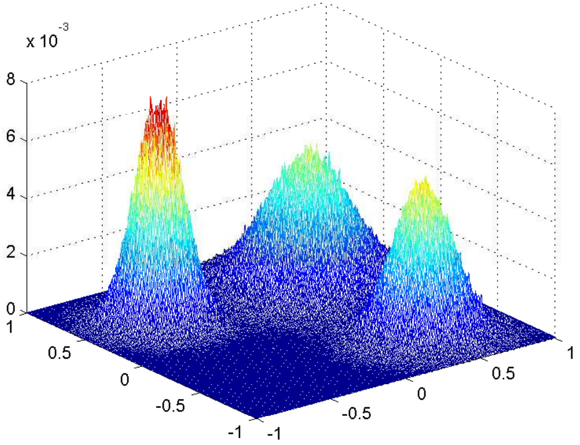
Compare

Continuum limit: PDE solution



Compare

Monte Carlo simulation result



Compare

- Convergence analysis explains why they are close.

Markov chain model

- N nodes indexed by $n = 1, \ldots, N$, placed over uniform grid points $V_N = \{v_N(1), \ldots, v_N(N)\} \subset \mathscr{D}$.
- $X_{N,M}(k,n) \in \mathbb{R}$: network state of node n at time k
- $X_{N,M}(k) = [X_{N,M}(k,1), \ldots, X_{N,M}(k,N)]^\top \in \mathbb{R}^N$: Markov chain
- Stochastic difference equation:
  -
    $$X_{N,M}(k+1) = X_{N,M}(k) + F_N(X_{N,M}(k)/M, U_N(k))$$
  - M: “averaging” parameter
  - $U_N(k)$: i.i.d. random variables independent of $X_{N,M}(k)$.

A concrete example: 1D network (for simplicity)

- Probabilities transmitting to right & left: $P_r(n)$ & $P_l(n)$ (sum up to 1).
- $U_N(k) = [Q(k,1),\ldots,Q(k,N),T(k,1),\ldots,T(k,N),G(k,1),\ldots,G(k,N)]^\top$, a random vector comprising independent random variables:
  - $Q(k,n) \in [0,1]$: uniform; determines if node is transmitter or not.
  - $T(k,n) \in \{R,L,S\}$: ternary; determines direction of transmission.
  - $G(k,n)$: number of messages generated at node n at time k.

For $x = [x_1,\ldots,x_N]^\top \in \mathbb{R}^N$, nth component of $F_N(x,U_N(k))$:

$$
\begin{cases}
1+G(k,n), \text{ if} \\
\quad Q(k,x_{n-1}) < W(n-1,x_{n-1}),\ T(k,n-1)=R,\ Q(k,x_n) > W(n,x_n),\ Q(k,x_{n+1}) > W(n+1,x_{n+1}); \\
\quad \text{or} \\
\quad Q(k,x_{n+1}) < W(n+1,x_{n+1}),\ T(k,n+1)=L,\ Q(k,x_n) > W(n,x_n),\ Q(k,x_{n-1}) > W(n-1,x_{n-1}) \\
-1+G(k,n), \text{ if} \\
\quad Q(k,x_n) < W(n,x_n),\ T(k,n)=L,\ Q(k,x_{n-1}) > W(n-1,x_{n-1}),\ Q(k,x_{n-2}) > W(n-2,x_{n-2}); \\
\quad \text{or} \\
\quad Q(k,x_n) < W(n,x_n),\ T(k,n)=R,\ Q(k,x_{n+1}) > W(n+1,x_{n+1}),\ Q(k,x_{n+2}) > W(n+2,x_{n+2}) \\
G(k,n), \text{ otherwise.}
\end{cases}
$$

x_n with $n \le 0$ or $n \ge N+1$ are defined to be 0.

