Ph.D. Preliminary Examination

Continuum Limits of Markov Chains with Application to Network Modeling

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What we do & Why

Networks modeled by Markov chains.

- Traditional approach: Monte Carlo simulation.
  For networks with a very large number of nodes — slow...
- Continuum limits of Markov chains: deterministic functions with continuous variables — can characterize as PDE solutions.
- Approximate networks by PDEs and solve on computers (e.g., Matlab) — fast!

Our work = theory + application

- Theory: a class of Markov chains converges to PDEs.
- Application: some of these Markov chains model networks $\Rightarrow$ approximate networks by PDEs.

Theory supports application.
Outline of talk:

1. Result of application: approximate network by PDE.
2. Motivational example: approximate multiple random walks by PDE.
   - Not a new result;
   - Not a special case of OUR class of Markov chains;
   - Easier than ours to deal with;
   - But has similar flavor — helps understand.
3. Theory: convergence analysis.
4. Back to network: use theory to explain application.

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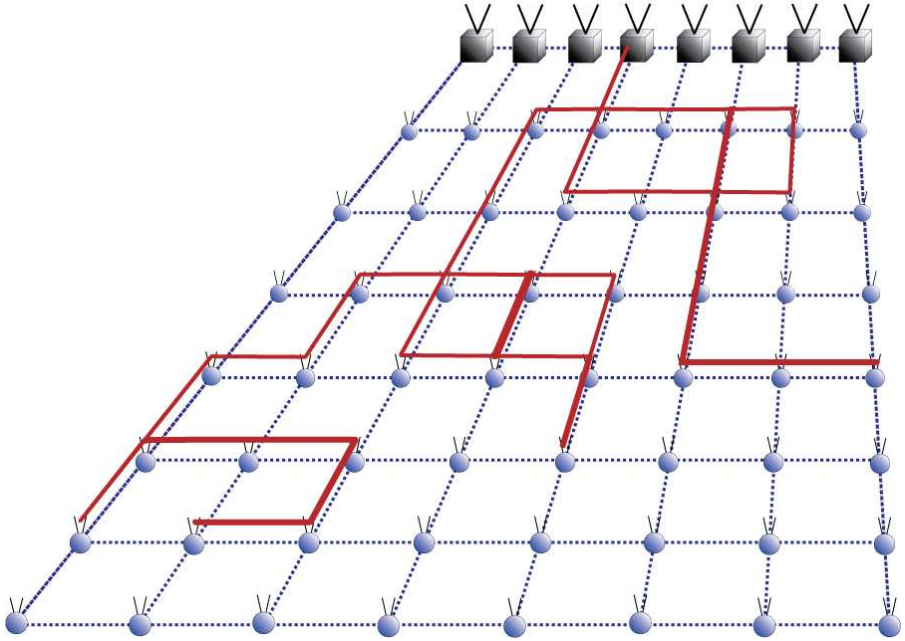
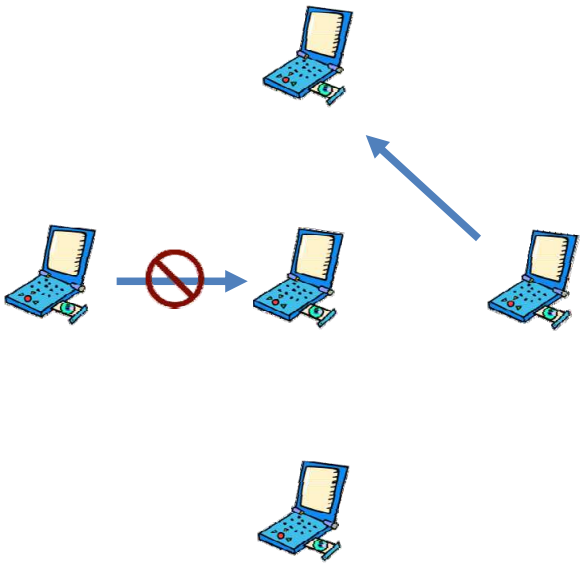
Wireless sensor network

- Sensor nodes: generate messages and relay them to destination nodes at boundary.
- All nodes have message queues.
- Nodes only communicate with immediate neighbors (1D: left and right; 2D: north, south, east, west).
- Nodes transmit or receive at each time step, but not both.



Channel model

- All nodes share the same wireless channel.
- A transmission from a transmitter to a neighboring receiver is successful iff none of the other neighbors of the receiver is a transmitter.



Reception at a node fails when more than one of its neighbors transmit.

Quantities of interest

- Interested in calculating message queue lengths at nodes.
- Nodes on uniform grid, indexed by (i,j); distance between neighboring nodes ds.
- Notation:
  - $X(k,i,j)$: queue length of node (i,j) at time k, normalized by M
  - M: maximum queue length — assume all queues $< M$
  - $F(i,j,X(k,i,j))$: probability that node (i,j) tries to transmit
  - Set $F(i,j,X(k,i,j)) = X(k,i,j)$: nodes transmit with probability proportional to queue length
  - $P_n(k,i,j), P_s(k,i,j), P_e(k,i,j), P_w(k,i,j)$: probabilities of transmitting to north, south, east, west (sum up to 1).
    $P_e(k,i,j) = \frac{1}{4} + c_e(k,i,j)ds$, $P_w(k,i,j) = \frac{1}{4} + c_w(k,i,j)ds$
    $P_n(k,i,j) = \frac{1}{4} + c_n(k,i,j)ds$, $P_s(k,i,j) = \frac{1}{4} + c_s(k,i,j)ds$
  - $G(k,i,j)$: number of messages generated at node (i,j) at time k

Stochastic difference equation

- From now on, things in GRAY means please do not look into details.

$$
\begin{aligned}
&X(k+1,i,j)-X(k,i,j) = \\
&\begin{cases}
\frac{1+U(k,i,j)}{M}, \text{ with probability} \\
(1-X(k,i,j)) \\
\times [P_w(k,i-1,j)X(k,i-1,j))(1-X(k,i+1,j))(1-X(k,i,j+1))(1-X(k,i,j-1)) \\
+P_e(k,i+1,j)X(k,i+1,j)(1-X(k,i-1,j))(1-X(k,i,j+1))(1-X(k,i,j-1)) \\
+P_n(k,i,j-1)X(k,i,j-1)(1-X(k,i+1,j))(1-X(k,i-1,j))(1-X(k,i,j+1)) \\
+P_s(k,i,j+1)X(k,i,j+1)(1-X(k,i+1,j))(1-X(k,i-1,j))(1-X(k,i,j-1))] \\
\frac{-1+G(k,i,j)}{M}, \text{ with probability} \\
X(k,i,j) \\
\times [P_w(k,i,j)(1-X(k,i-1,j))(1-X(k,i-1,j+1))(1-X(k,i-1,j-1))(1-X(k,i-2,j)) \\
+P_e(k,i,j)(1-X(k,i+1,j))(1-X(k,i+1,j+1))(1-X(k,i+1,j-1))(1-X(k,i+2,j)) \\
+P_s(k,i,j)(1-X(k,i,j-1))(1-X(k,i+1,j-1))(1-X(k,i-1,j-1))(1-X(k,i,j-2)) \\
+P_n(k,i,j)(1-X(k,i,j+1))(1-X(k,i+1,j+1))(1-X(k,i-1,j+1))(1-X(k,i,j+2))] \\
\frac{G(k,i,j)}{M}, \text{ otherwise}
\end{cases}
\end{aligned}
$$

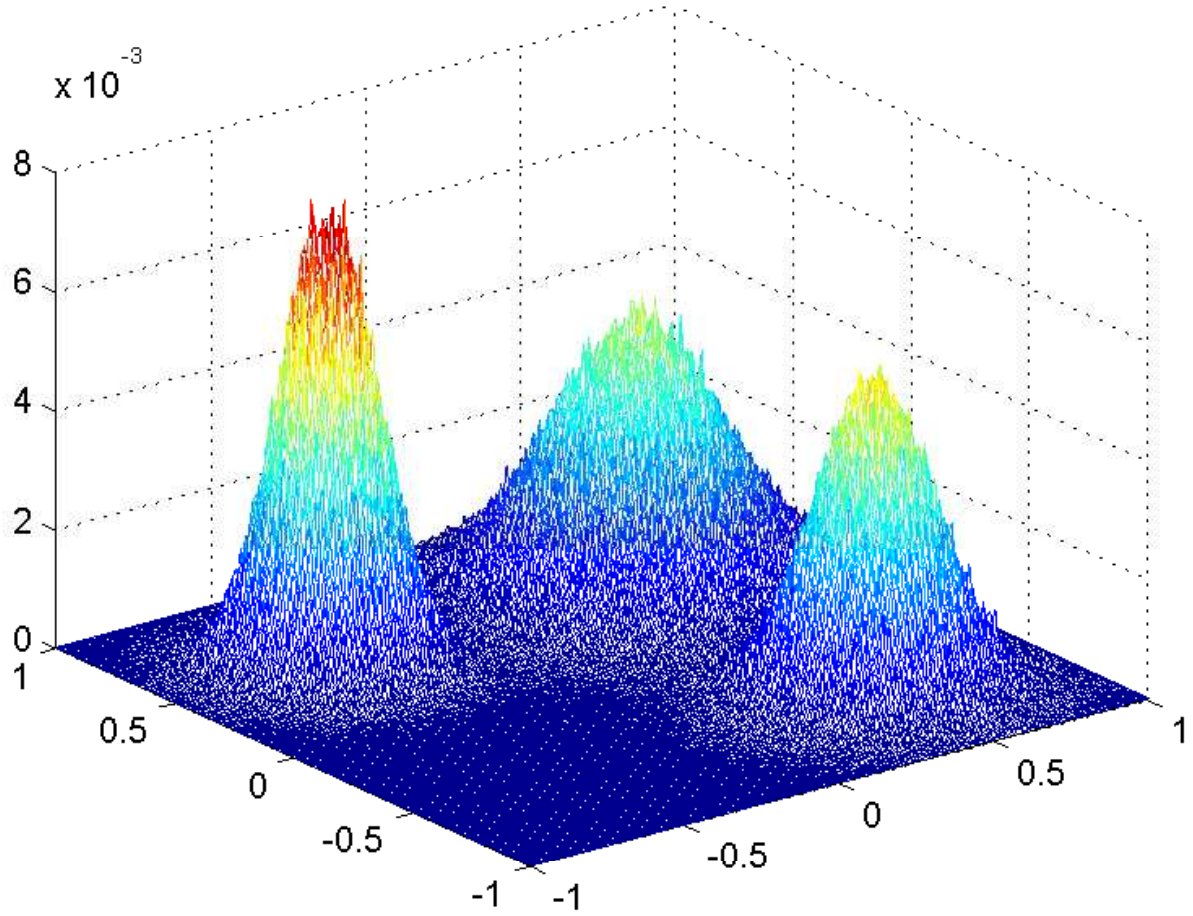
Stochastic difference equation

- From now on, things in GRAY means please do not look into details.

