Parameterization and Evaluation of Robotic Orientation Workspace: A Geometric Treatment

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***Abstract*—The volume of the orientation workspace is measured based on two invariant principles: It must be invariant with respect to the ground frame and invariant with respect to the orientation description. A method which is based on quaternions and differential geometry is developed for the measurement of the volume correctly. The method is extended for an arbitrary orientation description by means of a mapping theorem proposed for the first time. An example of a serial spherical wrist shows that the volumes that are obtained by the proposed method are consistent with the two invariant principles.**

***Index Terms*—Euler angles, metric, orientation, quaternion, workspace, 3-sphere.**

I. Introduction

ROBOTIC manipulators that are capable of spherical motion, i.e., arbitrary rotations around a fixed point, have many industrial and medical applications [1]. The research on 3-degree-of-freedom (3-DOF) robotic spherical motion generation can be categorized into two common topologies: serial (open-chained) mechanisms and parallel (closed-chained) mechanisms. Examples of serial mechanisms are the gimbal mechanism and the spherical wrist [2], while examples of parallel mechanisms are the robotic shoulder [3], [4], the parallel wrist [5], the 3-RUU wrist [6], the 3-URC wrist [7], the 3-DOF rotational parallel tripod [8], the Argos [9], and robotic ankle rehabilitation [10], [11].

The workspace evaluation of all such spherical robotic manipulators is fundamental in the design process, similar to the evaluation of the position workspace of many robotic systems. Correct evaluation of the orientation workspace can guide a designer to choose proper design parameters for a better workspace. The simplest way is to calculate the volume of the orientation workspace. The orientation workspace can be parameterized in different ways, such as the fixed angles, the Euler angles, the angle–axis parameterization, the Euler–Rodrigues parameters (unit quaternion), the exponential coordinates [12], and the tilt-and-torsion angles [13]. Previous effort on the orientation workspace includes [10], [12], [14]–[21]. A method to visualize a two-dimensional (2-D) projection of the orientation workspace as a subset of the 2-sphere was presented in [16]. In [14], a similar section of the orientation workspace was found, and its size is measured by the area of the bounding "orientation pyramid." These methods simplify the problem by only considering two dimensions of the orientation workspace, ignoring the third dimension (3-D). In [18] and [13], the tilt-and-torsion angles were used to visualize the orientation workspace as a subset of a solid cylinder. Although no calculation of the orientation workspace volume is given, a visualization can give an indication of the size of the orientation workspace. However, such a picture is dependent on both the reference frame selected and the orientation parameterization used. Only the simple volume of the orientation workspace was considered in [12] and [21], without taking the underlying geometry of the space into account. This and the aforementioned visualization method are susceptible to misinterpretation because the size of the workspace is misrepresented by the use of the simple Euclidean measure when the space is in fact curved. It was shown in [10] that the rotation can be represented in an image space with a curved leaf-surface. The integration measures for several parameterizations that include Euler angles were derived in [17], upon the Lie group tools of the Haar measure. The integration measure for exponential-coordinate parameterization was obtained in [22], by means of bi-invariant Riemannian metrics. These measures were summarized and used for computation of the volume of orientation workspace in [20]. Related discussion can also be found in [15] and [19]. An engineer-friendly and detailed introduction of the Haar measure and integral can be found in [17]. However, a simple, generic approach to derive the integration measure of an arbitrary parameterization of SO(3) is still missing. In this paper, we attempt to look at this problem from a geometric point of view, since the consideration of the orientation workspace as a patch area on the 3-sphere provides a straightforward geometric intuition. A generic approach upon a new mapping theorem is proposed for the derivation of a list of volume elements of different parameterizations, which are summarized and ready for use by engineers.

This paper is organized as follows. Section II describes the need for an invariant measure of orientation volume. Section III derives such a measure, by the use of the unit quaternion description of orientation. Section IV generalizes the calculations for some common orientation descriptions. Finally, Section V demonstrates the use of the calculation by a numerical example.

II. Measure Principles of Orientation Workspace

Similar to the position workspace of robotic manipulators, the simplest way to measure the orientation workspace is to calculate its volume. The orientation workspace of a spherical robotic system can be defined as the complete collection of

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all reachable orientations. This definition can be extended to a fixed-position orientation workspace of an arbitrary robotic system. The volume of the orientation workspace is the size of these collections. The orientation workspace is complicated, since it is not within a Euclidean space. Here, two principles are proposed to validate any computation of the volume of any orientation workspace.

The first principle comes from the Haar measure [17], [23], [24]. Let G be a locally compact group. Then, a measure μ on G, which satisfy the following conditions

1) $\mu(xE) = \mu(E)$ for every $x \in G$ and every measurable $E \subseteq G$;
2) $\mu(U) > 0$ for every nonempty open set $U \subseteq G$;
3) $\mu(K) < \infty$ for every compact set $K \subseteq G$;

is called the left-invariant Haar measure.

The set of all orientations forms a Lie group, denoted $SO(3)$ (special orthogonal matrices of size 3×3), which is locally compact. The left-invariant Haar measure must have the property of $\mu(RW) = \mu(W)$ for every $R \in SO(3)$ and every measurable $W \subseteq SO(3)$, where R is a rotation matrix, W represents the workspace of orientation, and μ is the volume of the workspace that we are investigating. According to Lie theory, there always exists a unique left-invariant integration measure on an arbitrary Lie group. The left-invariant Haar measure is the foundation of our first principle on volume computation in robotics.