Related deterministic difference equation

- Define $x_{N,M}(k)$ by:

$$x_{N,M}(k+1) = x_{N,M} + \frac{1}{M} f_N(x_{N,M}(k)), \quad x_{N,M}(0) = X_{N,M}(0)/M \text{ a.s.}$$

$$f_N(x) = EF_N(x, U_N(k)), \quad x \in \mathbb{R}^N.$$

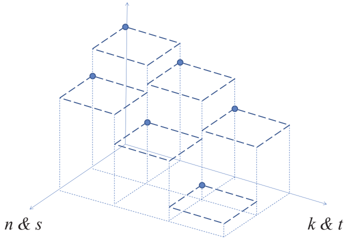
- Concrete example: 1D network: for $x = [x_1, \ldots, x_N]^\top \in [0,1]^N$,

$$\begin{aligned} f_N(x) =& (1-x_n)[P_r(n-1)x_{n-1}(1-x_{n+1}) + P_l(n+1)x_{n+1}(1-x_{n-1})] \\ & - x_n[P_r(n)(1-x_{n+1})(1-x_{n+2}) + P_l(n)(1-x_{n-1})(1-x_{n-2})] \\ & + g(n). \end{aligned}$$

x_n with $n \leq 0$ or $n \geq N+1$ are defined to be 0.

Continuous-time-space extensions of Markov chain

- $X^{(p)}_{N,M}(t,s)$: piece-wise constant continuous-time-space extension for $X_{N,M}(k)$.



$X_{N,M}(k)$: solid dots; $X^{(p)}_{N,M}(t,s)$: dashed-line rectangles

Main result

- Define the norm $\|\cdot\|^{(p)}$ for $X_{N,M}^{(p)}(t,s)$

$$\|X_{N,M}^{(p)}(t,s)\|^{(p)} = \sup_{t\in[0,T]} \int_{\mathscr{D}} |X_{N,M}^{(p)}(t,s)|\, ds.$$

- Under some conditions, if f_N, scaled in a certain way, converges to f as $N \to \infty$, and if z solves the PDE

$$\frac{\partial z}{\partial t}(s,t) = f\left(s, z(s,t), \frac{\partial z}{\partial s}(s,t), \frac{\partial^2 z}{\partial s^2}(s,t)\right),$$

  then as N and M go to ∞ in a dependent way, the Markov chain converges to the PDE solution z in the sense that $\|X_{N,M}^{(p)} - z\|^{(p)}$ converges to 0.

Main change in convergence result since prelim

- Old convergence result was under ∞-norm $\|\cdot\|_\infty^{(p)}$:

$$\|X_{N,M}^{(p)}(t,s)\|_\infty^{(p)} = \sup_{(t,s)\in[0,T]\times\mathscr{D}} |X_{N,M}^{(p)}(t,s)|.$$

- Compare current norm

$$\|X_{N,M}^{(p)}(t,s)\|^{(p)} = \sup_{t\in[0,T]} \int_{\mathscr{D}} |X_{N,M}^{(p)}(t,s)|\, ds.$$

- Under proper conditions (respectively), for general Markov model, for each of the two norms, have a similar convergence result.
- For this particular wireless sensor network model, able to prove convergence only for current norm – specific expression of deterministic difference equation of f_N near boundary.
- Big picture of proof similar to prelim presentation.

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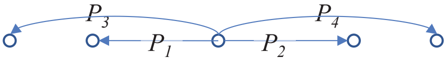
General transmission range

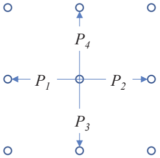
- Only considered communication and interference between immediate neighbors.
- Transmission range $L = 1, 2, \ldots$
- For 1D L-step N-node uniform network, node at $s \in V_N \subset \mathbb{R}$ communicate with nodes at $s \pm l ds_N$, where $1 \leq l \leq L$.



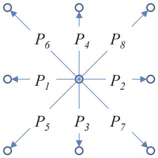
1D network with $L = 2$.