$$X(k+1,i,j)-X(k,i,j)=$$

$$\begin{cases} \frac{1+U(k,i,j)}{M}, \text{ with probability} \\ (1-X(k,i,j)) \\ \times[P_w(k,i-1,j)X(k,i-1,j))(1-X(k,i+1,j))(1-X(k,i,j+1))(1-X(k,i,j-1)) \\ +P_e(k,i+1,j)X(k,i+1,j)(1-X(k,i-1,j))(1-X(k,i,j+1))(1-X(k,i,j-1)) \\ +P_n(k,i,j-1)X(k,i,j-1)(1-X(k,i+1,j))(1-X(k,i-1,j))(1-X(k,i,j+1)) \\ +P_s(k,i,j+1)X(k,i,j+1)(1-X(k,i+1,j))(1-X(k,i-1,j))(1-X(k,i,j-1))] \\ \frac{-1+G(k,i,j)}{M}, \text{ with probability} \\ X(k,i,j) \\ \times[P_w(k,i,j)(1-X(k,i-1,j))(1-X(k,i-1,j+1))(1-X(k,i-1,j-1))(1-X(k,i-2,j)) \\ +P_e(k,i,j)(1-X(k,i+1,j))(1-X(k,i+1,j+1))(1-X(k,i+1,j-1))(1-X(k,i+2,j)) \\ +P_s(k,i,j)(1-X(k,i,j-1))(1-X(k,i+1,j-1))(1-X(k,i-1,j-1))(1-X(k,i,j-2)) \\ +P_n(k,i,j)(1-X(k,i,j+1))(1-X(k,i+1,j+1))(1-X(k,i-1,j+1))(1-X(k,i,j+2))] \\ \frac{G(k,i,j)}{M}, \text{ otherwise} \end{cases}$$

Monte Carlo simulation result



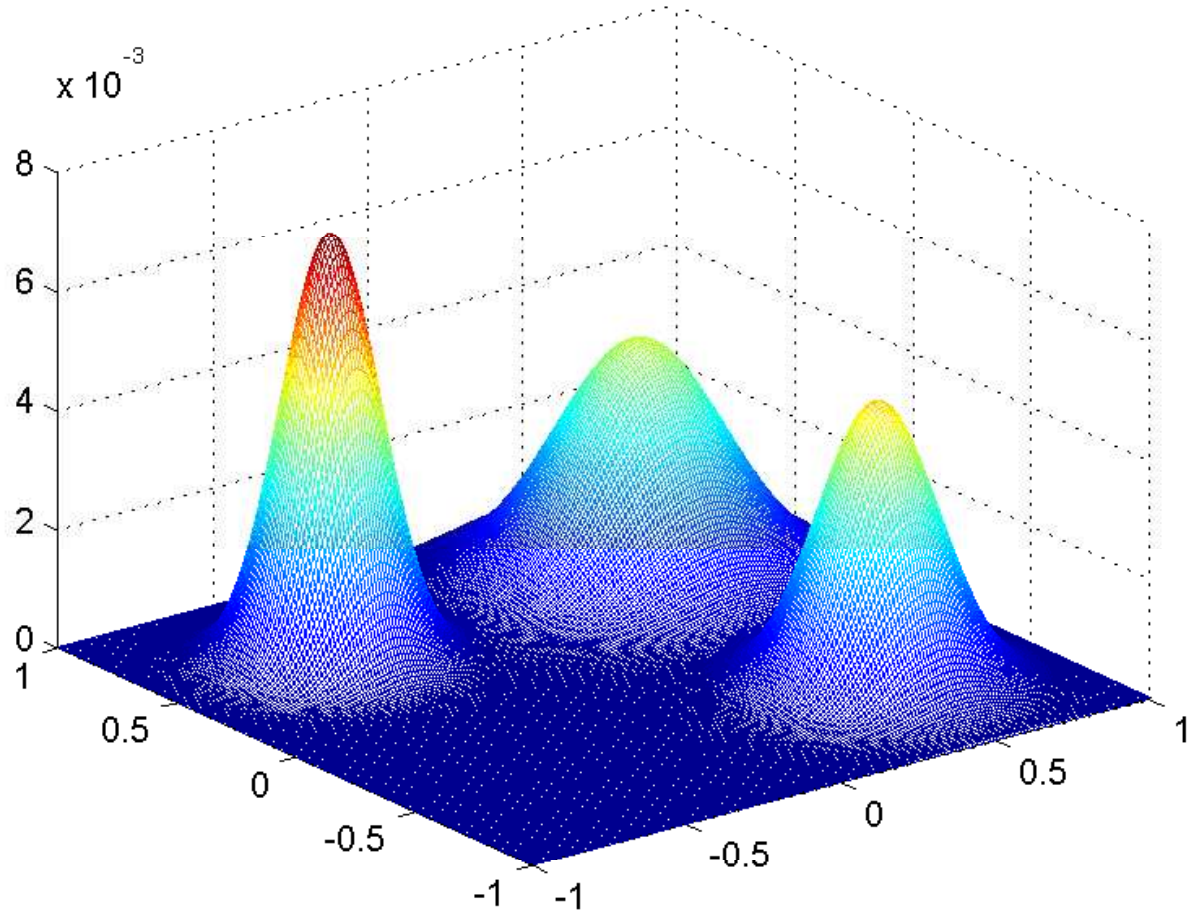
Normalized node queue length

Continuum limit: PDE solution

$$\frac{\partial z}{\partial t}(t,x,y) = \nabla \cdot \left(\frac{1}{4}(1-z(t,x,y))^3(1+5z(t,x,y))\nabla z(t,x,y) + \begin{pmatrix} c_w(t,x,y) - c_e(t,x,y) \\ c_s(t,x,y) - c_n(t,x,y) \end{pmatrix} z(t,x,y)(1-z(t,x,y))^4 \right) + g(x,y)$$

Compare

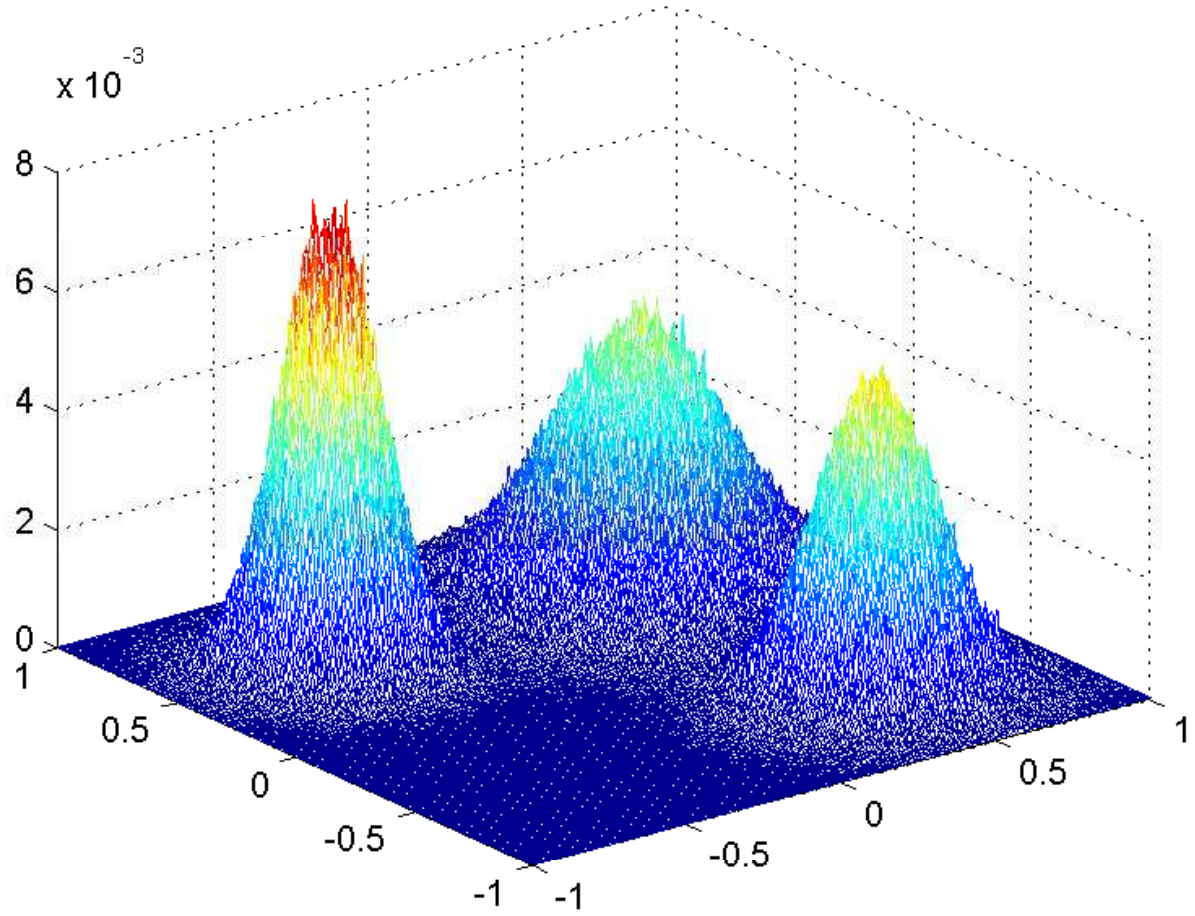
Continuum limit: PDE solution



Normalized node queue length

Compare

Monte Carlo simulation result



Normalized node queue length

Compare

- In what sense and under what conditions are they close?

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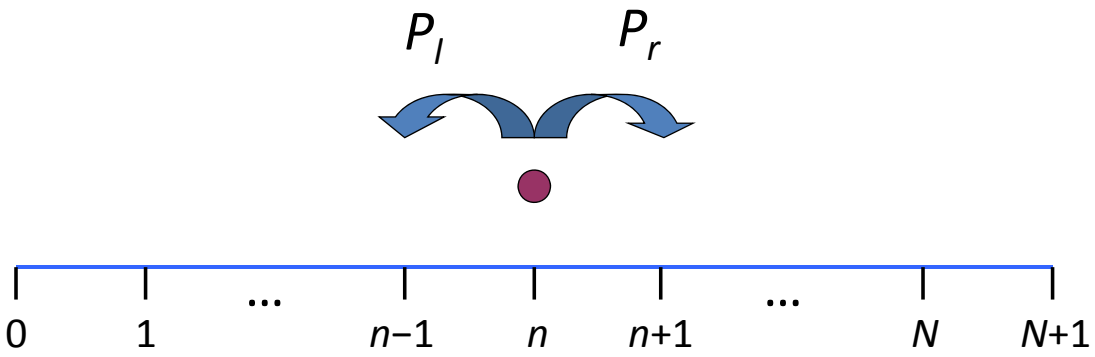
Random walk of a single particle



- Uniformly located points.
- $P_r + P_l = 1$.

Multiple random walks

- Consider M i.i.d. random walks on same domain.
- $X_N(k,n)$: number of particles at point n at time k:
- $X_N(k) = [X_N(k,1),\ldots,X_N(k,N)]^T \in \mathbb{R}^N$: Markov chain

$$X_N(k+1,n) = (\text{\# of particles coming from left point } n-1 \text{ at time } k) + (\text{\# of particles coming from right point } n+1 \text{ at time } k).$$

Multiple random walks

- Consider M i.i.d. random walks on same domain.
- $X_N(k,n)$: number of particles at point n at time k:
- $X_N(k) = [X_N(k,1),\ldots,X_N(k,N)]^T \in \mathbb{R}^N$: Markov chain

$$\begin{aligned} X_N(k+1,n) &= (\text{\# of particles coming from left point } n-1 \text{ at time } k) \\ &\quad + (\text{\# of particles coming from right point } n+1 \text{ at time } k) \\ &= Q_r(X_N(k,n-1),P_r(n-1),P_l(n-1),U(k,n-1)) \\ &\quad + Q_l(X_N(k,n+1),P_r(n+1),P_l(n+1),U(k,n+1)). \end{aligned} \tag{1}$$

(Description of functions Q_r and Q_l in backup slides.)

