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*Transport analogies*

**Momentum Transport and Fluid Dynamics**

*Lecture#1*

Nomenclature; dependent (v, τ, p) & independent variables [ (x,y,z); (r,Θ,z); (r,Θ,φ) ]

Definition of a vector: associates a scalar with each coordinate direction in a given coordinate system

Velocity vector---examples in rectangular, cylindrical & spherical coordinates

Mass flux vector for pure fluids; ρv, with dimensions of mass per area per time

The flux of any quantity has dimensions of that quantity per area per time

Flux elevates the tensorial rank of a quantity by one unit

Scalars have a rank of zero (i.e., mass, pressure, temperature, energy)

Vectors have a rank of 1 (i.e., velocity, mass flux, temp. & pressure grad.)

n-th-order tensors have a rank of n (i.e., momentum flux, for n=2)

"Del" or gradient operator in rectangular coordinates (i.e., spatial rates of change), with dimensions of inverse length;

\[ \nabla = \sum_{i=1}^{3} \delta_i \frac{\partial}{\partial x_i} = \delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z} \]
Examples of gradients; temperature ($\nabla T$), pressure ($\nabla p$) and velocity ($\nabla \mathbf{v}$) gradients

Balance on overall fluid mass—statement of the Equation of Continuity in words (each term has dimensions of mass per time)

“Rate of accumulation of overall fluid mass within a differential control volume (CV) = Net rate at which mass enters the control volume due to mass flux acting across all of the surfaces which bound fluid within the control volume”

Accumulation rate processes: time rate of change that depends on the nature of the control volume; (partial derivative for stationary CV, total or substantial derivative for moving CV)

$$\frac{\partial}{\partial t} \frac{d}{dt} \frac{D}{Dt}$$

Mathematical statement of the Equation of Continuity—analog with macroscopic mass balance (i.e., accumulation = rate of input – rate of output, or net rate of input)
After division by the size of the control volume, each term in EOC has dimensions of mass per volume per time;

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} = - (\rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho)$$

via the product rule for differentiation involving the “del” operator and the product of a scalar with a vector. The Equation of Continuity is derived using general vector notation in Problem#8.1 on pages 222-3, TPfCRD.

Steady state form of the microscopic Equation of Continuity for incompressible fluids
Comparison of steady state microscopic EOC, $\nabla \cdot \mathbf{v} = 0$, with the steady state macroscopic mass balance for (i) tube flow, and (ii) venturi & orifice meters for incompressible fluids, $\rho <\mathbf{v}> S_1 = \rho <\mathbf{v}> S_2$, where $S = \pi D^2/4$ and $\rho_1 = \rho_2$. If the flow cross-sectional area does not change from inlet to outlet, then $S_1 = S_2$ and the average fluid velocity is the same across the inlet plane and the outlet plane.
Hence, $<\mathbf{v}>_1 = <\mathbf{v}>_2$. An equivalent statement of the balance on overall fluid mass at the microscopic level, for steady state operation with one-dimensional flow in the z-direction, is $\partial \mathbf{v}/\partial z = 0$, in cylindrical coordinates.

Lecture#2
Expressions for incompressible EOC, $\nabla \cdot \mathbf{v} = 0$, in 3 different coordinate systems
See B.4 on page #846 in BSL’s *Transport Phenomena*

Comment about the additional geometric factors for EOC in cylindrical (1/r) and spherical (1/r²) coordinates.

Determine the rank of a vector/tensor operation;
- Sum the rank of each quantity involved
- Subtract 2 for the "dot" operation
- Subtract 1 for the "cross" operation

Use the following table to relate the resultant rank to a vector, tensor or scalar

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Rank</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>scalar</td>
<td>0</td>
<td>mass, pressure, temperature, energy</td>
</tr>
<tr>
<td>vector</td>
<td>1</td>
<td>velocity, mass flux, ∇T &amp; ∇p, ∇ × v</td>
</tr>
<tr>
<td>nᵗʰ-order tensor</td>
<td>n</td>
<td>momentum flux (i.e., n=2)</td>
</tr>
</tbody>
</table>

For solid-body rotation, the velocity vector of the solid is given by \( \mathbf{v} = \mathbf{\Omega} \times \mathbf{r} \), where \( \mathbf{\Omega} \) is the angular velocity vector and \( \mathbf{r} \) is the position vector from the axis of rotation. The vorticity vector is defined by \( (1/2) \nabla \times \mathbf{v} = (1/2) \nabla \times [\mathbf{\Omega} \times \mathbf{r}] = \mathbf{\Omega} \).

Applications of the microscopic EOC for steady state one-dimensional flow problems with no time dependence (i.e., \( \partial/\partial t = 0 \))

(a) Tube flow---v\(_z\)(r); w/ 1 inlet stream & 1 outlet stream
   Average velocity, \(<v_z>\) = constant, via macroscopic mass balance with no change in the flow cross sectional area
   \( \partial v_z/\partial z = 0 \), implies that \( v_z \neq f(z) \), and \( \Theta \) is the symmetry variable

(b) Flow on the shell side of the double-pipe heat exchanger; \( \partial v_z/\partial z = 0 \) & \( v_z(r) \)
   EOC is not affected by the boundaries. Same result for tube flow.

(c) Tangential flow between two rotating concentric solid cylinders; \( v_\Theta(r) \)
   \( \partial v_\Theta/\partial \Theta = 0 \), implies that \( v_\Theta \neq f(\Theta) \), & \( v_\Theta \neq f(z) \) if end effects are negligible

(d) Radial flow, horizontally, between two parallel circular plates; \( v_r(r,z) = f(z)/r \)
   \( 1/r \partial \div (rv_r)/\partial r = 0 \), implies that \( rv_r \neq f(r) \), and \( \Theta \) is the symmetry variable

(e) Diverging flow bounded by two stationary walls @ \( \Theta = \pm \alpha \); \( v_r(r,\Theta) = f(\Theta)/r \)
   \( 1/r \partial \div (rv_r)/\partial r = 0 \), implies that \( rv_r \neq f(r) \), \( z \) isn’t important, planar flow
Note: this is a two-dimensional flow problem in rectangular coordinates with \( v_x \) & \( v_y \), but only one-dimensional flow in polar or cylindrical coordinates (i.e., only \( v_r \)).

(f) Polar flow between two stationary concentric solid spheres; \( v_{\theta}(r,\theta) = f(r)/\sin \theta \)
\[
\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_{\theta} \sin \theta) = 0,
\]
implies that \( v_{\theta} \sin \theta = f(\theta) \), \( \phi \) is the symmetry variable.

(g) Transient radial flow in spherical coordinates, induced by an expanding bubble;
\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) = 0,
\]
yields \( v_r(r,t) = f(t)/r^2 \), because \( r^2 v_r \neq f(r) \), \( \theta \) & \( \phi \) are symmetry variables, and the time-varying radius of the bubble precludes a steady state analysis.

For analysis of simple 2-dimensional and 3-dimensional flows via the Equation of Continuity, see Problem#8-16 (p. 235, TPfCRD) and Problem#8-17 (p. 236, TPfCRD), respectively. Other complex examples of 2-dimensional flow can be found on pp. 284-287 and pp. 304-305.