Principle 1: The volume of an orientation workspace must be invariant with respect to the selection of the reference frame attached to the ground.

One physical meaning of Principle 1 is that the volume of the orientation workspace must be the same no matter how the base of a robot is oriented in the space. Furthermore, the volume of the orientation workspace is a geometric object and must not depend on the way we choose to parameterize it. Our second principle is as follows.

Principle 2: The volume of an orientation workspace must be invariant with respect to the selection of the parameterization of the orientation.

In differential geometry, this principle is known as the invariance of the volume under reparameterization, which can be proved by utilizing the change of variable formula for multiple integrals [25]. Since orientations can be considered to lie on a curved hypersurface, the volume of an orientation workspace can be considered as the volume of one subset of orientations on the hypersurface. Before the evaluation of the volume, the correct metric and volume element of this hypersurface must be found. The quaternions will be used to clarify this issue in the next section.

III. Quaternions

The use of quaternions to represent rotations is well known [26]. Given a unit quaternion $q = a + bi + cj + dk$ (where $a^2 + b^2 + c^2 + d^2 = 1$), the associate rotation matrix is given by

$$\mathbf{R} = \begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{bmatrix}. \tag{1}$$

Since the replacement of q with $-q$ does not affect $\mathbf{R}$, only half of all unit quaternions are necessary to specify an orientation (or rotation). Unit quaternions can be considered to lie on the 3-sphere ($\mathbb{S}^3$), by $\mathbf{q} = [a \quad b \quad c \quad d]^T$, in 4-D space ($\mathbb{R}^4$).[1] This aspect of unit quaternions makes them a powerful parameterization of orientation, since a sphere (in any dimension) is a perfectly symmetric object. That is, there is nothing special about any of its points; they are all weighted equally. Each point corresponds to a particular rotation, or orientation. Since all points are distributed evenly over $\mathbb{S}^3$, a measure of the volume of a subset of $\mathbb{S}^3$ must give a true indication of the corresponding volume of the orientation workspace.

To determine this volume, a coordinate system and the induced metric need to be determined. A metric is a tensor field on a Riemannian manifold that determines the dot product of tangent vectors to the manifold, the distance between points and can be used to measure volume [27], [28]. It can be represented as an $n \times n$ matrix, where n is the dimension of the manifold. A generalization of spherical polar coordinates may be used to parameterize $\mathbb{S}^3$:

$$a = \cos\psi \tag{2a}$$

$$b = \sin\psi\cos\phi \tag{2b}$$

$$c = \sin\psi\sin\phi\cos\theta \tag{2c}$$

$$d = \sin\psi\sin\phi\sin\theta. \tag{2d}$$

To cover $\mathbb{S}^3$, the angles ψ, ϕ, and θ would have the following ranges:

$$0 \le \psi \le \pi$$
$$0 \le \phi \le \pi$$
$$0 \le \theta < 2\pi.$$

However, only half of $\mathbb{S}^3$ is required to represent all possible orientations (and it does not matter what half we choose), so ψ can be further restricted as in (3). In addition, in order to use the necessary tools of differential geometry, we will from now on use "$<$" instead of "$\le$," so that the range of angles will determine an *open* subset of $\mathbb{R}^3$. The orientations that are left out can be safely ignored for the purpose of calculation of volume, because they represent a set of measure zero (they form a set of lesser dimension). Most importantly, the singular positions of the coordinate system ($\phi = 0, \pi$ and $\psi = 0, \pi$) are not part of the coordinate chart:

$$0 < \psi < \frac{\pi}{2} \tag{3a}$$

$$0 < \phi < \pi \tag{3b}$$

$$0 < \theta < 2\pi. \tag{3c}$$

[1]Homogeneous rotation matrix in the image space can be obtained from the quaternions without the constraint on normalization. However, it introduces a scaling factor, which is not suitable for the discussion here.

The new restriction on ψ means that $a > 0$. Upon (2), the metric tensor [27], [28] of $\mathbb{S}^3$ is then given by the symmetric matrix[2]

$$g_{\psi\phi\theta} = \begin{bmatrix} \frac{\partial \mathbf{q}}{\partial \psi}\cdot\frac{\partial \mathbf{q}}{\partial \psi} & \frac{\partial \mathbf{q}}{\partial \psi}\cdot\frac{\partial \mathbf{q}}{\partial \phi} & \frac{\partial \mathbf{q}}{\partial \psi}\cdot\frac{\partial \mathbf{q}}{\partial \theta} \\ \frac{\partial \mathbf{q}}{\partial \phi}\cdot\frac{\partial \mathbf{q}}{\partial \psi} & \frac{\partial \mathbf{q}}{\partial \phi}\cdot\frac{\partial \mathbf{q}}{\partial \phi} & \frac{\partial \mathbf{q}}{\partial \phi}\cdot\frac{\partial \mathbf{q}}{\partial \theta} \\ \frac{\partial \mathbf{q}}{\partial \theta}\cdot\frac{\partial \mathbf{q}}{\partial \psi} & \frac{\partial \mathbf{q}}{\partial \theta}\cdot\frac{\partial \mathbf{q}}{\partial \phi} & \frac{\partial \mathbf{q}}{\partial \theta}\cdot\frac{\partial \mathbf{q}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\psi & 0 \\ 0 & 0 & \sin^2\psi\sin^2\phi \end{bmatrix}. \quad (4)$$