- Define $U(k) = [U(k,1),\ldots,U(k,N)]^T \in \mathbb{R}^N$ and write (1) as

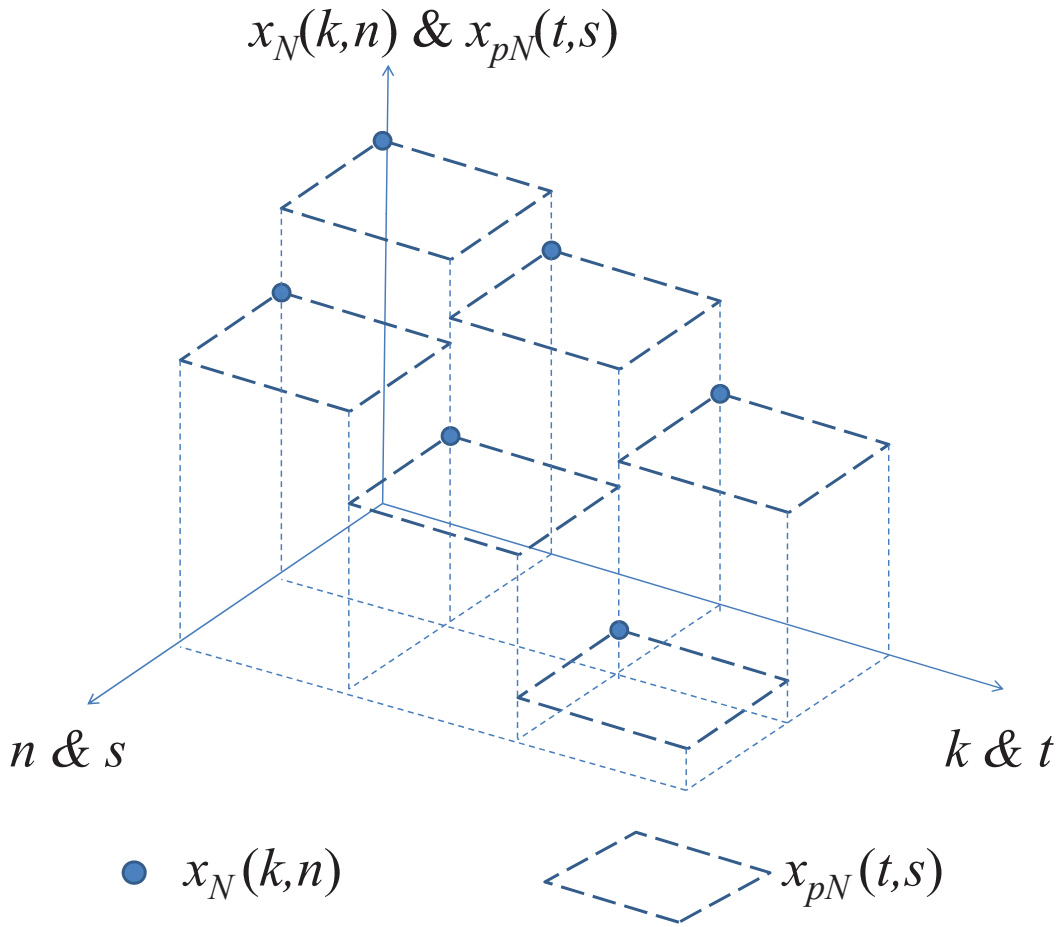
$$X_N(k+1) = X_N(k) + F_N(X_N(k),U(k)).$$

$x_N(k)$ defined by deterministic f_N

- Define
$$f_N(x) = EF_N(x, U(k)), \quad x \in \mathbb{R}^N.$$
- Define $x_N(k)$ by deterministic difference equation
$$x_N(k+1) = x_N(k) + f_N(x_N(k)), \quad x_N(0) = X_N(0)/M \text{ a.s.}$$
- By (1), can find expression of above equation (details in backup slides)
$$\begin{aligned} &x_N(k+1,n) - x_N(k,n) \\ &= P_r(n-1)x_N(k,n-1) + P_l(n+1)x_N(k,n+1) - x_N(k,n). \end{aligned} \tag{2}$$

$\Rightarrow f_N$ is linear

Continuous-time-space extensions of $X_N(k)$ and $x_N(k)$



Continuous-time-space extensions of $X_N(k)$ and $x_N(k)$ – cont'd

- $ds_N \sim 1/N$: distance between two neighboring points
- Set time step length $dt := 1/M$.
- First extend $X_N(k)$ and $x_N(k)$ in time and define piece-wise constant continuous-time extensions:
  - $X_{oN}(\tilde{t})$ for $X_N(k)$: $X_{oN}(\tilde{t}) = X_N(\lfloor \tilde{t}/dt \rfloor)/M, \quad \tilde{t} \in \mathbb{R}$.
  - $x_{oN}(\tilde{t})$ for $x_N(k)$: $x_{oN}(\tilde{t}) = x_N(\lfloor \tilde{t}/dt \rfloor), \quad \tilde{t} \in \mathbb{R}$.
  - Constant piece length dt.
- Further extend above two in space with constant piece length ds_N:
  - $X_{pN}(t,s)$: piece-wise constant continuous-time-space extension for $X_N(k)$.
  - $x_{pN}(t,s)$: piece-wise constant continuous-time-space extension for $x_N(k)$.

"$X_N(k)/M$ and $x_N(k)$ close for big M" $\Rightarrow$ "X_{pN} and x_{pN} close for big M"

- $B_i(k,n)$: Bernoulli r.v. representing presence of ith particle at point n at time k. $B_i(k) = [B_i(k,1),\ldots,B_i(k,N)]^T \in \mathbb{R}^N$.
- B_i's are i.i.d. $\Rightarrow EB_i(k) = EB(k)$.

$$X_N(k) = \sum_{i=1}^{M} B_i(k).$$

"$X_N(k)/M$ and $x_N(k)$ close for big M" $\Rightarrow$ "X_{pN} and x_{pN} close for big M"

- $B_i(k,n)$: Bernoulli r.v. representing presence of ith particle at point n at time k. $B_i(k) = [B_i(k,1),\ldots,B_i(k,N)]^T \in \mathbb{R}^N$.
- B_i's are i.i.d. $\Rightarrow EB_i(k) = EB(k)$.

$$X_N(k) = \sum_{i=1}^{M} B_i(k).$$

- X_N and $EX_N(k)$ close for big M because
  - $EX_N(k) = \sum_{i=1}^{M} EB_i(k) = MEB(k)$.
  - By SLLN, $X_N(k)/M$ and $EB(k)$ close for big M.

"$X_N(k)/M$ and $x_N(k)$ close for big M" $\Rightarrow$ "X_{pN} and x_{pN} close for big M"

- $B_i(k,n)$: Bernoulli r.v. representing presence of ith particle at point n at time k. $B_i(k) = [B_i(k,1),\ldots,B_i(k,N)]^T \in \mathbb{R}^N$.
- B_i's are i.i.d. $\Rightarrow EB_i(k) = EB(k)$.

$$X_N(k) = \sum_{i=1}^{M} B_i(k).$$

- X_N and $EX_N(k)$ close for big M.
- Linearity of $f_N \Rightarrow$ "$EX_N(k)/M$ and $x_N(k)$ satisfy the same equation." (in backup slides)

$$x_N(k+1) = x_N(k) + f_N(x_N(k)).$$

$$EX_N(k+1)/M = EX_N(k)/M + f_N(EX_N(k)/M).$$

"$X_N(k)/M$ and $x_N(k)$ close for big M" $\Rightarrow$ "X_{pN} and x_{pN} close for big M"

- $B_i(k,n)$: Bernoulli r.v. representing presence of ith particle at point n at time k. $B_i(k) = [B_i(k,1),\ldots,B_i(k,N)]^T \in \mathbb{R}^N$.
- B_i's are i.i.d. $\Rightarrow EB_i(k) = EB(k)$.

$$X_N(k) = \sum_{i=1}^{M} B_i(k).$$

- X_N and $EX_N(k)$ close for big M.
- Linearity of $f_N \Rightarrow$ "$EX_N(k)/M$ and $x_N(k)$ satisfy the same equation." (in backup slides)
- Above two $\Rightarrow$ $X_N(k)/M$ and $x_N(k)$ close for big M $\Rightarrow$ X_{pN} and x_{pN} close for big M.
- Next we derive a PDE for x_{pN}.

Getting PDE

- Assume

$$P_l(n) = p_l(nds_N) = 1/2 + c_l(nds_N)ds_N,$$
$$P_r(n) = p_r(nds_N) = 1/2 + c_r(nds_N)ds_N,$$

Define $c := c_l - c_r$.

- By difference equation of $x_N(k)$

$$\begin{aligned} &x_N(k+1,n) - x_N(k,n) \\ &= P_r(n-1)x_N(k,n-1) + P_l(n+1)x_N(k,n+1) - x_N(k,n), \end{aligned} \tag{2}$$

x_{pN} satisfies

$$\begin{aligned} &x_{pN}(t+dt,s) - x_{pN}(t,s) \\ &= p_r(s-ds_N)x_{pN}(t,s-ds_N) + p_l(s+ds_N)x_{pN}(t,s+ds_N) \\ &\quad - x_{pN}(t,s). \end{aligned} \tag{3}$$