Example of 2-dimensional incompressible fluid flow analysis for hollow fiber ultrafiltration via the Equation of Continuity. Consider axial flow through a tube with a permeable wall such that the radial component of the fluid velocity vector cannot be neglected. Cylindrical coordinates is most appropriate to exploit the symmetry of the macroscopic boundaries. Axial and radial flow occur, where the former is of primary importance and the latter is much smaller in magnitude relative to \( v_z \), but \( v_r \) is extremely important for membrane separations that employ hollow fibers. Constant fluid density within the tube is reasonable, particularly when another component external to the capillary is transported into the tube across the porous wall. Application of the Equation of Continuity for this problem yields;

\[
\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} = 0
\]

Order-of-magnitude analysis for \( \partial v_z/\partial z \) allows one to rearrange the previous equation and estimate the magnitude of the radial velocity component at the tube wall. Furthermore, multiplication of \( v_r(r=R) \) by the lateral surface area of the porous section of the tube (i.e., \( 2\pi rL \)) provides a calculation of the volumetric flowrate across the permeable wall that agrees with the steady state macroscopic mass balance. If fluid escapes across the tube wall, then the inlet volumetric flowrate \( Q(z=0) = Q_{in} \) will be reduced in the exit stream at \( z=L \) (\( Q_{in} > Q_{out} \)). Hence, unlike steady state flow through a straight tube with constant cross-sectional area and an impermeable wall, \( v_z \) depends on axial coordinate \( z \).
and the second term in the Equation of Continuity is estimated in “finite-difference” fashion using grid points at the tube inlet and tube outlet;

\[
\frac{\partial v_z}{\partial z} \approx \frac{1}{L} \left\{ \frac{Q_{out} - Q_{in}}{\pi R^2} \right\}.
\]

This “back-of-the-envelope” approximation for \( \frac{\partial v_z}{\partial z} \) is used to estimate \( v_r(r=R) \) via rearrangement and integration of the Equation of Continuity for 2-dimensional flow in cylindrical coordinates. This approach illustrates a general strategy for 2-dimensional flow problems, based solely on the Equation of Continuity. If one of the two velocity components can be approximated with reasonable accuracy, then the other important component of the fluid velocity vector is calculated such that one does not violate the balance on overall fluid mass. The calculation proceeds as follows;

\[
d(r v_r) = -\left( \frac{\partial v_z}{\partial z} \right) r dr \approx \left\{ \frac{Q_{in} - Q_{out}}{\pi R^2 L} \right\} r dr
\]

\[
\int_{r_v, r=0}^{r_v, r=R} d(r v_r) \approx \left\{ \frac{Q_{in} - Q_{out}}{\pi R^2 L} \right\} \int_{r=0}^{r=R} r dr
\]

\[
R v_r (r = R) \approx \left\{ \frac{Q_{in} - Q_{out}}{\pi R^2 L} \right\} \frac{1}{2} R^2
\]

Now, it is possible to (i) estimate the magnitude of the radial velocity component at the tube wall;

\[
v_r (r = R) \approx \frac{Q_{in} - Q_{out}}{2\pi RL}
\]

This “semi-quantitative” prediction from the microscopic balance on overall fluid mass agrees completely with the steady state macroscopic mass balance that equates rates of input to rates of output. Since mass flowrates can be replaced by volumetric flowrates if the fluid density does not change appreciably, the rate of input is \( Q_{in} \), whereas the rate of output is the sum of \( Q_{out} \) and \( 2\pi RL v_r (r=R) \).

**Problem**

What coordinate system is most appropriate to describe the basic information of fluid mechanics for incompressible laminar flow through a tapered tube (i.e., the tube radius is a linear function of axial position)?
Lecture#3

Forces due to momentum flux that act across surfaces with orientation defined by \( \mathbf{n} \)
Convective momentum flux, \( \rho \mathbf{v} \mathbf{v} \) is obtained from a product of the mass flux vector, \( \rho \mathbf{v} \), and the ratio of momentum to mass (i.e., \( \mathbf{v} \))

Viscous momentum flux----use Newton's law of viscosity for Newtonian fluids
Pressure contribution to momentum flux, only contributions are normal stresses

Each of the 9 scalar components of \( \rho \mathbf{v} \mathbf{v} \) and \( \tau \) needs 2 subscripts
Normal stress vs. shear stress----diagonal vs. off-diagonal matrix elements

Normal stresses act in the direction of the unit normal vector \( \mathbf{n} \) to surface \( S \)
Shear stresses act in the two coordinate directions which describe surface \( S \)
Surface \( S \) across which the stresses due to momentum flux act
the unit normal vector \( \mathbf{n} \) is oriented in the coordinate direction identified by the 1st subscript on \( \rho \mathbf{v} \mathbf{v} \) or \( \tau \), and the force or stress acts in the coordinate direction given by the 2nd subscript

Viscous Stress and Momentum Flux in Fluid Dynamics

The fluid velocity vector is one of the most important variables in fluid mechanics. Remember that a vector is best described as a quantity that has magnitude and direction. A more sophisticated description identifies a vector as a mathematical entity that associates a scalar with each coordinate direction in a particular coordinate system. Hence, there are three scalar velocity components that constitute the velocity vector, and they are typically written in the following manner in three different coordinate systems:

\[
\begin{align*}
    v_x & \quad v_y & \quad v_z & \quad \text{in rectangular cartesian coordinates} \\
    v_r & \quad v_\theta & \quad v_z & \quad \text{in cylindrical coordinates} \\
    v_r & \quad v_\theta & \quad v_\phi & \quad \text{in spherical coordinates}
\end{align*}
\]

It is important to mention here that each flow problem is solved in only one coordinate system---the coordinate system that best exploits the symmetry of the macroscopic boundaries. At the introductory level, the problems will be simple enough that the student should identify a primary direction of flow, and only consider the velocity component that corresponds to this flow direction. Hence, it will be acceptable to assume one-dimensional flow and disregard two of the three velocity components for simple problems in fluid dynamics.
Viscous stress is an extremely important variable, and this quantity is identified by the Greek letter, \( \tau \). Viscous stress represents molecular transport of momentum that is analogous to heat conduction and diffusion. All molecular transport mechanisms correspond to irreversible processes that generate entropy under realistic conditions. When fluids obey Newton’s law of viscosity, there is a linear relation between viscous stress and velocity gradients. All fluids do not obey Newton's law of viscosity, but almost all gases and low-molecular-weight liquids are Newtonian. In this course, we will discuss problems in Newtonian fluid dynamics, as well as non-Newtonian fluid dynamics. The problems in non-Newtonian fluid dynamics relate directly to the rheology laboratory experiments next semester.

Fluid pressure is the third important variable, and it is designated by the letter \( p \). The force balances that we will generate contain fluid pressure because pressure forces are exerted across surfaces, and there are, at most, six surfaces that completely enclose fluid within a so-called control volume. The fluid within the control volume is the system. Our force balances will apply to fluids in motion---hence, the name, fluid dynamics. However, our balances will be completely general to describe the situation when fluids are at rest. In other words, the force balances will be applicable to describe hydrostatics when the velocity vector and \( \tau \) vanish.

**Physical properties and transport analogies**

Physical properties of a fluid can be described within the context of transport analogies for all of the transport processes. Numerical solutions to fluid dynamics problems require that the viscosity \( \mu \) and the density \( \rho \) are known. If the fluid is Newtonian and incompressible, then both of these physical properties are constants that only depend on the fluid itself. The viscosity \( \mu \) is the molecular transport property that appears in the linear constitutive relation that equates the molecular transport of momentum with velocity gradients. The ratio of viscosity to density is called the kinematic viscosity, \( \nu = \mu/\rho \), or momentum diffusivity with units of (length)\(^2\)/time.

Numerical solutions to simple thermal energy transport problems in the absence of radiative mechanisms require that the viscosity \( \mu \), density \( \rho \), specific heat \( C_p \), and thermal conductivity \( k \) are known. Fourier's law of heat conduction states that the thermal conductivity is constant and independent of position for simple isotropic fluids. Hence, thermal conductivity is the molecular transport property that appears in the linear law that expresses molecular transport of thermal energy in terms of temperature gradients. The thermal diffusivity \( \alpha \) is constructed from the ratio of \( k \) and \( \rho C_p \). Hence, \( \alpha = k/\rho C_p \) characterizes diffusion of thermal energy and has units of (length)\(^2\)/time.
The binary molecular diffusion coefficient, $D_{AB}$, has units of (length)$^2$/time and characterizes the microscopic motion of species A in solvent B, for example. $D_{AB}$ is also the molecular transport property that appears in the linear law that relates diffusional fluxes and concentration gradients. In this respect, the same quantity, $D_{AB}$, represents a molecular transport property for mass transfer and a diffusion coefficient. This is not the case for the other two transport processes.