Given a subset U of $\mathbb{S}^3$, its volume by the use of the volume element is given by $\sqrt{\det g_{\psi\phi\theta}}\, d\psi\, d\phi\, d\theta$ [27], i.e.,

$$V_w = \int_U \sqrt{\det g_{\psi\phi\theta}}\, d\psi\, d\phi\, d\theta = \int_U \sin^2\psi \sin\phi\, d\psi\, d\phi\, d\theta \quad (5)$$

which can be called the weighted volume. Note that the restriction on ϕ given in (3) is used in (5).

IV. Generic Approach

Taking the rotation matrix for any orientation parameterization, with (i, j) entry R_{ij}, we consider it as a map that takes the three angle parameters into 9-D space $\mathbb{R}^9$, i.e.,

$$\mathbf{r} = [\, R_{11} \quad R_{12} \quad R_{13} \quad R_{21} \quad R_{22} \quad R_{23} \quad R_{31} \quad R_{32} \quad R_{33}\,]^T.$$

We first give a theorem.

Theorem 1: Considering the same subset of orientations being mapped onto a semi-3-sphere in the 4-D quaternion space and a 3-D submanifold in the 9-D matrix-component space, respectively, the volume measured on the semi-3-sphere in the 4-D space is proportional to the volume measured on the 3-D submanifold in the 9-D space, with a constant ratio of $1/(16\sqrt{2})$.

Proof: Consider an arbitrary parameterization of orientation with three parameters, $\mathbf{x} = [\, x_1 \quad x_2 \quad x_3\,]^T$. The weighted volumes of the same subset of orientations measured in the 4-D quaternion space and 9-D matrix-component space are ${}^qV_w$ and ${}^rV_w$, respectively. We have

$${}^qV_w = \int_U \sqrt{\det({}^qg_x)}\, dx_1 dx_2 dx_3$$

$${}^rV_w = \int_U \sqrt{\det({}^rg_x)}\, dx_1 dx_2 dx_3.$$

Both metrics, which are induced from the manifolds' embedding in a Euclidean space, ${}^qg_x$ and ${}^rg_x$ are 3×3 matrices. Their entries (i, j) are defined by

$${}^qg_x^{ij} = \left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T \frac{\partial \mathbf{q}}{\partial x_j}$$

$${}^rg_x^{ij} = \left(\frac{\partial \mathbf{r}}{\partial x_i}\right)^T \frac{\partial \mathbf{r}}{\partial x_j}$$

for $i, j = 1, 2, 3$.

The quantities $\partial\mathbf{q}/\partial x_j$ and $\partial\mathbf{r}/\partial x_j$ are tangent vectors to the coordinate curves on their respective manifolds. In particular, on the 3-sphere, the tangent vectors $\partial\mathbf{q}/\partial x_j$ are perpendicular to the vector $\mathbf{q} = [\, a, b, c, d\,]^T$ in $\mathbb{R}^4$ that defines the tangent point on $\mathbb{S}^3$, i.e.,

$$\mathbf{q}^T \frac{\partial \mathbf{q}}{\partial x_j} = 0. \quad (6)$$

Furthermore, by (1) and the chain rule, we have

$$\frac{\partial \mathbf{r}}{\partial x_i} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial x_i}$$

where

$$\frac{\partial \mathbf{r}}{\partial \mathbf{q}} = \begin{bmatrix} 2a & 2b & -2c & -2d \\ 2d & 2c & 2b & 2a \\ -2c & 2d & -2a & 2b \\ -2d & 2c & 2b & -2a \\ 2a & -2b & 2c & -2d \\ 2b & 2a & 2d & 2c \\ 2c & 2d & 2a & 2b \\ -2b & -2a & 2d & 2c \\ 2a & -2b & -2c & 2d \end{bmatrix}.$$

By means of some manipulations, we have

$$\begin{aligned} {}^rg_x^{ij} &= \left(\frac{\partial \mathbf{r}}{\partial x_i}\right)^T \frac{\partial \mathbf{r}}{\partial x_j} \\ &= \left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}}\right)^T \frac{\partial \mathbf{r}}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial x_j} \\ &= \left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T (8\mathbf{I} + 4\mathbf{q}\mathbf{q}^T) \frac{\partial \mathbf{q}}{\partial x_j} \\ &= \left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T 8\mathbf{I}\frac{\partial \mathbf{q}}{\partial x_j} + \left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T 4\mathbf{q}\mathbf{q}^T \frac{\partial \mathbf{q}}{\partial x_j} \\ &= 8\left(\frac{\partial \mathbf{q}}{\partial x_i}\right)^T \frac{\partial \mathbf{q}}{\partial x_j} + 0 \\ &= 8\,{}^qg_x^{ij} \end{aligned}$$

where $\mathbf{I}$ is the 4×4 identity matrix, and (6) is utilized.