Getting PDE – cont'd

- Set $dt = ds_N^2 \Rightarrow$ "dt and $M := 1/dt$ are now functions of N." Write M_N, dt_N.
- Put into R.H.S. of (3) Taylor expansions:

$$x_{pN}(t, s \pm ds_N) = x_{pN}(t,s) \pm \frac{\partial x_{pN}}{\partial s}(t,s) ds_N + \frac{\partial^2 x_{pN}}{\partial s^2}(t,s)\frac{ds_N^2}{2} + O(ds_N^3),$$

$$c(s \pm ds_N) = c(s) \pm c_s(s) ds_N + O(ds_N^2).$$

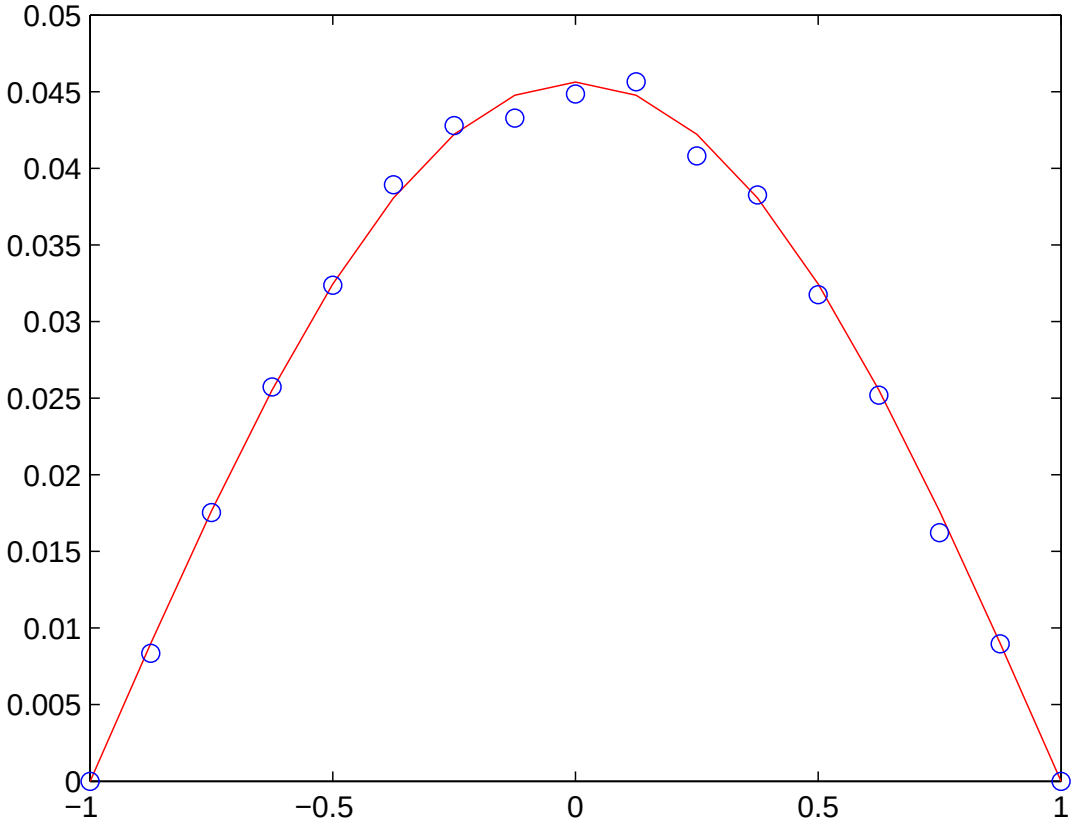
- Divide both sides by $dt_N = ds_N^2$,

$$\frac{x_{pN}(t+dt_N, s) - x_{pN}(t,s)}{dt_N} = \frac{1}{2}\frac{\partial^2 x_{pN}}{\partial s^2}(t,s) + c(s)\frac{\partial x_{pN}}{\partial s}(t,s) + c_s(s) x_{pN}(t,s) + \frac{O(ds_N^3)}{ds_N^2}.$$

- Take limit as $N \to \infty$ ($M_N \to \infty$, $ds_N, dt_N \to 0$), and get PDE for x_{pN}:

$$\frac{\partial x_{pN}}{\partial t}(t,s) = \frac{1}{2}\frac{\partial^2 x_{pN}}{\partial s^2}(t,s) + c(s)\frac{\partial x_{pN}}{\partial s}(t,s) + c_s(s) x_{pN}(t,s).$$

Monte Carlo simulation v.s. PDE solution



Particle number (normalized by M)

Monte Carlo simulation — PDE solution

Discussion

- Not new result, done by e.g. Einstein.
- This example gives us some basic ideas of convergence.
- Our Markov chains are different.
- No linearity or i.i.d. sum: need new technique to show "$X_N(k)/M$ close to $x_N(k)$ for big M".
- Getting PDE here not rigorous.
- We do convergence analysis for our Markov chains rigorously.
- We also study rate of convergence.

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General setting

- $\mathscr{D}$: fixed subset of $\mathbb{R}^d$.
- N nodes indexed by $n = 1, \ldots, N$, placed over uniform grid points $V_N = \{v_N(1), \ldots, v_N(N)\} \subset \mathscr{D}$.
- ds_N: distance between neighboring nodes
- $X_N(k,n) \in \mathbb{R}$: network state of node n at time k
- $X_N(k) = [X_N(k,1), \ldots, X_N(k,N)] \in \mathbb{R}^N$: Markov chain
- Stochastic difference equation:
  - $$X_N(k+1) = X_N(k) + F_N(X_N(k)/M, U(k))$$

    (Compare $X_N(k+1) = X_N(k) + F_N(X_N(k), U(k))$ from random walks.)
  - M: "normalizing" parameter
  - $U(k)$: i.i.d. random variables independent of $X_N(k)$

Recall network model

- X_N: message queue length
- M: maximum queue length
- U: number of messages generated by nodes and choices of nodes whether and where to transmit

General setting – cont'd

- Deterministic f_N:

$$f_N(x) = EF_N(x, U(k)), \quad x \in \mathbb{R}^N.$$

- Define $x_N(k)$ by deterministic difference equation:

$$x_N(k+1) = x_N(k) + \frac{1}{M} f_N(x_N(k)), \quad x_N(0) = X_N(0)/M \text{ a.s.}$$

(Compare $x_N(k+1) = x_N(k) + f_N(x_N(k))$ from random walks.)

Main result

- Under some conditions, if f_N, scaled in a certain way, converges to f as $N \to \infty$, and if z solves the PDE

$$\frac{\partial z}{\partial t}(s,t) = f\left(s, z(s,t), \frac{\partial z}{\partial s}(s,t), \frac{\partial^2 z}{\partial s^2}(s,t)\right),$$

then in some sense, the Markov chain $X_N(k)/M$, converges to the PDE solution z as N and M go to ∞ in a dependent way.

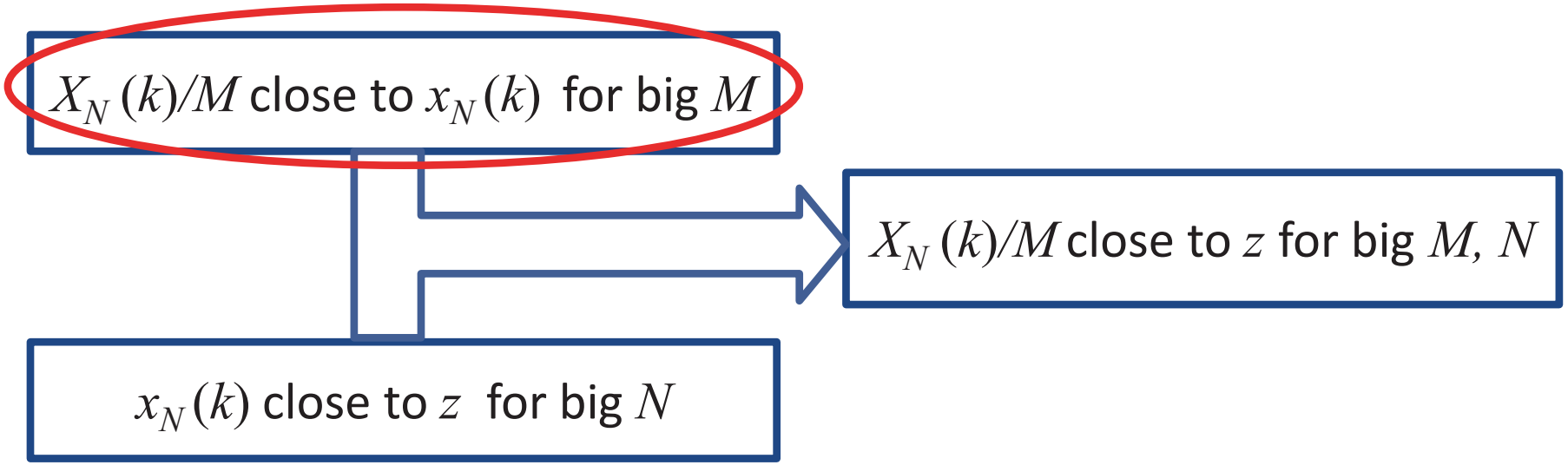
Outline of proof

$X_N(k)/M$ close to $x_N(k)$ for big M

$x_N(k)$ close to z for big N

$X_N(k)/M$ close to z for big M, N

Convergence to ODE: $X_N(k)/M$ and $x_N(k)$ close for big M



- Consider fixed N.

In contrast to multiple random walks...

- X_N no longer sum of i.i.d. r.v.'s.
- f_N no longer linear.

Use Kushner's convergence theorem [1] to show $X_N(k)/M$ and $x_N(k)$ close for big M.

[1] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory*. Cambridge, MA: MIT Press, 1984.

Convergence to ODE

- Using Kushner's theorem,

Lemma (ODE Convergence Lemma)

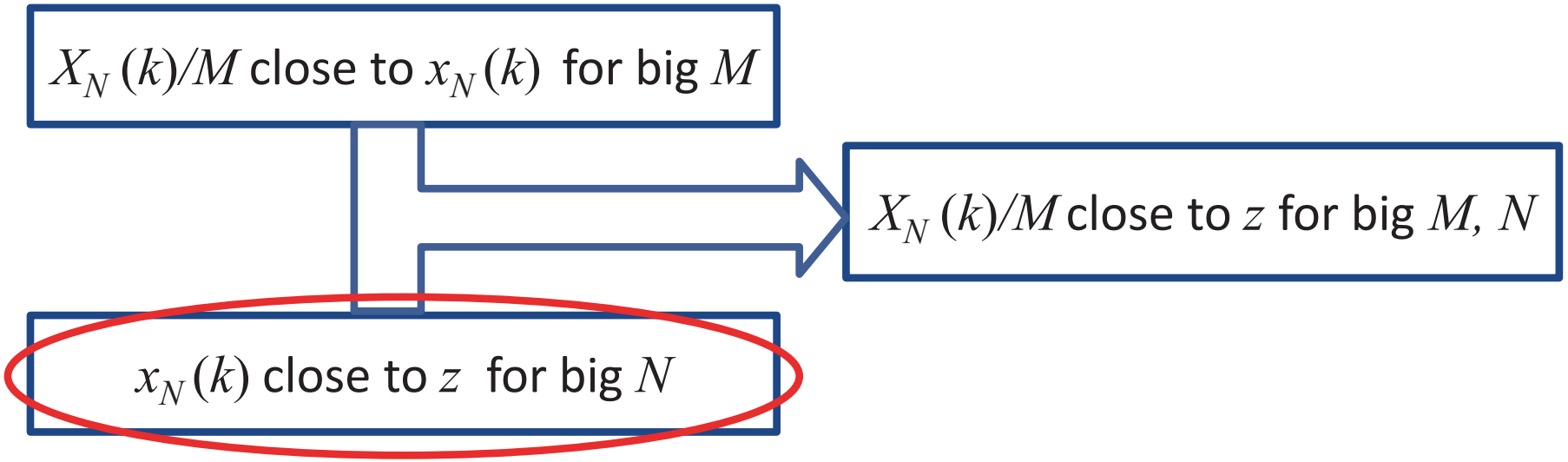
Under some conditions (on continuity and integrability of $F_N(x, U(k))$), for each N, as $M \to \infty$, $X_{oN}(\tilde{t})$ and $x_{oN}(\tilde{t})$, the continuous-time extensions of $X_N(k)/M$ and $x_N(k)$ both converge to solution of ODE $\dot{y} = f_N(y)$.