Before we leave this section on physical properties, it is instructive to construct the ratio of the diffusivities for thermal energy transfer and mass transfer with respect to momentum transport. In doing so, we will generate dimensionless numbers that appear in correlations for heat and mass transfer coefficients. The ratio of momentum diffusivity $v$ to thermal diffusivity $\alpha$ is equivalent to the Prandtl number, $Pr = v/\alpha = \mu C_p/\kappa$. The Prandtl number is simply a ratio of physical properties of a fluid. However, a very large value of the Prandtl number means that diffusion of thermal energy away from a hot surface, for example, is poor relative to the corresponding diffusion of momentum. This implies that the thermal boundary layer which contains all of the temperature gradients will remain close to the surface when the fluid flow problem is fully developed. Convective transport parallel to a hot surface maintains thin thermal boundary layers by "sweeping away" any thermal energy that diffuses too far from the surface. Fully developed laminar flow in a straight tube of circular cross-section means that the momentum boundary layer (containing all of the velocity gradients) next to the surface on one side of the tube has grown large enough to intersect the boundary layer from the surface on the other side of the tube. It should be no surprise that these boundary layers will meet in the center of the tube when fully developed flow is attained, and the thickness of the momentum boundary layer is actually the radius of the tube. Hence, a very large Prandtl number means qualitatively that under fully developed laminar flow conditions when the momentum boundary layer has filled the cross-section of the tube, the thermal energy or temperature boundary layer hugs the wall. As a consequence, high rates of heat transfer are prevalent because transport normal to a surface is inversely proportional to the thickness of the boundary layer adjacent to the surface in question. This boundary layer contains all of the gradients that generate molecular transport.

Analogously, the ratio of momentum diffusivity $v$ to mass diffusivity $D_{AB}$ is equivalent to the Schmidt number, $Sc = v/D_{AB} = \mu/\rho D_{AB}$. It follows directly from the discussion in the previous paragraph that for very large values of the Schmidt number, mass transfer boundary layers remain close to the adjacent surface and high rates of mass transfer are obtained. Hence, the Schmidt number is the mass transfer analog of the Prandtl number. The momentum transport analog of the Schmidt or Prandtl numbers is 1, because we take the ratio of momentum diffusivity to momentum diffusivity. The consequence of this statement is that if a heat transfer correlation containing the Prandtl
number can be applied to an analogous momentum transport problem, then the Prandtl number is replaced by 1 to calculate the friction factor. Of course, if the heat transfer problem is completely analogous to a posed mass transfer problem, then the Prandtl number in the heat transfer correlation is replaced by the Schmidt number to calculate the mass transfer coefficient.

The fundamental balances that describe momentum transport

In this section, we discuss the concepts that one must understand to construct force balances based on momentum rate processes. The fluid, the specific problem, and the coordinate system are generic at this stage of the development. If the discussion which follows seems quite vague, then perhaps it will become more concrete when specific problems are addressed. The best approach at present is to state the force balance in words, and then focus on each type of momentum rate process separately.

The strategy for solving fluid dynamics problems begins by putting a control volume within the fluid that takes advantage of the symmetry of the boundaries, and balancing the forces that act on the system. The system is defined as the fluid that is contained within the control volume. Since a force is synonymous with the time rate of change of momentum as prescribed by Newton's laws of motion, the terms in the force balance are best viewed as momentum rate processes. The force balance for an open system is stated without proof as, \( 1 = 2 - 3 + 4 \), where:

1. is the rate of accumulation of fluid momentum within the control volume
2. is the rate at which fluid momentum enters the control volume via momentum flux acting across the surfaces that bound the fluid within the control volume
3. is the rate at which fluid momentum leaves the control volume via momentum flux acting across the surfaces that bound the fluid within the control volume
4. is the sum of all external forces that act on the fluid within the control volume

It should be emphasized that force is a vector quantity and, hence, the force balance described qualitatively above is a vector equation. A vector equation implies that three scalar equations must be satisfied. This is a consequence of the fact that if two vectors are equal, then it must be true that they have the same x-component, the same y-component, and the same z-component, for example, in rectangular coordinates. At the introductory level, it is imperative that we choose the most important of the three scalar equations that represent the vector force balance. The most important scalar equation
is obtained by balancing forces in the primary direction of flow--and since we assume one-dimensional flow, it should be relatively straightforward to identify the important flow direction and balance forces in that direction.

The accumulation rate process

We are ready to associate mathematical quantities with each type of momentum rate process that is contained in the vector force balance. The fluid momentum vector is expressed as $\rho \mathbf{v}$, which is equivalent to the overall mass flux vector. This is actually the momentum per unit volume of fluid because mass is replaced by density in the vectorial representation of fluid momentum. Mass is an extrinsic property that is typically a linear function of the size of the system. In this respect, $m \mathbf{v}$ is a fluid momentum vector that changes magnitude when the mass of the system increases or decreases. This change in fluid momentum is not as important as the change that occurs when the velocity vector is affected. On the other hand, fluid density is an intrinsic property, which means that it is independent of the size of the system. Hence, $\rho \mathbf{v}$ is the momentum vector per unit volume of fluid that is not affected when the system mass increases or decreases. We are ready to write an expression for the rate of accumulation of fluid momentum within the control volume. This term involves the use of a time derivative to detect changes in fluid momentum during a period of observation that is consistent with the time frame during which the solution to the specified problem is required. If $\Delta V$ represents the size of the control volume, then

$$\frac{d}{dt}(\rho \mathbf{v} \Delta V)$$

is the mathematical representation of the accumulation term with units of momentum per time--hence, rate of momentum. A few comments are in order here before we proceed to the form of the other terms in the force balance. If the control volume is stationary, or fixed in space, then the spatial coordinates of $\Delta V$ are not functions of time. Consequently, the control volume $\Delta V$ can be moved to the left side of the derivative operator because $\Delta V$ is actually a constant and the total time derivative can be replaced by the partial time derivative. Hence, terms of type 1 in the force balance can be simplified as follows when the control volume is fixed in space;

$$\Delta V \frac{\partial}{\partial t}(\rho \mathbf{v})$$
It should be obvious that this term is volumetric, meaning that the accumulation mechanism applies to the entire system contained within the control volume. The stipulation that the control volume is stationary did simplify the mathematics to some extent, but the final form of the force balance does not depend on details pertaining to the movement of the control volume. Possibilities for this motion include a control volume that is stationary with fixed spatial coordinates, a control volume that moves at every point on its surface with the local fluid velocity, or a control volume that moves with a velocity that is different from the local fluid velocity. Of course, we chose the simplest case, but as I indicated above, the final form of all of the balances is not a function of this detail. Finally, if the fluid is a liquid, then the assumption of incompressibility is typically invoked. An incompressible liquid is characterized by a density that is not a strong function of pressure based on equation of state principles. Hence, for isothermal liquid systems, one assumes that density changes are negligible and the accumulation mechanism can be simplified further as;

$$ρΔV\frac{∂v}{∂t}$$

Remember that this "unsteady state" term is unimportant and will be neglected when we seek solutions that are independent of time---the so-called steady state behaviour of the system. In practice, we must wait until the transients decay and the measurable quantities are not functions of time if we hope to correlate steady state predictions with experimental results.

**Rate processes due to momentum flux**

The terms identified by 2 and 3 in the force balance are unique because they are surface-related and act across the surfaces that bound the fluid within the control volume. Surface-related is a key word, here, which indicates that flux is operative. The units of momentum flux are momentum per area per time.