Therefore, we have

$$\begin{aligned} {}^rV_w &= \int_U \sqrt{\det({}^rg_x)}\, dx_1 dx_2 dx_3 \\ &= \int_U \sqrt{8^3 \det({}^qg_x)}\, dx_1 dx_2 dx_3 \\ &= 16\sqrt{2}\int_U \sqrt{\det({}^qg_x)}\, dx_1 dx_2 dx_3 \\ &= 16\sqrt{2}\,{}^qV_w. \end{aligned}$$

■

Theorem 1 can be understood via intrinsic geometry. Construct another map $\mathbf{p} = \mathbf{r}/(2\sqrt{2})$ from the semi-3-sphere in the 4-D quaternion space to the 3-D submanifold in the 9-D matrix-component space, where $\mathbf{r}$ is defined in (1). Upon the

[2]The Riemannian metric tensor is usually denoted by g_{ij} in the differential geometry literature.

proof of Theorem 1, we can see that the dot products of tangent vectors are preserved by this constructed map $\mathbf{p}$. Hence, this map is isometric, which indicates that the semi-3-sphere in the 4-D quaternion space and the 3-D submanifold in the 9-D matrix-component space must have the same intrinsic geometry. Volume is apparently preserved under this isometry, or sometimes called an isometry invariant, because the volume integral depends only on the dot products of tangent vectors. The constant $16\sqrt{2}$ comes from the un-normalized map $\mathbf{r}$. In the ensuing discussion, we keep this un-normalized map and the constant, in order to simplify the derivations.

Corollary 1: Considering the same subset of orientations being mapped onto a semi-3-sphere in the 4-D quaternion space and a 3-D submanifold in the 9-D matrix-component space, respectively, the volume fraction measured on semi-3-sphere in the 4-D space equals the volume fraction measured on the 3-D submanifold in the 9-D space.

Proof: The complete weighted volumes in the 4-D space and the 9-D space are ${}^qV_w^c$ and ${}^rV_w^c$[3], respectively. The volume fraction ${}^rV_f$ in the 9-D space is given by

$${}^rV_f = \frac{{}^rV_w}{{}^rV_w^c} = \frac{16\sqrt{2}\,{}^qV_w}{16\sqrt{2}\,{}^qV_w^c} = \frac{{}^qV_w}{{}^qV_w^c} = {}^q V_f$$

where ${}^qV_f$ is the volume fraction in the 4-D space. ■

Theorem 1 provides a simple way to calculate the volume element as follows.

1) Calculate the three partial derivatives of each entry of the rotation matrix.
2) Form the metric by calculating the six dot products of the partial derivative vectors.
3) Take the square root of the determinant of the metric.
4) Divide the result by $16\sqrt{2}$ to obtain the volume element weighting.

A. Euler Angles

Given the *ZYX* Euler angles, we have

$$\mathbf{R} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} \quad (10)$$

where α, β, and γ are the Euler angles around *Z*-, *Y*-, and *X*-axes, respectively. From (12), we have

$$\mathbf{r} = \begin{bmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma \\ \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma \\ \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ \cos\beta\cos\gamma \end{bmatrix}.$$

[3]Although singularities exist, any mentioned parameterization of orientation in this paper is capable of describing all possible orientations, except for a set of lesser dimension (i.e., a set of measure zero) that will not affect volume calculations.

TABLE I
VOLUME ELEMENTS FOR THE 12 EULER ANGLE CONVENTIONS

Convention	β range	Volume element
XYZ, XZY, YXZ, YZX, ZXY, ZYX	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\frac{1}{8}\cos\beta\, d\alpha\, d\beta\, d\gamma$
XYX, XZX, YXY, YZY, ZXZ, ZYZ	$(0, \pi)$	$\frac{1}{8}\sin\beta\, d\alpha\, d\beta\, d\gamma$

Now, the metric is formed by the formula

$${}^rg_{\alpha\beta\gamma} = \begin{bmatrix} \frac{\partial\mathbf{r}}{\partial\alpha}\cdot\frac{\partial\mathbf{r}}{\partial\alpha} & \frac{\partial\mathbf{r}}{\partial\alpha}\cdot\frac{\partial\mathbf{r}}{\partial\beta} & \frac{\partial\mathbf{r}}{\partial\alpha}\cdot\frac{\partial\mathbf{r}}{\partial\gamma} \\ \frac{\partial\mathbf{r}}{\partial\beta}\cdot\frac{\partial\mathbf{r}}{\partial\alpha} & \frac{\partial\mathbf{r}}{\partial\beta}\cdot\frac{\partial\mathbf{r}}{\partial\beta} & \frac{\partial\mathbf{r}}{\partial\beta}\cdot\frac{\partial\mathbf{r}}{\partial\gamma} \\ \frac{\partial\mathbf{r}}{\partial\gamma}\cdot\frac{\partial\mathbf{r}}{\partial\alpha} & \frac{\partial\mathbf{r}}{\partial\gamma}\cdot\frac{\partial\mathbf{r}}{\partial\beta} & \frac{\partial\mathbf{r}}{\partial\gamma}\cdot\frac{\partial\mathbf{r}}{\partial\gamma} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2\sin\beta \\ 0 & 2 & 0 \\ -2\sin\beta & 0 & 2 \end{bmatrix}.$$

The square root of the determinant of ${}^rg_{\alpha\beta\gamma}$ is given by

$$\sqrt{\det({}^rg_{\alpha\beta\gamma})} = 2\sqrt{2}\cos\beta.$$

Finally, we have the volume element weighting

$$\sqrt{\det({}^qg_{\alpha\beta\gamma})} = \frac{1}{8}\cos\beta.$$

Since ${}^qg_{\alpha\beta\gamma}$ is the metric on the half 3-sphere with a reparameterization, the total volume remains the same, π^2. Hence, the resulting volume element can be divided by π^2 to produce the volume fraction.