- Extensions close, so are corresponding sequences $\Rightarrow$ $X_N(k)/M$ and $x_N(k)$ are close for big M: for any sequence $\{\zeta_N\}$, for each N and for M sufficiently large,

$$P\left\{\max_{k,n}\left|\frac{X_N(k,n)}{M} - x_N(k,n)\right| > \zeta_N\right\} \leq \frac{1}{N^2}.$$

- This finishes first step.

Convergence to PDE: $x_N(k)$ and z close for big N



- Consider fixed M.

Basic idea

- Recall: $x_N(k)$ defined by

$$x_N(k+1) - x_N(k) = \frac{1}{M} f_N(x_N(k)).$$

- Set $dt_N = ds_N^2/M$. (Compare $dt_N = ds_N^2$ from random walks)
- Intuition: divide both sides by ds_N^2 and let $N \to \infty$,

$$\begin{array}{ccc} \frac{x_N(k+1)-x_N(k)}{dt_N(:=\frac{ds_N^2}{M})} & = & \frac{f_N(x_N(k)/M)}{ds_N^2} \\ \downarrow & & \downarrow \\ \dot{z} & = & f \end{array}$$

- PDE: $\dot{z} = f$.

f_N converges to f

- Rigorous description in backup slides.
- For each N, at "corresponding" points,

$$\frac{f_N}{ds_N^2} = f + O(\gamma_N).$$

- $\gamma_N \in \mathbb{R}^N$: truncation error, function of N, goes to 0 as $N \to \infty$.
- How fast γ_N goes to 0 depends on f_N: for example, 1D network $\gamma_N = O(ds_N)$.

Define sequence z_N from PDE solution z

- Fix $T > 0$. Assume there is unique $z : [0,T] \times \mathscr{D} \to \mathbb{R}$ that solves PDE

$$\dot{z}(t,s) = f(s, z(t,s), \nabla z(t,s), \nabla^2 z(t,s)).$$

- Define total time step

$$K_N = \lfloor T/dt_N \rfloor.$$

- Define time points

$$t_N(k) = k dt_N.$$

  Time indices $k = 0, \ldots, K_N$ correspond to $t_N(k) \in [0,T]$.

- Define

$$z_N(k,n) = z(t_N(k), v_N(n)),$$

  and $z_N(k) = [z_N(k,1), \ldots, z_N(k,N)]^T \in \mathbb{R}^N$.

- Goal: z_N and x_N close for big N.

Relationship between z_N and f_N

- Define $u_N(k) \in \mathbb{R}^N$ by

$$z_N(k+1) - z_N(k) = \frac{1}{M} f_N(z_N(k)) - dt_N u_N(k).$$

- Define $u_N = [u_N(0), \ldots, u_N(K_N - 1)] \in \mathbb{R}^{N \times K_N}$.

Lemma

Assume z is continuously differentiable in t. Then

$$\|u_N\|_\infty^{(N)} = O(\max\{\gamma_N, dt_N\}).$$

(γ_N: truncation error)
$\|\cdot\|_\infty^{(N)}$: ∞-norm (uniform norm), takes maximum absolute value of all components.

Error function ε_N

- Define

$$\varepsilon_N(k,n) = x_N(k,n) - z_N(k,n),$$

$\varepsilon_N(k) = [\varepsilon_N(k,1), \ldots, \varepsilon_N(k,N)]^T \in \mathbb{R}^N$,
$\varepsilon_N = [\varepsilon_N(1), \ldots, \varepsilon_N(K_N)] \in \mathbb{R}^{N \times K_N}$.

- Goal: uniform convergence: $\|\varepsilon_N\|_\infty^{(N)} \to 0$ as $N \to \infty$.
- Assume zero initial error: $\|\varepsilon_N(0)\|_\infty^{(N)} = 0$.
- Then for fixed $z_N(k)$, there is function $H_N : \mathbb{R}^{N \times K_N} \to \mathbb{R}^{N \times K_N}$ s.t.

$$\varepsilon_N = H_N(u_N).$$

$$u_N \longrightarrow \boxed{H_N} \longrightarrow \varepsilon_N$$

- Define

$$\mu_N = \lim_{\alpha \to 0} \sup_{\|u\|_\infty^{(N)} \le \alpha} \frac{\|H_N(u)\|_\infty^{(N)}}{\|u\|_\infty^{(N)}}.$$

- μ_N characterizes how much H_N amplifies its input near 0.

Main result of PDE convergence step

- Under some conditions $\|\varepsilon_N\|_\infty^{(N)} \to 0$ with same rate as $O(\|u_N\|_\infty^{(N)}) = O(\max\{\gamma_N, dt_N\})$.

Lemma (PDE Convergence Lemma)

Assume that

- *z is continuously differentiable in t;*
- *for each N, $\|\varepsilon_N(0)\|_\infty^{(N)} = 0$; and*
- *sequence $\{\mu_N\}$ is bounded.*

Then

$$\|\varepsilon_N\|_\infty^{(N)} = O(\|u_N\|_\infty^{(N)}) = O(\max\{\gamma_N, dt_N\}).$$

- Key assumption: boundedness of $\{\mu_N\}$.
- Hard to directly check in practice.

A sufficient condition for key assumption

- $Df_N(x)$: derivative matrix of f_N at x. Define

$$A_N(k) = I_N + \frac{1}{M} Df_N(z_N(k)),$$

Lemma (Bounded μ_N Lemma)

If $\|A_N(k)\|_\infty^{(N)} = 1 + O(dt_N)$, then $\{\mu_N\}$ is bounded.

(Outline of proof in backup slides.)

- $\|\cdot\|_\infty^{(N)}$: induced ∞-norm on $\mathbb{R}^{N\times N}$ — for $A = (a_{i,j}) \in \mathbb{R}^{N\times N}$, $\|A\|_\infty^{(N)} = \max_{1\le i\le N} \sum_{j=1}^N |a_{i,j}|$, (maximum absolute row sum of the matrix A).
- Helps check in practice; will use to prove convergence for network.

Main result

$X_N(k)/M$ close to $x_N(k)$ for big M

$x_N(k)$ close to z for big N

$X_N(k)/M$ close to z for big M, N

Main result

- Let M be a function of N and rewrite as M_N.
- Define $X_N = [X_N(1)/M, \ldots, X_N(K_N)/M]$, $z_N = [z_N(1), \ldots, z_N(K_N)] \in \mathbb{R}^{N \times K_N}$.
- As N and M_N go to ∞ in a dependent way, X_N converges uniformly to z_N a.s., and at least with same rate as $\max\{\gamma_N, dt_N\}$.

Theorem

Suppose assumptions in ODE Convergence Lemma and PDE Convergence Lemma hold. Then a.s., there exist $0 < c < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N > N_0$ and for each $M_N > \hat{M}_N$,

$$\|X_N - z_N\|_\infty^{(N)} \leq c \max\{\gamma_N, dt_N\}.$$

- Idea of proof: previous two steps and triangle inequality.

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1D network

- For simplicity, analyze 1D case ($\mathscr{D} \subset \mathbb{R}$). (2D is similar.)
- By f_N of 1D network, truncation error $\|\gamma_N\|_\infty^{(N)} = O(ds_N)$.
- PDE:

$$\dot{z}(t,s) = \frac{\partial}{\partial s}\left(\frac{1}{2}(1-z(t,s))(1+3z(t,s))z_s(t,s) + c(s)z(t,s)(1-z(t,s))^2\right) + g_p(s).$$

- To prove convergence, only assumption in main theorem hard to check is key assumption: boundedness of $\{\mu_N\}$.

Lemma on key assumption (bnded $\{\mu_N\}$) in 1D network

Lemma (Network Lemma)

Assume $\max\{|z|,|z_s|,|z_{ss}|,|c|,|c_s|\}$ *is bounded, then* $\{\mu_N\}$ *is bounded.*

- To prove, calculate $\|A_N(k)\|_\infty^{(N)}$:

$$\|A_N(k)\|_\infty^{(N)} = \max_{n=1,\dots,N} \frac{1}{M}(|P_l(n)z_N(k,n)(1-z_N(k,n-1))| + |(1-z_N(k,n))[P_r(n-1)(1-z_N(k,n+1)) - P_l(n+1)z_N(k,n+1)]$$
$$+ P_l(n)z_N(k,n)(1-z_N(k,n-2))| + |M - [P_r(n-1)z_N(k,n-1)(1-z_N(k,n+1)) + P_l(n+1)z_N(k,n+1)(1-z_N(k,n-1))]$$
$$- [P_r(n)(1-z_N(k,n+1))(1-z_N(k,n+2)) + P_l(n)(1-z_N(k,n-1))(1-z_N(k,n-2))]| + |(1-z_N(k,n))[P_l(n+1)(1-z_N(k,n-1))$$
$$- P_r(n-1)z_N(k,n-1)] + P_r(n)z_N(k,n)(1-z_N(k,n+2))]| + |P_r(n)z_N(k,n)(1-z_N(k,n+1))|).$$

- Put in Taylor's expansions and rearrange:

$$\|A_N(k)\|_\infty^{(N)} \le \max_{n=1,\dots,N} |-c_s(v_N(n)) - (v_N(n))z_{ss}(t_N(k),v_N(n)) + 4c_s(v_N(n))z(t_N(k),v_N(n)) + 4c(v_N(n))z_s(t_N(k),v_N(n))$$
$$+ 3z_s(t_N(k),v_N(n))^2 - 3c_s(v_N(n))z(t_N(k),v_N(n))^2 + 3z(t_N(k),v_N(n))z_{ss}(t_N(k),v_N(n))$$
$$- 6c(v_N(n))z(t_N(k),v_N(n))z_s(t_N(k),v_N(n))|\frac{ds_N^2}{M} + 1 + c_1\frac{ds_N^3}{M}.$$

- By $dt_N = ds_N^2/M$, get $\|A_N(k)\|_\infty^{(N)} = 1 + O(dt_N)$. Satisfies condition of Bounded μ_N Lemma.

Recall Bounded μ_N Lemma

Lemma (Bounded μ_N Lemma)

If $\|A_N(k)\|_\infty^{(N)} = 1 + O(dt_N)$, *then* $\{\mu_N\}$ *is bounded.*

Proposition on convergence in 1D network

- Use main theorem.

Proposition

Suppose assumption in Network Lemma holds. Then a.s., $\|X_N - z_N\|_\infty^{(N)} \to 0$ *as* N *and* M_N *go to* ∞ *in a dependent way, with rate* $O(\|u_N\|_\infty^{(N)}) = O(\max(dt_N, \gamma_N)) = O(\max(dt_N, ds_N)) = O(ds_N)$.

- This finishes convergence analysis for network.