Two types of communicating neighbors in 2D

- For 2D L-step N-node uniform network, consider two types of communicating neighbors. Node at $s = (s_1, s_2) \in V_N \subset \mathbb{R}^2$ communicates with nodes at $(s_1 \pm l_1 ds_N, s_2 \pm l_2 ds_N)$, where
  - for Type-I networks, $0 \le l_1, l_2 \le L, l_1 + l_2 > 0$, and $l_1 l_2 = 0$; and
  - for Type-II networks, $0 \le l_1, l_2 \le L$ and $l_1 + l_2 > 0$.



Type-I



Type-II

- Interference can now occur between communicating neighbors.

General uniform PDE

- Now let $P_i(k,n)$ be probability of node n transmit to ith communicating neighbor at time k. Assume $P_i(k,n) = p_i(kdt_N, v_N(n))$, and $p_i(t,s) = b_i(t,s) + c_i(t,s)ds_N$; b_i and c_i: diffusion and convection.
- General uniform PDE for J-dimensional uniform network with transmission range L:

$$\dot{z} = \sum_{j=1}^{J} \left(b^{(j)} \frac{\partial}{\partial s_j} \left((1 + (\lambda_{(J,L)} + 1)z)(1-z)^{(\lambda_{(J,L)}-1)} \frac{\partial z}{\partial s_j} \right) \right.$$
$$+ 2(1-z)^{(\lambda_{(J,L)}-1)} \frac{\partial z}{\partial s_j} \frac{\partial b^{(j)}}{\partial s_j} + z(1-z)^{\lambda_{(J,L)}} \frac{\partial^2 b^{(j)}}{\partial s_j^2}$$
$$\left. + \frac{\partial}{\partial s_j} \left(c^{(j)} z (1-z)^{\lambda_{(J,L)}} \right) \right) + g_p,$$

where

$$\lambda_{(J,L)} := \begin{cases} 2LJ, & \text{for Type-I networks;} \\ (1+2L)^J - 1, & \text{for Type-II networks,} \end{cases}$$

$b^{(j)} = \sum_l^{\lambda_{(J,L)}} \frac{\rho_{jl}^2 b_l}{2}$, and $c^{(j)} = \sum_l^{\lambda_{(J,L)}} \rho_{jl} c_l$, where $\{e_1, \ldots, e_J\}$ is the natural basis of $\mathbb{R}^J$ and $\rho_{jl} = (v_N(n,l) - v_N(n))^T(e_j)$.

Transformation function of nonuniform networks

- Consider nonuniform (mobile) network with N nodes over $\mathscr{D}$.
- $\tilde{v}_N(k,n)$: location of node n at time k in nonuniform network.
- Transformation function $\phi(t,s)$ of nonuniform network maps uniform node locations to nonuniform ones:

$$\tilde{v}_N(k,n) = \phi(kdt, v_N(n)).$$

- ϕ characterizes node locations of nonuniform network.

Continuum limits of mirroring networks

- Network behavior of a certain network, for all possible m, n and k:
  - probability that node m sends to node n at time k;
  - the fact of whether node m and n interfere.
- Two seq.'s of networks (not necessarily uniform; both indexed by N) are said to mirror each other if, for each N, the N-node networks in both seq.'s have the same network behavior (assume same initial state and incoming traffic function g_p).

Theorem (Mirroring Networks Theorem)

Let $\mathbf{A}$ be a seq. of uniform networks. Suppose seq. $\mathbf{B}$ mirrors $\mathbf{A}$ and has transformation function ϕ. Then $\mathbf{A} \to q(t,s)$ iff $\mathbf{B} \to u(t,s) =: q(t,\theta(t,s))$.

(θ: inverse of ϕ w.r.t space variable s, i.e., $\theta(t,\phi(t,s)) = s$.)