The aforementioned method was used to calculate the correct weighting for each of the 12 Euler angle conventions, and the results are summarized in Table I. Note that, these results apply equally to the fixed angle convention, since the rotation matrices are the same as those for Euler angles, except with an opposite order in the notation.

B. Angle–Axis Parameterization

The rotation matrix for the angle–axis parameterization is given by (8), shown at the bottom of the next page. To describe it by the use of only three parameters, the standard spherical polar coordinates are used to describe the unit vector u:

$$u_x = \sin\psi_1\cos\psi_2$$
$$u_y = \sin\psi_1\sin\psi_2$$
$$u_z = \cos\psi_1.$$

By the use of $\alpha = \psi_3$, the resulting matrix in (8) is written in the vector form

$$\mathbf{r} = \begin{bmatrix} \sin^2\psi_1 \cos^2\psi_2 (1-\cos\psi_3) + \cos\psi_3 \\ \sin^2\psi_1 \cos\psi_2 \sin\psi_2 (1-\cos\psi_3) + \cos\psi_1 \sin\psi_3 \\ \sin\psi_1 \cos\psi_2 \cos\psi_1 (1-\cos\psi_3) - \sin\psi_1 \sin\psi_2 \sin\psi_3 \\ \sin^2\psi_1 \cos\psi_2 \sin\psi_2 (1-\cos\psi_3) - \cos\psi_1 \sin\psi_3 \\ \sin^2\psi_1 \sin^2\psi_2 (1-\cos\psi_3) + \cos\psi_3 \\ \sin\psi_1 \sin\psi_2 \cos\psi_1 (1-\cos\psi_3) + \sin\psi_1 \cos\psi_2 \sin\psi_3 \\ \sin\psi_1 \cos\psi_2 \cos\psi_1 (1-\cos\psi_3) + \sin\psi_1 \sin\psi_2 \sin\psi_3 \\ \sin\psi_1 \sin\psi_2 \cos\psi_1 (1-\cos\psi_3) - \sin\psi_1 \cos\psi_2 \sin\psi_3 \\ \cos^2\psi_1 (1-\cos\psi_3) + \cos\psi_3 \end{bmatrix}.$$

The metric for this description is

$$^{r}g_{\psi_1\psi_2\psi_3} = \begin{bmatrix} \sin^2\left(\frac{\psi_3}{2}\right) & 0 & 0 \\ 0 & \sin^2\psi_1 \sin^2\left(\frac{\psi_3}{2}\right) & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

and therefore, the volume element used is

$$\frac{1}{2}\sin\psi_1 \sin^2\left(\frac{\psi_3}{2}\right) d\psi_1\, d\psi_2\, d\psi_3.$$

C. Exponential Coordinates

The rotation matrix for the exponential coordinates [12], [29] is given by

$$\mathbf{R} = \mathbf{I} + \frac{\sin\|\omega\|}{\|\omega\|}\hat{\omega} + \frac{1-\cos\|\omega\|}{\|\omega\|^2}\hat{\omega}^2$$

where $\mathbf{I}$ is the 3×3 identity matrix, $\hat{\omega}$ is the cross-product matrix given by

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

and, when considered as a vector, $\omega = [\omega_x, \omega_y, \omega_z]^T$ lies in a ball of radius π so that $\|\omega\|^2 \le \pi^2$.

Written as a vector, we have

$$\mathbf{r} = \begin{bmatrix} \frac{\omega_x^2 + \cos\|\omega\|\left(\omega_y^2+\omega_z^2\right)}{\|\omega\|^2} \\ \frac{(1-\cos\|\omega\|)\,\omega_x\omega_y}{\|\omega\|^2} - \frac{\sin\|\omega\|\,\omega_z}{\|\omega\|} \\ \frac{(1-\cos\|\omega\|)\,\omega_x\omega_z}{\|\omega\|^2} + \frac{\sin\|\omega\|\,\omega_y}{\|\omega\|} \\ \frac{(1-\cos\|\omega\|)\,\omega_x\omega_y}{\|\omega\|^2} + \frac{\sin\|\omega\|\,\omega_z}{\|\omega\|} \\ \frac{\omega_y^2 + \cos\|\omega\|\left(\omega_x^2+\omega_z^2\right)}{\|\omega\|^2} \\ \frac{(1-\cos\|\omega\|)\,\omega_y\omega_z}{\|\omega\|^2} - \frac{\sin\|\omega\|\,\omega_x}{\|\omega\|} \\ \frac{(1-\cos\|\omega\|)\,\omega_x\omega_z}{\|\omega\|^2} - \frac{\sin\|\omega\|\,\omega_y}{\|\omega\|} \\ \frac{(1-\cos\|\omega\|)\,\omega_y\omega_z}{\|\omega\|^2} + \frac{\sin\|\omega\|\,\omega_x}{\|\omega\|} \\ \frac{\omega_z^2 + \cos\|\omega\|\left(\omega_x^2+\omega_y^2\right)}{\|\omega\|^2} \end{bmatrix}.$$