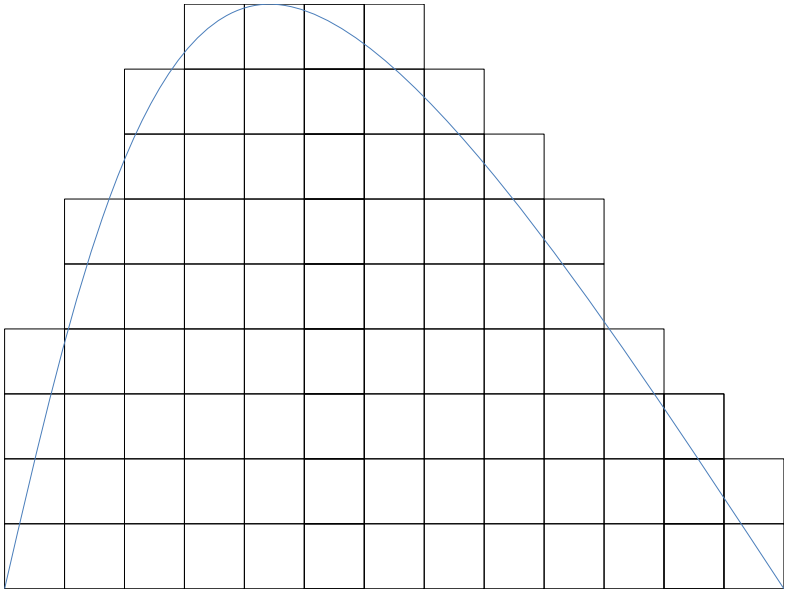
Interpretation of PDE

- $z(t_N(k), v_N(n)) \approx \frac{X_N(k,n)}{M}$: PDE solution at corresponding points approximates normalized value of Markov chain.
- Area under curve of PDE solution:

$$\int_{\mathscr{D}} z(t_N(k), s) ds \approx \sum_{n=1}^{N} \frac{X_N(k_o, n)}{M} ds_N.$$

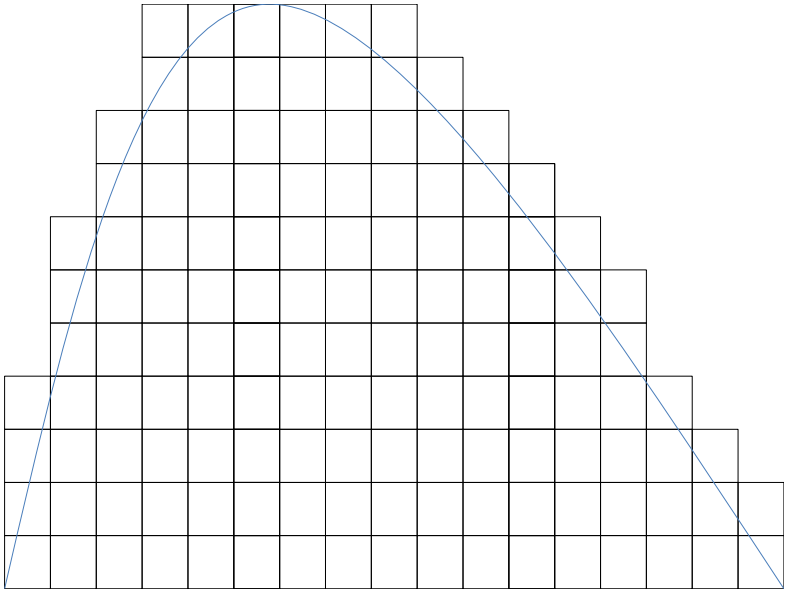
This area measures sum of "data-coverage product" of all nodes: (normalized queue length of each node $\times$ space occupied by each node) summed all over network.

Interpretation of PDE – cont'd



- Each column: message queue of a node.
- Width of each rectangle: space occupied by node.
- Hight of each rectangle: bits in a message.
- N and M grow: more nodes and more messages per node, but each node occupies less space and each message has fewer bits $\Rightarrow$ sum of areas of rectangles remains approximately area under curve, so normalized total queue length in network will not explode.

Interpretation of PDE – cont'd



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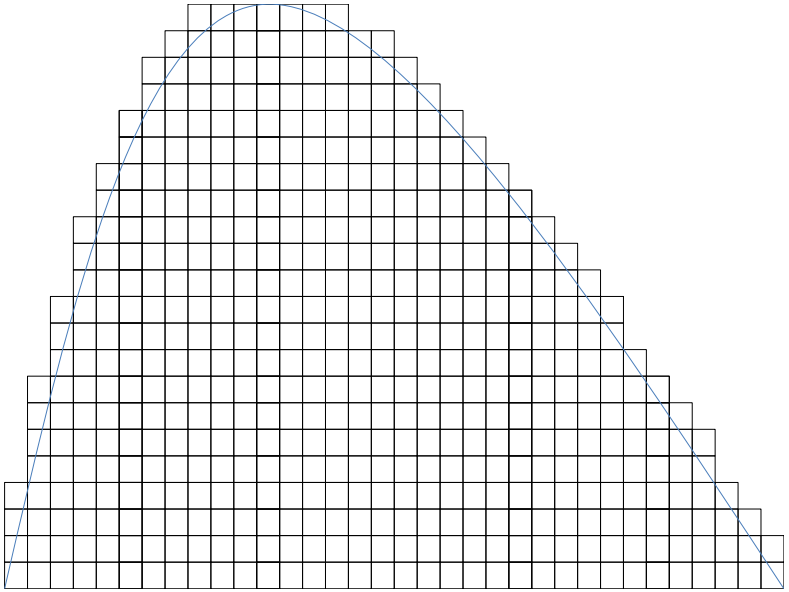
Interpretation of PDE – cont'd



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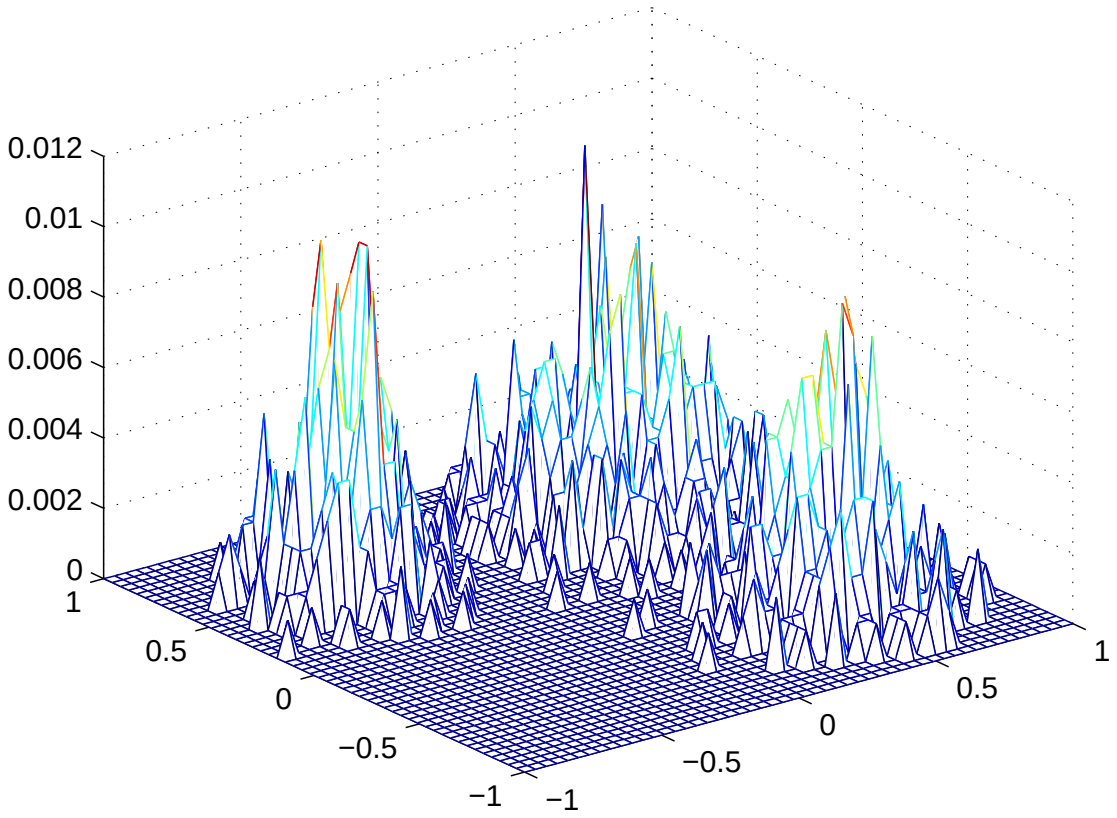
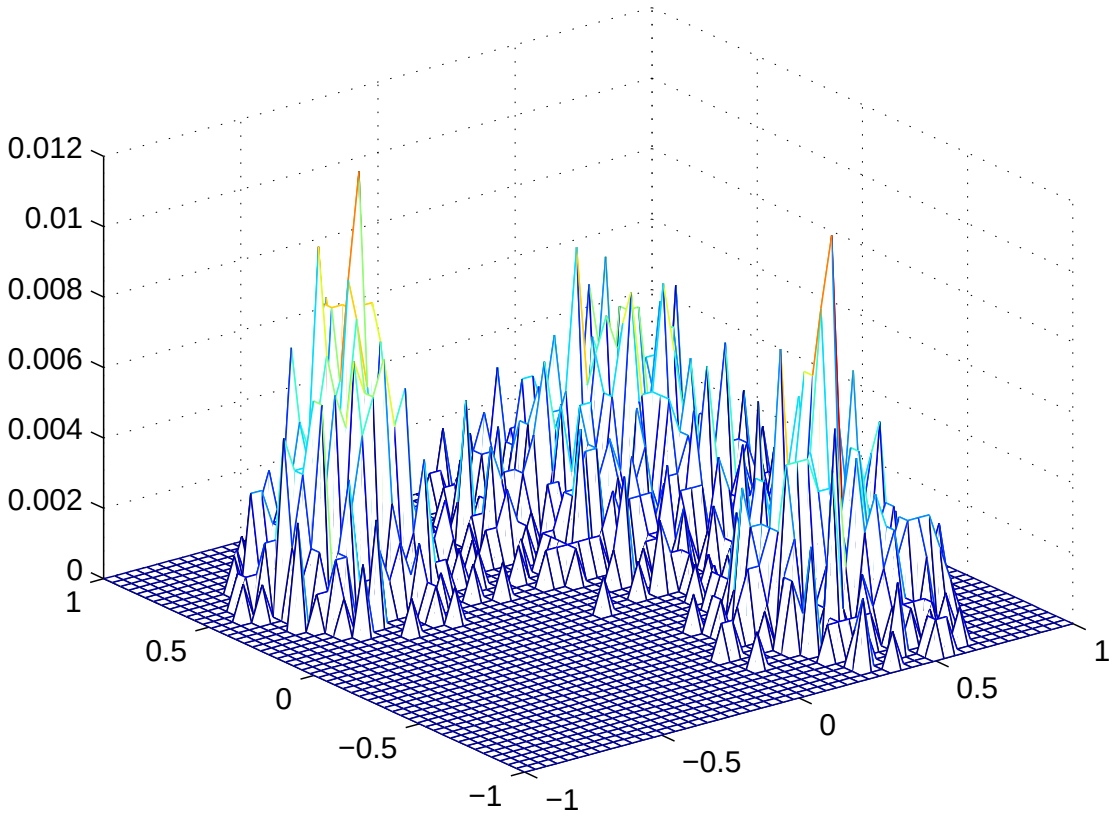
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 50, M = 1000$