There are three contributions to momentum flux that have units of momentum per area per time. Since these units are the same as force per unit area, one of the flux mechanisms is pressure. Remember that pressure is a scalar quantity, which means that there is no directional nature to fluid pressure. In other words, fluid pressure acts similarly in all coordinate directions. However, pressure forces are operative in a fluid, and they act perpendicular to any surface that contacts the fluid. These forces act along the direction of the unit normal vector that characterizes the orientation of the surface and, for this reason, pressure forces are classified as normal forces. In general, a normal
force is one that acts perpendicular to the surface across which the force is transmitted. Choose any well-defined simple surface in one of the coordinate systems mentioned above (rectangular, cylindrical, or spherical) and the student should be able to identify two orthogonal coordinate directions within the surface, and one coordinate that is normal to the surface. Consider the walls, floor, or ceiling of a room in rectangular coordinates, for example. An alternative viewpoint is a follows—as one moves on a simple surface, two coordinates change and one remains fixed. This simple surface is typically defined as one with a constant value of the coordinate that remains fixed in the surface. The coordinate that remains fixed is also in the direction of the unit normal vector. In summary, forces due to momentum flux act across surfaces and can be classified as normal forces or shear forces. As mentioned above, normal forces act perpendicular to a surface along the unit normal vector. Shear forces act parallel to the surface along one of the two coordinate directions that make up the surface. Hence, momentum flux initially identifies a simple surface with a unit normal vector that is coincident with one of the unit vectors of an orthogonal coordinate system. Then momentum flux identifies a vector force per unit area that acts across this surface, and this vector force has three scalar components. One of these scalar force components acts co-linear with the unit normal vector to the surface, and this force is designated as a normal force. The other two scalar force components act along coordinate directions within the surface itself, and these forces are called shear forces because the surface area across which the force acts is parallel to the direction of the force.

Another important contribution to momentum flux is due to convective fluid motion, and this mechanism is called convective momentum flux—designated by \( \rho \mathbf{vv} \). The mathematical form of convective momentum flux is understood best by initially constructing the total mass flux vector for a pure or multicomponent fluid, and then generating the product of mass flux with momentum per unit mass. Mass flux is a vectorial quantity that has units of mass per area per time, and \( \rho \mathbf{v} \) is the mathematical representation of the total mass flux vector. Of course, \( \rho \mathbf{v} \) also represents the momentum vector per unit volume of fluid as introduced above for the accumulation rate process. The total mass flux vector represents an important contribution to the balance on overall fluid mass. If one accepts \( \rho \mathbf{v} \) as a vectorial representation of the convective flux of overall fluid mass, then it is possible to construct the product of \( \rho \mathbf{v} \) with the momentum vector per unit mass of fluid, the latter of which is analogous to the velocity vector \( \mathbf{v} \). This product of \( \rho \mathbf{v} \) and \( \mathbf{v} \) is not the scalar ("dot") product or the vector ("cross") product that the student should be familiar with from vector calculus. Convective momentum flux is a quantity that generates nine scalars. This should be obvious if one chooses the rectangular coordinate system for illustration and multiplies the three scalar components of the mass flux vector \( (\rho v_x, \rho v_y, \rho v_z) \) by the three scalar components of the velocity vector \( (v_x, v_y, v_z) \). Using rigorous mathematical terminology,
convective momentum flux $\rho v \mathbf{v}$ is a second-rank tensor that associates a vector with each coordinate direction. Since there are three orthogonal coordinate directions that are identified by the unit vectors of the chosen coordinate system, $\rho v \mathbf{v}$ identifies a vector with each of the three coordinate directions. Remember that a vector associates a scalar with each of three coordinate directions, also in the chosen coordinate system. Hence, the student should rationalize that there are nine scalars that one can generate from three distinct vectors, and these nine scalars constitute a second-rank tensor such as convective momentum flux. At this stage in our discussion of momentum flux, which is unique to the study of fluid dynamics, it is instructive to write all nine scalars of $\rho v \mathbf{v}$ and comment about the subscripts on the scalar velocity components. It should be emphasized that the discussion which follows is applicable to the most complex flow problems and actually contains much more detail than that which is necessary to analyze one-dimensional flow. This claim is substantiated by the fact that eight of the nine scalars of $\rho v \mathbf{v}$ are identically zero for a simple one-dimensional flow problem with a velocity vector given by $\mathbf{v} = \delta_x v_x + \delta_y (0) + \delta_z (0)$. If fluid motion is restricted to the $x$-direction in rectangular coordinates as illustrated above, then the only non-vanishing scalar of convective momentum flux is $\rho v_x v_x$, which has units of momentum per time per area or force per unit area. Hence, $\rho v \mathbf{v}$ represents force per unit area that is transmitted across the surfaces that bound fluid within the control volume, and terms of this nature due to convection motion of a fluid must be included in a force balance. As mentioned above, one must construct the product of each of the nine scalars generated from this second-rank tensor with the surface area across which the force (or stress) is transmitted. Information about these surfaces and the coordinate direction in which the force acts is contained in the subscripts of the velocity components. For the most general type of fluid flow in rectangular coordinates, the nine scalars that one can generate from convective momentum flux are given below:

\[
\begin{align*}
\rho v_x v_x & \quad \rho v_x v_y & \quad \rho v_x v_z \\
\rho v_y v_x & \quad \rho v_y v_y & \quad \rho v_y v_z \\
\rho v_z v_x & \quad \rho v_z v_y & \quad \rho v_z v_z
\end{align*}
\]

It should be obvious that the nine scalars illustrated above for convective momentum flux fit nicely in a three-by-three matrix. All second-rank tensors generate nine scalars, and it is acceptable to represent the tensor by the matrix of scalars. If the matrix is symmetric, then the tensor is classified as a symmetric tensor. This is true for convective momentum flux because the product of two velocity components $v_i v_j$ does not change if the second component is written first. Another positive test for symmetry is obtained by interchanging the rows and columns of the three-by-three matrix to generate a second matrix that is indistinguishable from the original matrix.
As an illustrative example, let us focus on the element in the first row and second column, \( \rho v_x v_y \), for the matrix representation of the convective momentum flux tensor. The subscript \( x \) on the first velocity component indicates that \( \rho v_x v_y \) is a force per unit area acting across a simple surface oriented with a unit normal vector in the \( \pm x \)-direction. The subscript \( y \) on the second velocity component tells us that this force acts in the \( y \)-direction. If we perform this analysis for all nine components in the matrix for \( \rho v \) above, then the three entries in the first row represent \( x \)-, \( y \)-, and \( z \)-components, respectively, of the force per unit area that is transmitted across the simple surface defined by a constant value of the \( x \) coordinate, which means that the unit normal vector to the surface is co-linear with the \( x \)-direction. Likewise, the three entries in the second row of the matrix represent \( x \)-, \( y \)-, and \( z \)-components, respectively, of the force per unit area that is transmitted across the simple surface defined by a constant value of the \( y \) coordinate, which means that the unit normal vector to the surface is co-linear with the \( y \)-direction. Finally, the three entries in the third row of the matrix represent \( x \)-, \( y \)-, and \( z \)-components, respectively, of the force per unit area that is transmitted across the simple surface defined by a constant value of the \( z \) coordinate, which means that the unit normal vector to the surface is co-linear with the \( z \)-direction. Notice that the matrix components on the main diagonal from upper left to lower right have the same two subscripts and can be written in general as \( \rho(v_i)^2 \). These forces satisfy the requirement for normal forces. Each one acts in the \( i \)th coordinate direction (where \( i = x, y, \) or \( z \)) and the unit normal vector to the surface across which the force is transmitted is also in the \( i \)th direction. In summary, when a momentum flux tensor is expressed in matrix form, the main diagonal entries from upper left to lower right represent forces per unit area that act along the direction of the normal vector to the surface across which the force is transmitted. The off-diagonal elements represent shearing forces because these forces act in one of the two coordinate directions that define the surface across which the force is transmitted.