The metric in this case is given by

$$^{r}g_{\omega_x\omega_y\omega_z} = \begin{bmatrix} 2\omega_x^2 k_1 + k_2 & 2\omega_x\omega_y k_1 & 2\omega_x\omega_z k_1 \\ 2\omega_x\omega_y k_1 & 2\omega_y^2 k_1 + k_2 & 2\omega_y\omega_z k_1 \\ 2\omega_x\omega_z k_1 & 2\omega_y\omega_z k_1 & 2\omega_z^2 k_1 + k_2 \end{bmatrix}$$

where

$$k_1 = \frac{\|\omega\|^2 - 4\sin^2\frac{\|\omega\|}{2}}{\|\omega\|^4}$$

$$k_2 = \frac{8\sin^2\frac{\|\omega\|}{2}}{\|\omega\|^2}.$$

Hence, the volume element is simply

$$\frac{\sin^2\frac{\|\omega\|}{2}}{2\,\|\omega\|^2} d\omega_x\, d\omega_y\, d\omega_z.$$

To perform the integration in this case, the following limits were used:

$$-\pi < \omega_x < \pi$$
$$-\sqrt{\pi^2 - \omega_x^2} < \omega_y < \sqrt{\pi^2 - \omega_x^2}$$
$$-\sqrt{\pi^2 - \omega_x^2 - \omega_y^2} < \omega_z < \sqrt{\pi^2 - \omega_x^2 - \omega_y^2}.$$

It is interesting to note that if a change of variables is used to take advantage of the symmetry of the space and perform the integration by the use of spherical polar coordinates, the new

$$\mathbf{R} = \begin{bmatrix} u_x^2(1-\cos\alpha) + \cos\alpha & u_x u_y(1-\cos\alpha) - u_z\sin\alpha & u_x u_z(1-\cos\alpha) + u_y\sin\alpha \\ u_x u_y(1-\cos\alpha) + u_z\sin\alpha & u_y^2(1-\cos\alpha) + \cos\alpha & u_y u_z(1-\cos\alpha) - u_x\sin\alpha \\ u_x u_z(1-\cos\alpha) - u_y\sin\alpha & u_y u_z(1-\cos\alpha) + u_x\sin\alpha & u_z^2(1-\cos\alpha) + \cos\alpha \end{bmatrix} \quad (8)$$

variables (and the resulting volume element) correspond exactly to those of the angle–axis parameterization.

D. Tilt-and-torsion Angles

The tilt-and-torsion angles are very similar to the Euler angles. Here, instead of ϕ, θ, and σ as in [13], ϕ_1 (azimuth), ϕ_2 (tilt), and ϕ_3 (torsion) are used to denote the angles. The rotation matrix, which is written as a vector, is then

$$\mathbf{r} = \begin{bmatrix} \cos\phi_2\cos\phi_1\cos(\phi_1-\phi_3)+\sin\phi_1\sin(\phi_1-\phi_3) \\ \cos\phi_2\cos\phi_1\sin(\phi_1-\phi_3)-\sin\phi_1\cos(\phi_1-\phi_3) \\ \sin\phi_2\cos\phi_1 \\ \cos\phi_2\sin\phi_1\cos(\phi_1-\phi_3)-\cos\phi_1\sin(\phi_1-\phi_3) \\ \cos\phi_2\sin\phi_1\sin(\phi_1-\phi_3)+\cos\phi_1\cos(\phi_1-\phi_3) \\ \sin\phi_2\sin\phi_1 \\ -\sin\phi_2\cos(\phi_1-\phi_3) \\ -\sin\phi_2\sin(\phi_1-\phi_3) \\ \cos\phi_2 \end{bmatrix}.$$

The metric is

$$^{r}g_{\phi_1\phi_2\phi_3} = \begin{bmatrix} 4-4\cos\phi_2 & 0 & 2\cos\phi_2-2 \\ 0 & 2 & 0 \\ 2\cos\phi_2-2 & 0 & 2 \end{bmatrix}$$

and the volume element is

$$\frac{1}{8}\sin\phi_2 d\phi_1\, d\phi_2\, d\phi_3.$$

The range of the angles is

$$-\pi < \phi_1 < \pi$$
$$0 < \phi_2 < \pi$$
$$-\pi < \phi_3 < \pi.$$

E. Cayley Map

The Cayley map [26], [30] is a well-known parameterization of rotation without trigonometric functions. For $SO(3)$, the Cayley map yields a rotation matrix given by

$$\mathbf{R} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$$

where $\mathbf{I}$ is the 3×3 identity matrix, and

$$\mathbf{A} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

Writing $\mathbf{R}$ into a vector, we can have

$$\mathbf{r} = \begin{bmatrix} 1+x^2-y^2-z^2 \\ 2xy-2z \\ 2xz+2y \\ 2xy+2z \\ 1-x^2+y^2-z^2 \\ -2x+2yz \\ -2y+2xz \\ 2yz+2x \\ 1-x^2-y^2+z^2 \end{bmatrix} /(1+x^2+y^2+z^2).$$