Continuum limits of nonuniform networks

- Goal: for seq. $\mathbf{B}$ of nonuniform networks with given network behavior and transformation function ϕ, find its continuum limit u.
- By Mirroring Networks Theorem, if seq. $\mathbf{A}$ of uniform networks mirrors $\mathbf{B}$, then
  "$\mathbf{A} \to q(t,s)$" $\Rightarrow$ "$\mathbf{B} \to u(t,s) =: q(t,\theta(t,s))$" ($\theta$: inverse of ϕ).
- Uniform $\mathbf{A}$ with given network behavior $\to$ Known uniform PDE:

$$\dot{q} = Q\left(s, q, \frac{\partial q}{\partial s_j}, \frac{\partial^2 q}{\partial s_j^2}\right).$$

- By $u(t,s) =: q(t,\theta(t,s))$ and chain rule, $\mathbf{B} \to$ PDE:

$$\dot{u} = Q\left(\theta, u, \frac{\frac{\partial u}{\partial s_j}}{\frac{\partial \theta}{\partial s_j}}, \frac{\frac{\partial^2 u}{\partial s_j^2}}{\left(\frac{\partial \theta}{\partial s_j}\right)^2} - \frac{\frac{\partial^2 \theta}{\partial s_j^2}\frac{\partial u}{\partial s_j}}{\left(\frac{\partial \theta}{\partial s_j}\right)^3}\right).$$

Sensitivity of uniform continuum models to location perturbation

- Problem: how sensitive are uniform continuum models to location perturbation? i.e., if use uniform continuum models to approximate nonuniform networks, what is the approximation error from ignoring nonuniformity.
- Use Mirroring Networks Theorem: such approximation error = $O(\|\phi(t,s)-s\|)$.

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Network control based on continuum model

- Network behavior depends on transmission range L and prob. P_i of transmitting to each neighbor.

$$P_i(k,n) = p_i(kdt_N, v_N(n)), \qquad p_i(t,s) = b_i(t,s) + c_i(t,s)ds_N.$$

- General uniform PDE

$$\dot{z} = \sum_{j=1}^{J} \left(b^{(j)} \frac{\partial}{\partial s_j} \left((1 + (\lambda_{(J,L)} + 1)z)(1-z)^{(\lambda_{(J,L)}-1)} \frac{\partial z}{\partial s_j} \right) + 2(1-z)^{\left(\lambda_{(J,L)}-1\right)} \frac{\partial z}{\partial s_j} \frac{\partial b^{(j)}}{\partial s_j} \right.$$
$$\left. + z(1-z)^{\lambda_{(J,L)}} \frac{\partial^2 b^{(j)}}{\partial s_j^2} + \frac{\partial}{\partial s_j} \left(c^{(j)} z(1-z)^{\lambda_{(J,L)}} \right) \right) + g_p,$$

$\Rightarrow$ Continuum limit also depends on L, b_i, and c_i.

For uniform networks:

Control network behavior.

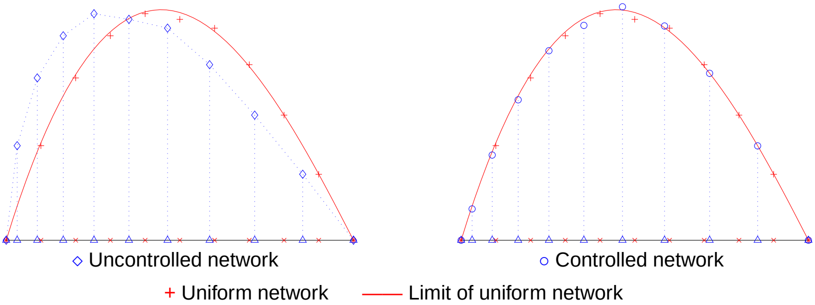
$\Uparrow$

Set L and b_i and c_i.

$\Downarrow$

Control global characteristic (continuum limit) of network.

Idea of controlling nonuniform networks



Motivation to control nonuniform networks

- For example, we can control a uniform network in order to achieve a steady state with well-balanced traffic load over the domain.
- Now the nodes start to move, and we still want similar global network characteristic.
- How to control the nonuniform (mobile) network?