Using this formalism, it is also possible to represent the pressure contribution to momentum flux in matrix notation. However in this case, all of the entries have the same magnitude (i.e., \( p \)) and they lie on the main diagonal from upper left to lower right. There are no off-diagonal components because fluid pressure generates surface forces that act in the direction of the unit normal vector to the surface across which the force or stress is transmitted---they are all normal forces. In each coordinate system, the matrix representation of the pressure contribution to momentum flux can be written in the following form;

\[
\begin{pmatrix}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & p \\
\end{pmatrix}
\]
Before we depart from the discussion of rate processes due to momentum flux, it is necessary to consider the molecular mechanism that relates viscous stress to linear combinations of velocity gradients via Newton's law of viscosity, if the fluid is classified as a Newtonian fluid. Viscous momentum flux is also a second-rank tensor that identifies a vector force per unit area with each of the three coordinate directions. These forces are not due to inertia or bulk fluid motion like those that are generated from $\rho v v$, but they are best viewed as frictional forces that arise when fluid parcels on adjacent streamlines slide past one another because they are moving at different relative speeds. A simple analogy of the shearing forces generated by viscous momentum flux is the action that one performs with a piece of sandpaper to make a wood surface smoothe. The wood surface is analogous to the wall of a pipe, for example, and the motion of the sandpaper is representative of the fluid layers that are adjacent to the wall. The surface forces under consideration definitely meet the requirements of shearing forces because the surface is oriented parallel to the direction of fluid motion, the latter of which coincides with the direction of the force. In polymer processing operations, if the fluid viscosity is large enough and the flow is fast enough, then thermal energy will be generated by frictional shear at the interface between the fluid and the wall. This is analogous to the fact that a wood surface is slightly warmer after it is sanded, and the surface temperature is higher when the sanding is performed more vigorously. When we generate the matrix representation of viscous momentum flux $\tau$, it is necessary to put two subscripts on the letter $\tau$ to facilitate the row and column for each entry. Unlike the convective momentum flux tensor where we inserted a single subscript on each velocity component in the product $vv$, we must now put both subscripts on $\tau$. However, it is acceptable to analyze two subscripts on $\tau$ in the same manner that we analyzed $\rho v v_j$ above. In rectangular coordinates, the matrix representation for viscous momentum flux is written as follows:

$$
\begin{bmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{bmatrix}
$$

The interpretation of these nine scalars follows directly from the discussion of the nine scalars that are generated by $\rho v v$. The only difference is that the forces result from a molecular mechanism that is analogous to heat conduction and mass diffusion, rather than bulk fluid motion. For example, the second row of scalars represents $x$-, $y$-and $z$-components, respectively, of the viscous force per unit area that is transmitted across the simple surface defined by a constant value of the $y$ coordinate, which means that the unit normal vector to the surface is co-linear with the $y$-direction.
We have now introduced a total of 21 scalar quantities; 9 from $\rho \mathbf{v} \mathbf{v}$, 9 from $\tau$, and 3 from the pressure contribution to momentum flux; that identify all of the possible surface force components which can be generated from the total momentum flux tensor. When each of these scalars is multiplied by the surface area across which the force acts, a quantity with units of momentum per time is obtained that represents a term of type 2 or 3 in the force balance. Inputs are identified as type 2, and outputs are classified as type 3. It should be no surprise that the 21 scalar surface forces are distributed equally among the three scalar balances that constitute the total vector force balance. Based on the introductory statements above with respect to the double-subscript nature of the scalar forces generated by $\rho \mathbf{v} \mathbf{v}$ and $\tau$, the student should be able to identify the surface across which the force acts from the first subscript, and the direction in which the force acts from the second subscript.

**Momentum rate processes due to external body forces**

All terms in the momentum balance have units of momentum per unit time, which is synonymous with the units of force. In this respect, it is necessary to include terms of type 4 in the force balance because they account for all of the external forces that act on the fluid within the control volume. These terms are different than those categorized by types 2 and 3 because they do not act across surfaces that bound the fluid within the control volume. Type 4 forces are usually called body forces because they act volumetrically like the accumulation rate process, which means that each fluid parcel within the system experiences the same effect due to a body force. The primary body force that we will encounter is gravity. The external vector force due to gravity is written intrinsically via the fluid density in the following manner;

$$\rho \mathbf{g} \Delta \mathbf{V}$$

where $\mathbf{g}$ is the gravitational acceleration vector. Once again, we have used the fluid density instead of the total fluid mass within the system to insure that our external force does not have to be modified if the mass or size of the system changes. It is true that the size of the control volume $\Delta V$ could change in response to an increase or decrease in system mass at constant density. However, one of the last steps in deriving the force balance for a specific problem is division by the control volume which generates a completely intrinsic equation that is independent of system size or mass. There are other types of external body forces in addition to gravity that should be included in a compete study of fluid dynamics. For example, fluid particles that have permanent electric dipoles will experience body forces in the presence of an electric field, and particles with magnetic moments experience forces and torques due to magnetic fields. These forces are important and must be considered in a study of ferrohydrodynamics and
magnetohydrodynamics. Unfortunately, fluid flow in the presence of electric and magnetic fields is rarely covered in undergraduate as well as graduate courses because the complexity of the problems increases several-fold, limiting discussion to the simplest examples for which exact solutions require the use of advanced mathematical techniques. Even though surface tension forces cannot be classified as body forces, they play an important role in the operation of viscosity-measuring devices, like the parallel-plate and cone-and-plate viscometers, where a thin film of fluid is placed between two closely spaced horizontal surfaces---the lower surface being stationary and the upper one rotating at constant angular velocity. In the absence of surface tension, the test fluid would spread and completely wet the solid surfaces in response to rotation which generates centrifugal forces. Then the fluid would "fall off the table" since there are no restraining walls. Of course, surface tension plays the role of restraining walls and keeps the fluid from exiting the viscometer if the rotational speeds are slow enough.

**General objectives for solving problems in fluid dynamics**

Most flow problems that we will encounter involve a fluid in motion adjacent to a stationary wall---the wall of a tube, for example---or a fluid that is set in motion by a moving surface---this is the case in a viscosity-measuring device. In general, the bulk fluid and the solid surface are moving at different relative speeds, and this generates velocity gradients and viscous stress at the interface. Macroscopic correlations in fluid dynamics focus on the fluid-solid interface and calculate the force exerted by the fluid on the solid, or vice versa, via the fluid velocity gradient "at the wall". These macroscopic momentum transport correlations contain the friction factor and the Reynolds number. Hence, one calculates the Reynolds number from the characteristics of the flow problem and uses these dimensionless correlations to determine the value of the friction factor. Frictional energy losses in straight sections of a tube are estimated from the friction factor. The size of a pump required to offset all of the dissipative processes that reduce fluid pressure can be estimated from the non-ideal macroscopic mechanical energy balance (i.e., Bernoulli equation) that incorporates friction loss via the friction factor. In some cases, macroscopic momentum transfer correlations relate torque and angular velocity for the viscosity-measuring devices, allowing one to calculate the viscosities of Newtonian and non-Newtonian fluids from measurements of torque vs. angular velocity.