TABLE II
SUMMARY OF INTEGRAL VOLUME ELEMENTS

Convention	Range of variables	Volume Element
ZYX Euler	$0 < \alpha < 2\pi$ $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ $0 < \gamma < 2\pi$	$\frac{1}{8}\cos\beta\, d\alpha\, d\beta\, d\gamma$
XZX Euler	$0 < \alpha < 2\pi$ $0 < \beta < \pi$ $0 < \gamma < 2\pi$	$\frac{1}{8}\sin\beta\, d\alpha\, d\beta\, d\gamma$
Quaternions	$0 < \psi < \frac{\pi}{2}$ $0 < \phi < \pi$ $0 < \theta < 2\pi$	$\sin^2\psi\sin\phi\, d\psi\, d\phi\, d\theta$
Angle-axis	$0 < \psi_1 < \pi$ $0 < \psi_2 < \pi$ $0 < \psi_3 < 2\pi$	$\frac{\sin\psi_1\sin^2\left(\frac{\psi_3}{2}\right)}{2} d\psi_1\, d\psi_2\, d\psi_3$
Exponential	$\|\omega\| < \pi$	$\frac{\sin^2\left(\frac{\|\omega\|}{2}\right)}{2\,\|\omega\|^2} d\omega_x\, d\omega_y\, d\omega_z$
T&T	$-\pi < \theta_1 < \pi$ $0 < \theta_2 < \pi$ $-\pi < \theta_3 < \pi$	$\frac{1}{8}\sin\theta_2 d\theta_1\, d\theta_2\, d\theta_3$
Cayley map	$-\infty < x, y, z < \infty$	$\frac{1}{(1+x^2+y^2+z^2)^2} dx\, dy\, dz$

The metric is

$$^{r}g_{xyz} = c\begin{bmatrix} 1+y^2+z^2 & -xy & -xz \\ -xy & 1+x^2+z^2 & -yz \\ -xz & -yz & 1+x^2+y^2 \end{bmatrix}$$

where $c = 8/(1+x^2+y^2+z^2)^2$. Therefore, the volume element is

$$\frac{1}{(1+x^2+y^2+z^2)^2} dxdydz.$$

The range of the parameters is

$$-\infty < x, y, z < \infty. \quad (9)$$

The complete results are shown in Table II. Each convention listed in Table II is a useful representation of orientation. Apparently, our derived volume elements agree completely with those in [17], [20], and [22]. One is free to choose which convention to analyze the orientation of a robotic system. However, to calculate the correct volume of the orientation workspace, the corresponding volume element must be applied.

V. Algorithm and Example

Since an analytical expression of the boundary of a practical orientation workspace is often complex (sometimes, it even does not exist), numerical integration to obtain the volume of the workspace is suggested here. The algorithm for the application of the derived volume elements is summarized in the following steps.

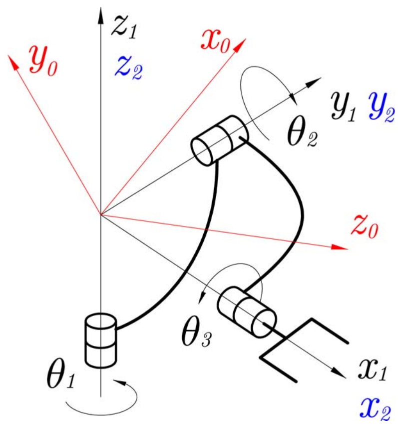

Fig. 1. Diagram of serial spherical manipulator, which shows a ground-fixed frame $\mathcal{F}_1$, a tool-attached frame $\mathcal{F}_2$ initially coincident, and an alternate ground-fixed frame $\mathcal{F}_0$.

1) Choose a parameterization of orientation for the robot.
2) Create equispaced grids in the space of the corresponding parameters.
3) Determine those grid points within the workspace, according to certain criteria, such as limits of joint angles, singularities, and mechanical interferences.
4) Integrate (add) the grid points within the workspace, by the multiplication of the corresponding volume element, to obtain the volume of workspace.

Note that the direct computation of orientation workspace is to integrate the grid points without multiplication of the volume element, such as [12] and [21].

Consider a spherical wrist with joint angles θ_1, θ_2, and θ_3, as shown in Fig. 1. These three joints have limited ranges, as given by

$$0 \leq \theta_1 \leq \frac{\pi}{6} \tag{10a}$$

$$-\frac{\pi}{12} \leq \theta_2 \leq \frac{\pi}{12} \tag{10b}$$

$$0 \leq \theta_3 \leq \frac{\pi}{6}. \tag{10c}$$

Two cases are considered. In Case 1, the rotation matrix to represent any orientation of this wrist in $\mathcal{F}_1$ is given by

$$ {}^1_2\mathbf{R} = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - s_1 c_3 & c_1 s_2 c_3 + s_1 s_3 \\ s_1 c_2 & s_1 s_2 s_3 + c_1 c_3 & s_1 s_2 c_3 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix} \tag{11}$$

where $c_i = \cos\theta_i$.