Control of nonuniform networks

- Goal: for a seq. $\mathbf{B}$ of nonuniform (mobile) networks with given transformation function ϕ, find its network behavior s.t. $\mathbf{B} \to$ given continuum limit u that solves known PDE:

$$\dot{u} = \Gamma\left(s, u, \frac{\partial u}{\partial s_j}, \frac{\partial^2 u}{\partial s_j^2}\right).$$

- By Mirroring Networks Theorem, if seq. $\mathbf{A}$ of uniform networks converges to $q(t,s) := u(t, \phi(t,s))$, then "$\mathbf{B}$ mirrors $\mathbf{A}$" $\Rightarrow$ "$\mathbf{B} \to u$".
- New goal: find network behavior of $\mathbf{A}$ (since $\mathbf{B}$ mirrors $\mathbf{A}$).

Control of nonuniform networks

- By chain rule, by $q(t,s) = u(t,\phi(t,s))$, and by $\dot{u} = \Gamma\left(s, u, \frac{\partial u}{\partial s_j}, \frac{\partial^2 u}{\partial s_j^2}\right)$, $\mathbf{A} \rightarrow$ PDE

$$\dot{q} = \Gamma\left(\phi, q, \frac{\frac{\partial q}{\partial s_j}}{\frac{\partial \phi}{\partial s_j}}, \frac{\frac{\partial^2 q}{\partial s_j^2}}{\left(\frac{\partial \phi}{\partial s_j}\right)^2} - \frac{\frac{\partial^2 \phi}{\partial s_j^2}\frac{\partial q}{\partial s_j}}{\left(\frac{\partial \phi}{\partial s_j}\right)^3}\right).$$

- Meanwhile, PDE of $\mathbf{A}$ of uniform networks should be a special case of general uniform PDE presented before:

$$\begin{aligned}\dot{q} = \sum_{j=1}^{J}\Bigg(b^{(j)}\frac{\partial}{\partial s_j}\left((1+(\lambda_{(J,L)}+1)q)(1-q)^{(\lambda_{(J,L)}-1)}\frac{\partial q}{\partial s_j}\right) \\ +2(1-q)^{(\lambda_{(J,L)}-1)}\frac{\partial q}{\partial s_j}\frac{\partial b^{(j)}}{\partial s_j} + q(1-q)^{\lambda_{(J,L)}}\frac{\partial^2 b^{(j)}}{\partial s_j^2} \\ +\frac{\partial}{\partial s_j}\left(c^{(j)}q(1-q)^{\lambda_{(J,L)}}\right)\Bigg) + g_p,\end{aligned}$$

Control of nonuniform networks

- Combine two equations above and get the comparison equation:

$$\Gamma\left(\phi, q, \frac{\frac{\partial q}{\partial s_j}}{\frac{\partial \phi}{\partial s_j}}, \frac{\frac{\partial^2 q}{\partial s_j^2}}{\left(\frac{\partial \phi}{\partial s_j}\right)^2} - \frac{\frac{\partial^2 \phi}{\partial s_j^2}\frac{\partial q}{\partial s_j}}{\left(\frac{\partial \phi}{\partial s_j}\right)^3}\right) = \sum_{j=1}^{J}\left(b^{(j)}\frac{\partial}{\partial s_j}\left((1+(\lambda_{(J,L)}+1)q)(1-q)^{(\lambda_{(J,L)}-1)}\frac{\partial q}{\partial s_j}\right)\right.$$

$$+2(1-q)^{\left(\lambda_{(J,L)}-1\right)}\frac{\partial q}{\partial s_j}\frac{\partial b^{(j)}}{\partial s_j} + q(1-q)^{\lambda_{(J,L)}}\frac{\partial^2 b^{(j)}}{\partial s_j^2}$$

$$\left. + \frac{\partial}{\partial s_j}\left(c^{(j)} q (1-q)^{\lambda_{(J,L)}}\right)\right) + g_p,$$

- Solve comparison equation for L, b_i and c_i.
- Find desired network behavior.
- We show two examples.