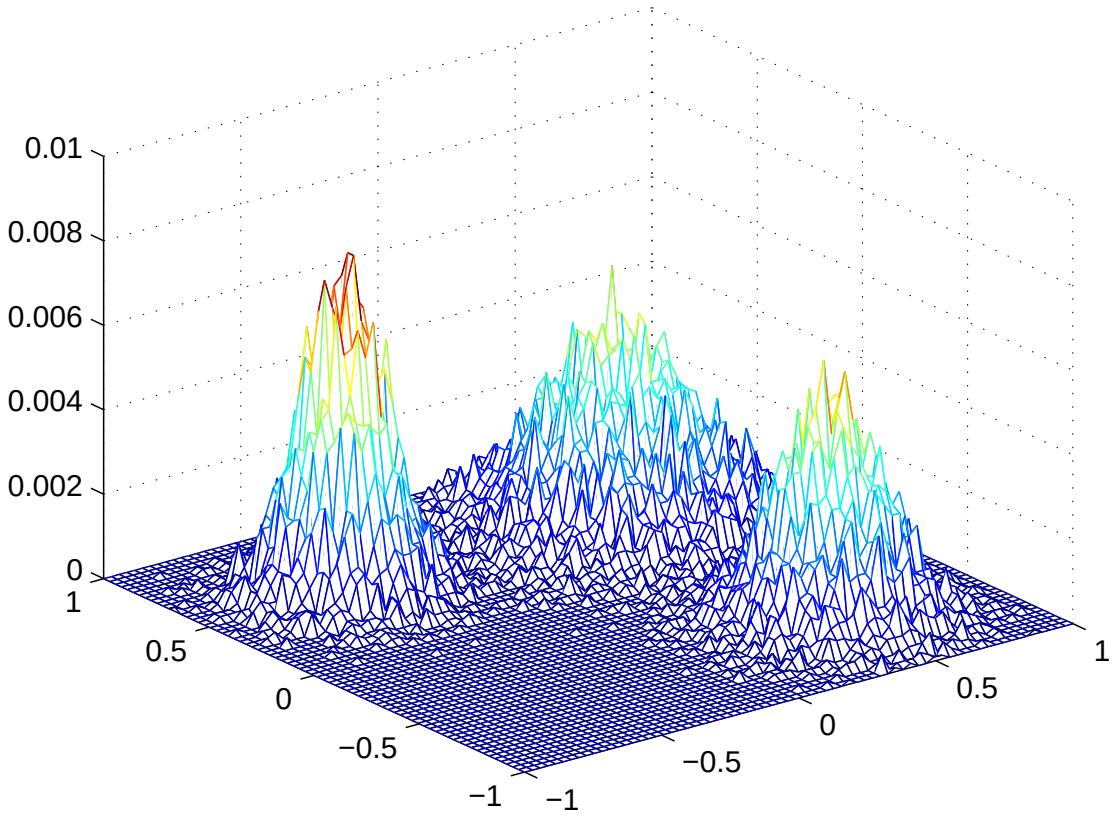
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 60, M = 5000$

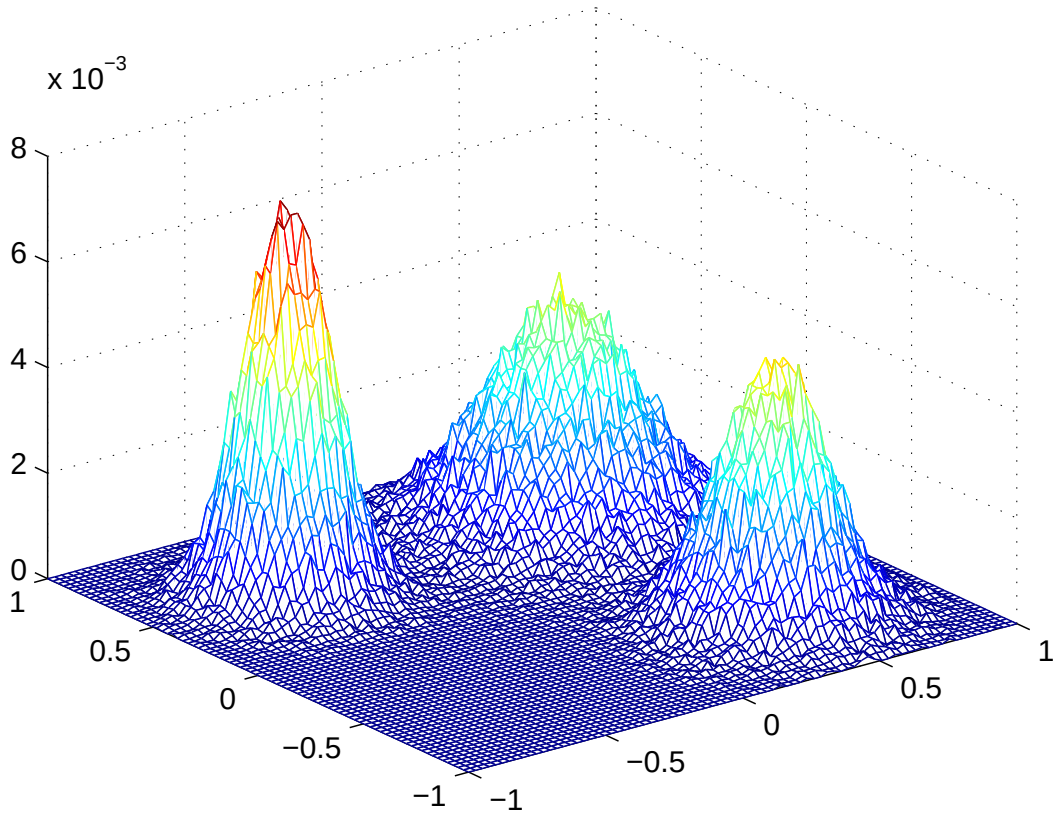
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 70, M = 10000$

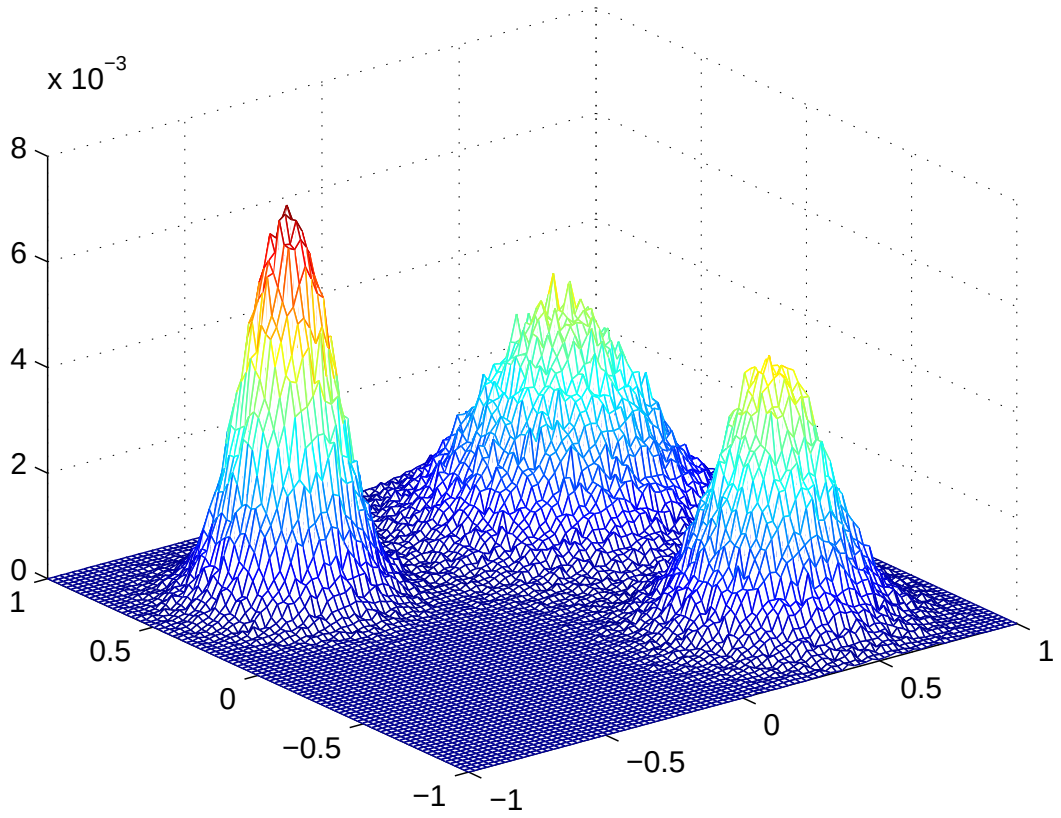
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 80, M = 50000$

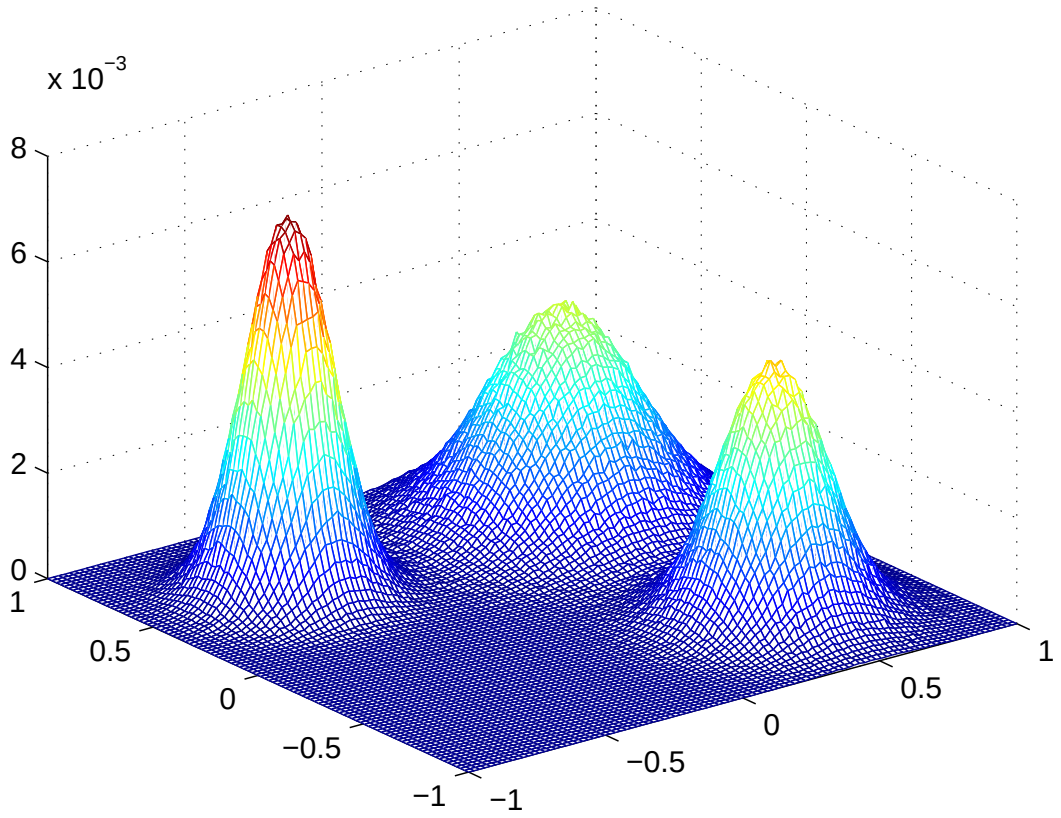
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 90, M = 100000$