**Example:**
The lateral surface of a straight tube with radius R and length L. Consider one dimensional flow in the z-direction induced by some combination of a pressure gradient and gravitational forces. Hence, steady state application of the Equation of Continuity yields, \( v_z = f(r) \).
The unit normal is $\mathbf{n} = \delta_r$. Hence, the 1\textsuperscript{st} subscript on $\rho \mathbf{v} \mathbf{v}$ and $\tau$ must be $r$. Pressure force acts in the $r$-direction, $\delta_r p$ pressure forces are normal forces Forces due to convective momentum flux vanish, if no-slip is a valid assumption

The only nonzero force due to $\rho \mathbf{v} \mathbf{v}$ is $\rho \mathbf{v}_r \mathbf{v}_r$, which acts across a surface where $\mathbf{n} = \delta_z$. For 3-dimensional flow, there are $\delta_r \rho \mathbf{v}_r \mathbf{v}_r$, $\delta_\theta \rho \mathbf{v}_\theta \mathbf{v}_\theta$, and $\delta_z \rho \mathbf{v}_z \mathbf{v}_z$. The primary force due to viscous momentum flux is $\tau_{rz}$, if $\mathbf{v}_z(r)$ via NLV

All forces or stresses due to total momentum flux act across the same surface area, $2\pi RL$;

<table>
<thead>
<tr>
<th>Convective forces</th>
<th>Viscous forces</th>
<th>Pressure forces</th>
<th>Direction in which the forces act</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho \mathbf{v}_r \mathbf{v}_r$</td>
<td>$\tau_{rr}$</td>
<td>$p$</td>
<td>$\delta_r$</td>
</tr>
<tr>
<td>$\rho \mathbf{v}<em>\theta \mathbf{v}</em>\theta$</td>
<td>$\tau_{r\theta}$</td>
<td>$-$</td>
<td>$\delta_\theta$</td>
</tr>
<tr>
<td>$\rho \mathbf{v}_z \mathbf{v}_z$</td>
<td>$\tau_{rz}$</td>
<td>$-$</td>
<td>$\delta_z$</td>
</tr>
</tbody>
</table>

Steady state one-dimensional fluid flow of an incompressible Newtonian fluid

If the functional dependence of the fluid velocity profile is $v_i(x_j)$, then balance forces in the $i$\textsuperscript{th}-coordinate direction. Hence, the 2\textsuperscript{nd} subscript on $\rho \mathbf{v} \mathbf{v}$ and $\tau$ is $i$. The control volume which contains the fluid should be differentially thick only in the $j$\textsuperscript{th}-coordinate direction. This yields an ordinary differential equation for the dependence of $\tau_{ji}$ on $x_j$ via $d\tau_{ji}/dx_j$. If the velocity profile is $v_i(x_j)$, then for incompressible Newtonian fluids, one should expect the following nonzero elements of the viscous stress tensor; $\tau_{ij}$ and $\tau_{ji}$.

Qualitative statement of the momentum shell balance, or the force balance

There are 3 scalar components because this is a vector equation

Accumulation of fluid momentum in control volume (CV) =

+ Input due momentum flux acting across surfaces which bound fluid in the CV
- Output due to momentum flux acting across surfaces which bound fluid in the CV
+ Sum of all external forces acting on the fluid in the CV (i.e., gravitational force)

Lecture#4

>Forces are represented by arrows
>By convention, draw all arrows in the positive coordinate directions; + forces
>Inputs due to momentum flux act across surfaces defined by smaller values of the independent variable that remains constant within the surface
>Outputs due to momentum flux act across surfaces defined by larger values of the independent variable that remains constant within the surface
Steady state analysis; laminar flow of an incompressible Newtonian fluid through a tube; The classic problem in fluid dynamics; \( v_z(r) \); tube is horizontal, no gravity forces in z-dir. The control volume is represented by a cylindrical shell at radius \( r \) with length \( L \) Size of the control volume is \( 2\pi rLdr \); differentially thick in the radial direction, only. There are four different surfaces that bound fluid within the control volume. The system is chosen as the fluid that occupies the differential control volume.

**z-Component of the force balance**---the objective is to obtain an ODE for \( \tau_{rz}(r) \) z-component forces are represented by arrows. Only consider forces or stress due to total momentum flux that act in the z-direction. The 2\(^{nd} \) subscript on \( \rho \mathbf{v} \mathbf{v} \) & \( \tau \) is \( z \). The only stresses that must be considered are:

<table>
<thead>
<tr>
<th>Stress</th>
<th>across which the stress acts</th>
<th>Surface area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure ( p )</td>
<td>( \delta_z ) normal stress</td>
<td>( 2\pi r dr )</td>
</tr>
<tr>
<td>Convective momentum flux</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho \mathbf{v}_r \mathbf{v}_z )</td>
<td>( \delta_r ) shear stress</td>
<td>( 2\pi rL )</td>
</tr>
<tr>
<td>( \rho \mathbf{v}_\theta \mathbf{v}_z )</td>
<td>( \delta_\theta ) shear stress</td>
<td>( Ldr )</td>
</tr>
<tr>
<td>( \rho \mathbf{v}_z \mathbf{v}_z )</td>
<td>( \delta_z ) normal stress</td>
<td>( 2\pi r dr )</td>
</tr>
<tr>
<td>Viscous momentum flux</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tau_{rz} )</td>
<td>( \delta_r ) shear stress</td>
<td>( 2\pi rL )</td>
</tr>
<tr>
<td>( \tau_{\theta z} )</td>
<td>( \delta_\theta ) shear stress</td>
<td>( Ldr )</td>
</tr>
<tr>
<td>( \tau_{zz} )</td>
<td>( \delta_z ) normal stress</td>
<td>( 2\pi r dr )</td>
</tr>
</tbody>
</table>

Use Newton's law of viscosity to determine which components of \( \tau \) are important. Only \( \tau_{rz} = \tau_{zr} \) survives if \( v_z(r) \) and \( \nabla \cdot \mathbf{v} = 0 \). The nine elements of \( \tau \) can be represented by a symmetric 3x3 matrix. There are only 6 independent equations for \( \tau \) via NLV, due to symmetry. All of the 2\(^{nd} \)-rank tensors in NLV are symmetric.

**Lecture#5**

**z-component force balance for one-dimensional laminar tube flow**

Accumulation rate process is volumetric; \[ \frac{\partial}{\partial t} (\rho v_z dV) = \rho dV \frac{\partial v_z}{\partial t} \]

Input due to convective momentum flux occurs at \( z=0 \); \[ \rho v_z v_z 2\pi r dr \] \( z=0 \)

Input due to fluid pressure occurs at \( z=0 \); \[ p 2\pi r dr \] \( z=0 \)

Input due to viscous momentum flux occurs at \( r \); \[ \tau_{rz} 2\pi rL \] \( r \)
Output due to convective momentum flux occurs at $z=L$; $[\rho v_z v_z 2\pi r \, dr]_{z=L}$
Output due to fluid pressure occurs at $z=L$; $[p \, 2\pi r \, dr]_{z=L}$
Output due to viscous momentum flux occurs at $r+dr$; $[\tau_{rz} 2\pi rL \, r+dr]$

External force due to gravity is volumetric; $\rho g_z \, dV$

Surface area $2\pi r dr$ is perpendicular to the $z$-direction
Surface areas $2\pi rL$ and $2\pi (r+dr)L$ are perpendicular $r$-direction

Accumulation = Input - Output + Sum of external forces

At steady state, Accumulation = 0
Input = forces due to momentum flux exerted by the surroundings on the system
Output = forces due to momentum flux exerted by the system on the surroundings
Input = Output, for convective momentum flux, because $v_z \neq f(z)$ via EOC
There is no horizontal $z$-component of the force due to gravity; $g_z = 0$

The $z$-component force balance represents a balance between pressure and viscous forces;
$$[p \, 2\pi r \, dr]_{z=0} + [\tau_{rz} 2\pi rL \, r] = [p \, 2\pi r \, dr]_{z=L} + [\tau_{rz} 2\pi rL \, r+dr]$$

Divide the previous equation by the size of the control volume, $dV = 2\pi rLdr$, which is differentially thick in the radial direction, rearrange the equation and combine terms;
$$\frac{(r\tau_{rz})_{r+dr} - (r\tau_{rz})_r}{r\, dr} = \frac{p(z = 0) - p(z = L)}{L} = \frac{\Delta p}{L}$$

Take the limit as $dr \to 0$, and recognize the definition of the 1st derivative. The result is the $z$-component of the Equation of Motion in cylindrical coordinates in terms of viscous stress;
$$\frac{1}{r} \frac{d}{dr} (r\tau_{rz}) = \frac{\Delta p}{L}$$