1D example

- Want nonuniform $\mathbf{B} \to$ PDE:

$$\dot{u} = \Gamma\left(s, u, \frac{\partial u}{\partial s_j}, \frac{\partial^2 u}{\partial s_j^2}\right) = \frac{\partial}{\partial s}\left(\frac{1}{2}(1-u)(1+3u)\frac{\partial u}{\partial s}\right) + g_p.$$

- Then uniform $\mathbf{A}$ with desired network behavior $\to$ PDE:

$$\dot{q} = \Gamma\left(\phi, q, \frac{\frac{\partial q}{\partial s_j}}{\frac{\partial \phi}{\partial s_j}}, \frac{\frac{\partial^2 q}{\partial s_j^2}}{\left(\frac{\partial \phi}{\partial s_j}\right)^2} - \frac{\frac{\partial^2 \phi}{\partial s_j^2}\frac{\partial q}{\partial s_j}}{\left(\frac{\partial \phi}{\partial s_j}\right)^3}\right)$$

$$= \frac{(1-q)(1+3q)}{2\left(\frac{\partial \phi}{\partial s}\right)^2}\frac{\partial^2 q}{\partial s^2} + \frac{(1-3q)}{\left(\frac{\partial \phi}{\partial s}\right)^2}\left(\frac{\partial q}{\partial s}\right)^2 + \frac{1}{4}(1-q)(1+3q)\frac{\partial}{\partial s}\left(\frac{1}{\left(\frac{\partial \phi}{\partial s}\right)^2}\right)\frac{\partial q}{\partial s} + g_p(\phi).$$

- Solve the comparison equation for L, b_i, and c_i:

$$b\frac{\partial}{\partial s}\left((1+5q)(1-q)^3\frac{\partial q}{\partial s}\right) + 2(1-q)^3\frac{\partial q}{\partial s}\frac{\partial b}{\partial s} + q(1-q)^4\frac{\partial^2 b}{\partial s^2} + \frac{\partial}{\partial s}(cq(1-q)) + g_p'$$

$$= \frac{(1-q)(1+3q)}{2\left(\frac{\partial \phi}{\partial s}\right)^2}\frac{\partial^2 q}{\partial s^2} + \frac{(1-3q)}{\left(\frac{\partial \phi}{\partial s}\right)^2}\left(\frac{\partial q}{\partial s}\right)^2 + \frac{1}{4}(1-q)(1+3q)\frac{\partial}{\partial s}\left(\frac{1}{\left(\frac{\partial \phi}{\partial s}\right)^2}\right)\frac{\partial q}{\partial s} + g_p(\phi).$$

2D example

- Want nonuniform $\mathbf{B} \to$ PDE:

$$\dot{u} = \Gamma\left(s, u, \frac{\partial u}{\partial s_j}, \frac{\partial^2 u}{\partial s_j^2}\right) = \dot{u} = \frac{3}{8}\sum_{j=1}^{2}\frac{\partial}{\partial s_j}\left((1+9u)(1-u)^7\frac{\partial u}{\partial s_j}\right) + g_p.$$

- Then uniform $\mathbf{A}$ with desired network behavior $\to$ PDE:

$$\dot{q} = \Gamma\left(\phi, q, \frac{\frac{\partial q}{\partial s_j}}{\frac{\partial \phi}{\partial s_j}}, \frac{\frac{\partial^2 q}{\partial s_j^2}}{\left(\frac{\partial \phi}{\partial s_j}\right)^2} - \frac{\frac{\partial^2 \phi}{\partial s_j^2}\frac{\partial q}{\partial s_j}}{\left(\frac{\partial \phi}{\partial s_j}\right)^3}\right) = \sum_{j=1}^{2}\left(\frac{3}{8}\frac{(1-q)^7(1+9q)}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\frac{\partial^2 q}{\partial x_j^2}\right.$$
$$\left. + \frac{3}{16}(1-q)^7(1+9q)\frac{\partial}{\partial x_j}\left(\frac{1}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\right)\frac{\partial q}{\partial x_j} + \frac{3}{4}\frac{(1-36q)(1-q)^6}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\left(\frac{\partial q}{\partial x_j}\right)^2\right) + g_p(\phi).$$