In Case 2, we introduce a new rotation ${}^0_1\mathbf{R}$, which describes $\mathcal{F}_1$ in the $\mathcal{F}_0$ frame, specified by a set of *ZYX* Euler angles, $\alpha' = \frac{5\pi}{6}, \beta' = -\frac{\pi}{3}$, and $\gamma' = \frac{\pi}{4}$. Note that ${}^0_1\mathbf{R}$ can be obtained by the substitution $\theta_1 = \alpha', \theta_2 = \beta'$, and $\theta_3 = \gamma'$ into (11). In the new frame $\mathcal{F}_0$, the orientation of the wrist is described by *ZYX* Euler angles $\alpha, \beta,$ and γ. The orientation of the wrist in the new frame $\mathcal{F}_0$ is given by

$$ {}^0_2\mathbf{R} = {}^0_1\mathbf{R}\,{}^1_2\mathbf{R}.$$

TABLE III
SUMMARY OF RESULTS

Convention	Frame	Direct volume	Weighted volume
ZYX Euler	$\mathcal{F}_1$	0.143548*	0.0177392*
	$\mathcal{F}_0$	0.4179	0.0178
XZX Euler	$\mathcal{F}_1$	0.6653	0.0177
	$\mathcal{F}_0$	0.1553	0.0178
Quaternions	$\mathcal{F}_1$	0.0182	0.0176
	$\mathcal{F}_0$	0.0536	0.0176
Angle-axis	$\mathcal{F}_1$	2.4875	0.0178
	$\mathcal{F}_0$	0.0198	0.0177
Exponential	$\mathcal{F}_1$	0.6610	0.0177
	$\mathcal{F}_0$	0.1506	0.0177
T&T	$\mathcal{F}_1$	0.1446	0.0178
	$\mathcal{F}_0$	0.2936	0.0177
Cayley map	$\mathcal{F}_1$	0.0196	0.0177
	$\mathcal{F}_0$	19123	0.0178

Starred entries have increased precision because an analytical solution exists.

Let us choose an arbitrary parameterization of the orientation of the spherical wrist, say the *XZX* Euler angles $\alpha, \beta,$ and γ, which yields a rotation matrix $\mathbf{R}(\alpha, \beta, \gamma)$.

In Case 1, we have

$$ {}^1_2\mathbf{R} = \mathbf{R}(\alpha, \beta, \gamma)$$

from which the joint angles can be extracted by the following formulas:

$$\theta_1 = \arctan 2\,(R_{21}, R_{11}) \tag{12a}$$

$$\theta_2 = \arctan 2\left(\sqrt{R_{11}^2 + R_{21}^2}, -R_{31}\right) \tag{12b}$$

$$\theta_3 = \arctan 2\,(R_{32}, R_{33}) \tag{12c}$$

where R_{ij} is the entry (*i,j*) of ${}^1_2\mathbf{R}$. Then, the grid points on α, β, γ within workspace can be determined readily by (10). The direct volume and weighted volume of the region in the space of $\alpha, \beta,$ and γ are found by integration to be 0.6653 and 0.0177, respectively.

For Case 2, we have

$$ {}^1_2\mathbf{R} = {}^1_0\mathbf{R}\,\mathbf{R}(\alpha, \beta, \gamma).$$

By the application of the same formulas (12), we can determine the grid points on α, β, γ within workspace by joint limits. The direct volume and weighted volume of the region in the space of α, β, γ are found by integration to be 0.1553 and 0.0178, respectively.

The aforementioned procedure for different parameterizations are repeated to produce Table III. It is apparent that the weighted volumes based on the derived volume elements obey both Principles 1 and 2. Note that the proposed algorithm is generic to the orientation workspace of different robotic systems. The only difference occurs in Step 3. Apparently, the direct calculation of orientation workspace gives incorrect volumes. Furthermore, the visualization of orientation workspace in any three-parameter space provides a similarly misleading implication on volume, since the projection from the 4-D 3-sphere to the three-parameter coordinate system is not invariant.

Note that equispaced grids in the space of chosen parameters are not equispaced on the 3-sphere. This can yield a problem of precision in integration. Ideally, infinitesimal grids will solve this problem. In practice, this is not possible due to computational cost. However, decreasing the size of grids will improve the precision in general. Another approach is to use improved Monte-Carlo integration, such as importance sampling [31], where sampling points are concentrated in the regions that make the largest contribution to an integral. This integration technology is available in scientific computation software, such as Mathematica.

VI. Conclusion

An approach to correctly calculate the volume of the orientation workspace of any robotic manipulator was proposed. The approach satisfies two fundamental principles: The result is invariant with respect to the selection of the reference frame, and it is invariant with respect to the selection of parameterization of orientation.

The method was derived by the utilization of the symmetry of the 3-sphere, and its use through unit quaternions to describe the orientation space and can be used for any orientation parameterization, where the rotation matrix can be determined by three parameters. The weighted integration volume elements were found for all discussed orientation parameterizations by means of a mapping theorem proposed for the first time.

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