"Divide and conquer", or "separate and integrate". Integrate the previous equation with respect to independent variable $r$. The result is;
$$r\tau_{rz} = C_1 + \left(\frac{\Delta p}{2L}\right)r^2$$
\[ \tau_{rz}(r) = \frac{C_1}{r} + \left(\frac{\Delta p}{2L}\right)r \]

Since the range of \( r \) is from the centerline @ \( r=0 \) to the wall @ \( r=R \), \( C_1=0 \) because viscous stress is finite. Hence;

\[ \tau_{rz}(r) = \left(\frac{\Delta p}{2L}\right)r \]

Now, relate \( \tau_{rz} \) to the velocity gradient via Newton's law of viscosity if the fluid is Newtonian;

\[ \tau_{rz}(r) = -\mu \frac{dv_z}{dr} = \left(\frac{\Delta p}{2L}\right)r \]

Integration yields;

\[ v_z(r) = C_2 - \left(\frac{\Delta p}{L}\right)\frac{r^2}{4\mu} \]

and integration constant \( C_2 \) is evaluated from the “no-slip” boundary condition at the stationary wall, where \( v_z=0 \). Finally, one obtains the quadratic velocity profile for incompressible flow of a Newtonian fluid through a straight tube of radius \( R \) and length \( L \);

\[ v_z(r) = \left(\frac{R^2}{4\mu}\right)\left(\frac{\Delta p}{L}\right)(1 - \eta^2) \]

where \( \eta = r/R \). The fluid velocity \( v_z \) is maximum along the centerline of the tube and minimum at the stationary solid wall (i.e., no slip). Viscous shear stress \( \tau_{rz} \) is maximum at the solid wall and minimum at the centerline (i.e., \( r=0 \)) where symmetry is invoked.

**Lecture#6**

Equation of Motion (EOM) represents the vector force balance in fluid dynamics

A vector equation implies that 3 scalar equations must be satisfied

Consider the classic problem of 1-dimensional incompressible laminar flow through a straight tube with circular cross section
Make the control volume differentially thick in both the \( r \)-direction and the \( z \)-direction.
Replace the inlet plane at \( z=0 \) with an inlet plane at \( z \).
Replace the outlet plane at \( z=L \) with an outlet plane at \( z + dz \).

Now, \( dV = 2\pi r dr dz \) (i.e., replace \( L \) by \( dz \)).

The "shell balance" approach for the \( z \)-component force balance for fluids in which only \( \tau_{rz} \) is important yields:

\[
\text{Accumulation} = \text{Input} - \text{Output} + \text{Sum of external forces}
\]

\[
\frac{\partial}{\partial t} \{ \rho v_z \, dV \} = + \left[ \rho v_z v_z \, 2\pi r \, dr \right]_z - \left[ \rho v_z v_z \, 2\pi r \, dr \right]_{z+dz}
+ \left[ \tau_{rz} \, 2\pi r \, dz \right]_r - \left[ \tau_{rz} \, 2\pi r \, dz \right]_{r+dr}
+ \left[ p \, 2\pi r \, dr \right]_z - \left[ p \, 2\pi r \, dr \right]_{z+dz}
+ \rho g_z \, dV
\]

Divide by \( dV = 2\pi r \, dr \, dz \):

\[
\frac{\partial}{\partial t} \{ \rho v_z \} = \left\{ + \left[ \rho v_z v_z \, 2\pi r \, dr \right]_z - \left[ \rho v_z v_z \, 2\pi r \, dr \right]_{z+dz} \right\} / 2\pi r \, dr \, dz
+ \rho g_z
\]

Take the limit as both \( dr \to 0 \) and \( dz \to 0 \):

\[
\frac{\partial}{\partial t} \{ \rho v_z \} = \left\{ + \left[ \rho v_z v_z \right]_z - \left[ \rho v_z v_z \right]_{z+dz} \right\} / dz
+ \rho g_z
\]

One obtains the \( z \)-component of the Equation of Motion (EOM) for 1-dimensional laminar flow of any type of fluid that exhibits no normal viscous stress (i.e., \( \tau_{zz} \approx 0 \)) through a straight tube in the entrance region, prior to the establishment of fully developed flow;
\[
\frac{\partial}{\partial t}(\rho v_z) = -\frac{\partial}{\partial z}(\rho v_z v_z) - \frac{1}{r} \frac{\partial}{\partial r}(r \tau_{rz}) - \frac{\partial p}{\partial z} + \rho g_z
\]

Manipulate the accumulation and convective momentum flux terms with assistance from the Equation of Continuity;

\[
\frac{\partial}{\partial t}(\rho v_z) + \frac{\partial}{\partial z}(\rho v_z v_z) = \rho \frac{\partial v_z}{\partial t} + v_z \frac{\partial \rho}{\partial t} + \rho v_z \frac{\partial v_z}{\partial z} + v_z \frac{\partial}{\partial z}(\rho v_z)
\]

\[
= v_z \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v_z) \right\} + \rho \left\{ \frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right\}
\]

EOC for 1-dimensional flow (i.e., z-direction) of any type of fluid, with \( v_r = v_\theta = 0; \)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v_z) = 0
\]

Hence;

\[
\frac{\partial}{\partial t}(\rho v_z) + \frac{\partial}{\partial z}(\rho v_z v_z) = \rho \left\{ \frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} \right\} = - \frac{1}{r} \frac{\partial}{\partial r}(r \tau_{rz}) - \frac{\partial p}{\partial z} + \rho g_z
\]

The previous equation is not restricted to incompressible fluids with constant density, even though \( \rho \) is outside of the derivative operators. Consider the last two terms on the right side of the previous z-component force balance;

\[
z\text{-component forces due to pressure and gravity} = - \frac{\partial p}{\partial z} + \rho g_z
\]

In vector notation, these two forces can be expressed as \(- \nabla p + \rho \mathbf{g}\)

Define the gravitational potential energy per unit mass \( \Phi \), such that \( \Phi = gh \), where \( g \) is the gravitational acceleration constant and \( h \) is a spatial coordinate which increases in the coordinate direction opposite to gravity (i.e., \( h = z \), measured vertically upward). The gravitation acceleration vector can be written as;

\[
\mathbf{g} = g \left( - \delta_h \right) = - \delta_h \frac{\partial \Phi}{\partial h} = - \nabla \Phi
\]

For incompressible fluids, where the fluid density \( \rho \) is constant, the gravitational force per unit volume can be written as;
\[ \rho g = -\rho \nabla \Phi = -\nabla \rho \Phi \]

Pressure and gravity forces in the Equation of Motion can be written as follows for incompressible fluids;

\[ -\nabla p + \rho g = -\nabla p - \nabla \rho \Phi = -\nabla (\rho + \rho \Phi) = -\nabla P \]

where P represents "dynamic pressure", which is a combination of fluid pressure and gravitational potential energy.

**Lecture#7**

As a consequence of the previous vector algebra, it is acceptable to neglect the gravitational force in each scalar component of the Equation of Motion and replace fluid pressure p by dynamic pressure P. The previous steady state z-component force balance for fully developed 1-dimensional laminar tube flow of an incompressible fluid \([i.e., v_z(r)]\) reduces to;

- **z-component:** \( 0 = -\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) - \frac{\partial P}{\partial z} \)

- **r-component:** \( 0 = -\frac{\partial P}{\partial r}, \) implies that \( P \neq f(r) \)

- **\( \Theta \)-component:** \( 0 = -\frac{1}{r} \frac{\partial P}{\partial \Theta}, \) implies that \( P \neq f(\Theta) \)

Since steady state analysis implies that \( P \neq f(t) \), it is reasonable to consider that the functional dependence of dynamic pressure is \( P = f(z) \), and one replaces \( \partial P/\partial z \) by \( dP/dz \). Furthermore, for 1-dimensional flow with \( v_z = f(r) \), Newton's law of viscosity reveals that \( \tau_{rz} \) is only a function of \( r \). Hence, one replaces \( \partial/\partial r (r \tau_{rz}) \) by \( d/dr (r \tau_{rz}) \) if the fluid is Newtonian. Now, the z-component force balance can be written as;