- Solve the comparison equation for L, b_i, and c_i:

$$\sum_{j=1}^{2}\left(b^{(j)}\frac{\partial}{\partial s_j}\left((1+25q)(1-q)^{23}\frac{\partial q}{\partial s_j}\right) + 2(1-q)^{23}\frac{\partial q}{\partial s_j}\frac{\partial b^{(j)}}{\partial s_j} + q(1-q)^{24}\frac{\partial^2 b^{(j)}}{\partial s_j^2} + \frac{\partial}{\partial s_j}\left(c^{(j)}q(1-q)^{24}\right)\right) + g_p'$$
$$= \sum_{j=1}^{2}\left(\frac{3}{8}\frac{(1-q)^7(1+9q)}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\frac{\partial^2 q}{\partial x_j^2} + \frac{3}{16}(1-q)^7(1+9q)\frac{\partial}{\partial x_j}\left(\frac{1}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\right)\frac{\partial q}{\partial x_j} + \frac{3}{4}\frac{(1-36q)(1-q)^6}{\left(\frac{\partial \phi_j}{\partial s}\right)^2}\left(\frac{\partial q}{\partial x_j}\right)^2\right)$$
$$+ g_p(\phi)$$

(ϕ_j is the jth component of ϕ).

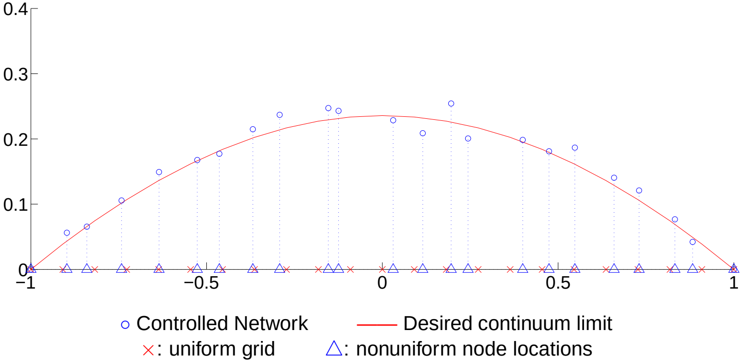
Distributed control using local information

- Above control scheme is centralized: requires knowledge of transformation function ϕ over entire $\mathscr{D}$.
- In practice, each node may not know location of all other nodes.
- A distributed control scheme: information needed to solve comparison equation can be approximated locally. In particular:
  - can approximate derivatives of ϕ from the locations of neighboring nodes using finite difference method. e.g., in 1D:

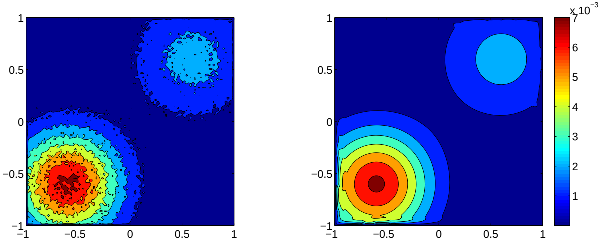
$$\frac{\partial \phi}{\partial s}(t,s) \approx \frac{\phi(kdt_N, v_N(n+1)) - \phi(kdt_N, v_N(n-1))}{2ds_N} = \frac{\tilde{v}_N(k,n+1) - \tilde{v}_N(k,n-1)}{2ds_N},$$

    where $t \approx kdt_N$ and $s \in [v_N(n-1), v_N(n+1))$.
  - can solve differential equation of b or c based on local information using numerical methods such as Euler's method.
- Meanwhile, can use location information of further neighbors to get more accurate approximation: adjustable trade-off between locality and accuracy.

1D example: simulation result



2D example: simulation result



Nonuniform network

Desired continuum limit

Table of contents

Summary

- Analyzed the convergence of a sequence of Markov chains to its continuum limit, solution of a PDE.
- Modeled uniform, nonuniform, and possibly mobile networks via PDEs.
- Developed distributed method to control behavior of nonuniform networks to achieve certain global characteristics.