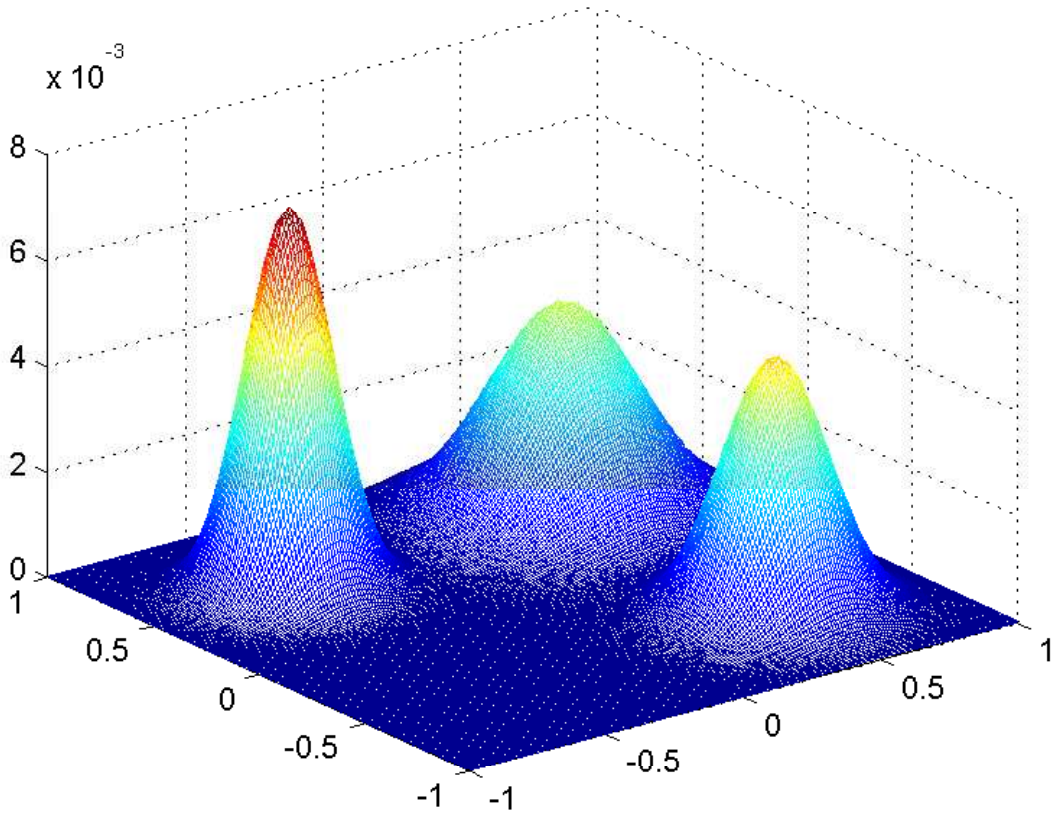
Monte Carlo simulation with increasing M and N



Normalized node queue length

$N = 100, M = 1000000$

Monte Carlo simulation with increasing M and N

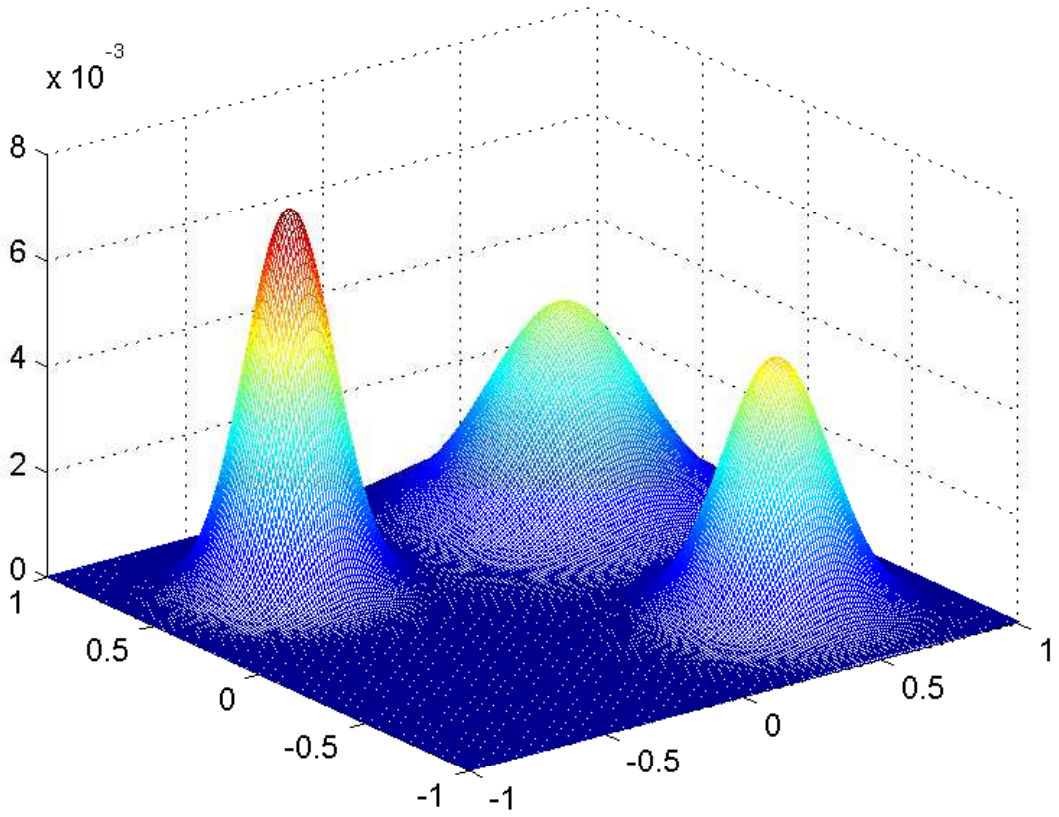


Normalized node queue length

$N = 200, M = 5000000$

This took weeks!

PDE Solution



Normalized node queue length

This took seconds.

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Summary

- Analyzed the convergence of a sequence of Markov chains to its continuum limit, solution of a PDE, in a two-step procedure.
- Provided precise sufficient conditions for the convergence and explicit rate.
- Approximated a network by a PDE.

Publications & Presentations

- Y. Zhang, E. K. P. Chong, J. Hannig, and D. Estep, "On continuum limits of Markov chains and network modeling," in *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, Georgia, December 15–17, 2010, pp. 6779–6784.
- Y. Zhang, E. K. P. Chong, J. Hannig, and D. Estep, "Continuum Limits of Markov Chains with Application to Network Modeling," in *arXiv*:1106.4288v1 [cs.NI].
- Y. Zhang, E. K. P. Chong, J. Hannig, and D. Estep, “Continuum limits of Markov chains and network modeling,” presented at *The Mathematical Association of America Rocky Mountain Section Meeting*, 2010
- Y. Zhang and E. K. P. Chong, “Continuum models of large networks,” presented in the poster session of the Graduate Student Visit Day of Dept. of Electrical and Computer Engineering, Colorado State University, 2010

Remaining work: study more general transmissions

- Ideas:
- 1D, 2-step

Remaining work: study more general transmissions

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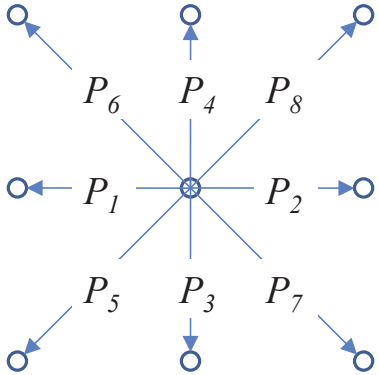
- 2D, new transmission range, 1-step,



Remaining work: study more general transmissions

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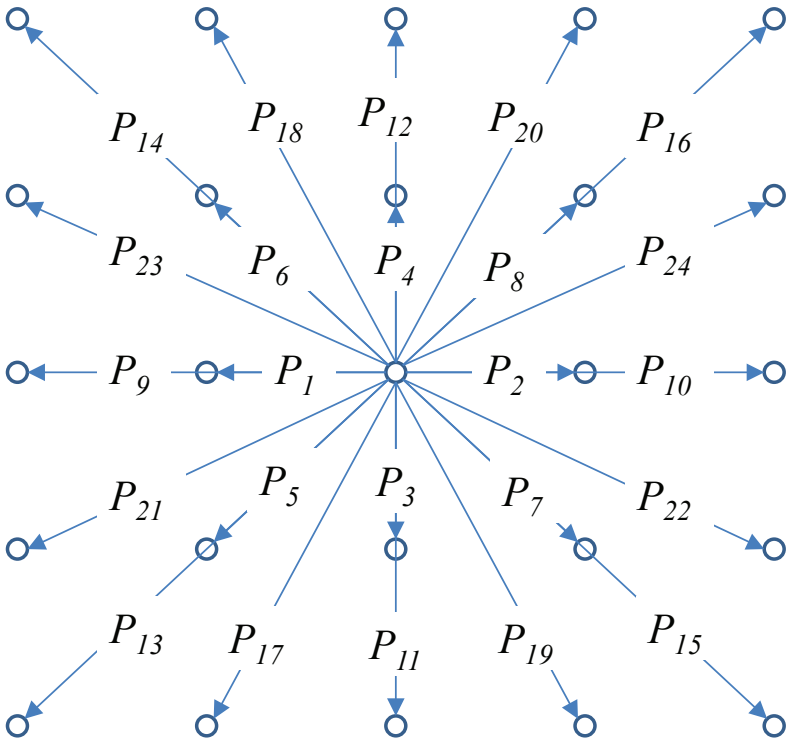
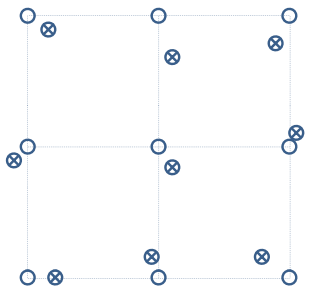


Remaining work: study more general transmissions

- Ideas:
- probability of transmitting: general function of queue length;
- power control: interference range $\neq$ transmission range;

Remaining work: continuum limit of nonuniform networks

- Study nonuniform, even mobile, networks.



$\otimes$ nonuniform nodes $\circ$ uniform nodes

- $\tilde{v}_N(k,n)$: location of nodes in nonuniform network.
- Transformation function $\phi(t,s)$ s.t.:

$$\tilde{v}_N(k,n) = \phi(t_N(k), v_N(n)),$$

- Know $\phi(t,s)$ of nonuniform network + know node behavior $\Rightarrow$ PDE of nonuniform network?

Remaining work: control of nonuniform networks

- Control transmissions of nodes s.t. continuum limit invariant under node locations, i.e.,
  Know $\phi(t,s)$ of nonuniform network + desired continuum limit $z(t,s)$: find node behavior of nonuniform network $\Rightarrow$ continuum limit of nonuniform network remains z.

Remaining work: other topics

- Current assumption: queue lengths $\leq$ max queue length M – study situation where queue lengths $\geq M$.
- Other systems, e.g., crowd behavior of people (large number of components interacting with each other).