- **z-component:** \[ \frac{1}{r} \frac{d}{dr} (r \tau_{rz}) = -\frac{dP}{dz} = Constant (C_1) \]

via separation of variables, because the viscous stress term involving \( \tau_{rz} \) is only a function of \( r \), whereas the dynamic pressure term is only a function of \( z \). If;

\[ -\frac{dP}{dz} = \text{Constant} \ (C_1) \]

then;

\[ P(z) = -C_1 \ z + C_2 \]
represents a linear dynamic pressure distribution along the tube axis. Integration constants $C_1$ and $C_2$ can be determined from the following boundary conditions;

\begin{align*}
  z = 0 & \quad P = P_0, \text{ which implies that } C_2 = P_0 \\
  z = L & \quad P = P_L, \text{ which implies that } C_1 = \frac{(P_0 - P_L)}{L} = \frac{\Delta P}{L}
\end{align*}

Now, one solves for the viscous shear stress distribution $\tau_{rz}(r)$ from;

\begin{align*}
  \frac{1}{r} \frac{d}{dr} (r \tau_{rz}) &= -\frac{dP}{dz} = C_1 = \frac{\Delta P}{L} \\
  d(r \tau_{rz}) &= \left(\frac{\Delta P}{L}\right) rdr \\
  r \tau_{rz} &= \left(\frac{\Delta P}{2L}\right) r^2 + C_3 \\
  \tau_{rz}(r) &= \left(\frac{\Delta P}{2L}\right) r + \frac{C_3}{r}
\end{align*}

Lecture#8
Boundary value problems;
The previous generic result for the viscous shear stress distribution is valid for;

\begin{enumerate}
  \item One-dimensional (i.e., only $v_z$) incompressible Newtonian fluid flow in the laminar regime through a tube at any orientation angle with respect to gravity
\end{enumerate}

**Boundary conditions based on the range of the independent variable, $0 \leq r \leq R$;**

$\tau_{rz}$ is finite along the symmetry axis of the tube @ $r=0$. Hence, $C_3 = 0$.

"No slip" at the solid-liquid interface, $v_z(@r=R) = 0$. Let $\eta = r/R$;

\begin{align*}
  \tau_{rz}(r) &= -\mu \frac{dv_z}{dr} = \left(\frac{\Delta P}{2L}\right) r \\
  v_z(r) &= C_4 - \left(\frac{\Delta P}{4\mu L}\right) r^2 \\
  v_z(r) &= \left(\frac{R^2 \Delta P}{4\mu L}\right) \{1 - \eta^2\}
\end{align*}
(b) One-dimensional (i.e., only \( v_z \)) incompressible Newtonian fluid flow in the laminar regime between two concentric cylinders: axial annular flow at any orientation angle with respect to gravity.

**Boundary conditions based on the range of the independent variable, \( R_{\text{inner}} \leq r \leq R_{\text{outer}}; \)**

"No slip" at the inner and outer stationary tube walls
\[ v_z = 0 \text{ at } r=R_{\text{inner}} \text{ and } r=R_{\text{outer}} \]

\[
\tau_{rz}(r) = -\mu \frac{dv_z}{dr} = \left( \frac{\Delta P}{2L} \right) r + \frac{C_3}{r}
\]

\[
v_z(r) = C_5 - \frac{C_3}{\mu} \ln r - \left( \frac{\Delta P}{4\mu L} \right) r^2
\]

This is essentially a “distorted” quadratic profile for \( v_z(r) \) due to the presence of the logarithmic term when axial flow occurs between two concentric cylinders. The final result for \( v_z(r) \) in these previous two examples could have been obtained directly via two integrations of the \( z \)-component of the Equation of Motion in terms of velocity gradients for incompressible Newtonian fluids, without solving for the viscous shear stress distribution as an intermediate step. It is only justified to use the tabulated form of the Equation of Motion on page#848 when the fluid is incompressible and Newtonian, and the flow is laminar. The Equation of Motion in terms of \( \tau \) on page#847 should be employed for compressible fluids and non-Newtonian fluids.

**Problem**

Consider radius ratio \( \kappa = R_{\text{inner}}/R_{\text{outer}} = 0.1 \) and 0.9 for laminar flow of an incompressible Newtonian fluid on the shell side of the double pipe heat exchanger. At which of these two radius ratios does a quadratic approximation to the actual velocity profile provide a better fit to the exact solution obtained by solving the \( z \)-component of the Equation of Motion?

**Answer:** For each value of the radius ratio \( \kappa \), one finds the best values for \( a_0, a_1, \) and \( a_2 \) that minimize the difference between the dimensionless “distorted” quadratic velocity profile, given by the previous equation after integration constants \( C_3 \) and \( C_5 \) are evaluated via two no-slip boundary conditions, and the following dimensionless second-order polynomial;
\[
\left\{ v_z(r) \right\}_{\text{AnnularFlow}} = a_0 + a_1 \eta + a_2 \eta^2
\]
\[
\left\{ v_{z,\text{maximum}} \right\}_{\text{TubeFlow}} = a_0 + a_1 \eta + a_2 \eta^2
\]

where \( \eta = r/R \). Results are summarized in the table below;

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.31</td>
<td>1.60</td>
<td>-1.97</td>
</tr>
<tr>
<td>0.05</td>
<td>0.08</td>
<td>1.94</td>
<td>-2.06</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.09</td>
<td>2.12</td>
<td>-2.07</td>
</tr>
<tr>
<td>0.20</td>
<td>-0.34</td>
<td>2.37</td>
<td>-2.05</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.57</td>
<td>2.58</td>
<td>-2.03</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.78</td>
<td>2.79</td>
<td>-2.02</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.99</td>
<td>3.00</td>
<td>-2.01</td>
</tr>
<tr>
<td>0.60</td>
<td>-1.19</td>
<td>3.20</td>
<td>-2.01</td>
</tr>
<tr>
<td>0.70</td>
<td>-1.40</td>
<td>3.40</td>
<td>-2.00</td>
</tr>
<tr>
<td>0.80</td>
<td>-1.60</td>
<td>3.60</td>
<td>-2.00</td>
</tr>
<tr>
<td>0.90</td>
<td>-1.80</td>
<td>3.80</td>
<td>-2.00</td>
</tr>
<tr>
<td>0.95</td>
<td>-1.90</td>
<td>3.90</td>
<td>-2.00</td>
</tr>
<tr>
<td>0.99</td>
<td>-1.98</td>
<td>3.98</td>
<td>-2.00</td>
</tr>
</tbody>
</table>

The effects of curvature are much less significant when \( \kappa = 0.9 \) relative to smaller values of the radius ratio. In fact, the axial quadratic velocity profile is not significantly different from the “distorted” quadratic profile when \( \kappa = 0.9 \) because flow between two concentric cylinders with very similar radii is essentially the same as flow between two flat plates, the latter of which is given by a quadratic profile.

**Numerical Analysis**

Consider the axial velocity profile for flow of an incompressible Newtonian fluid on the shell side of the double-pipe heat exchanger. The expression for \( v_z \) is given as equation 2.4-14 on page 55 [i.e., *Transport Phenomena, 2nd edition*, RB Bird, WE Stewart, EN Lightfoot (2002)], and the corresponding volumetric flow rate, \( Q \), is given as equation 2.4-17 on the same page.

**Problem**

Use the Newton-Raphson numerical root-finding technique to calculate the dimensionless radial position, \( \eta = r/R \), in the annular flow configuration where the streamline velocity is a specified fraction (\( \beta \)) of the centerline (i.e., maximum) fluid velocity for tube flow. Use
double precision and report your answer as $\eta_{\text{root}}$ that satisfies the following convergence criterion; $F(\eta_{\text{root}}) < 10^{-6}$, for the cases given below; $[\beta = 1 - \eta^2 + (1 - \kappa^2)\ln(\eta)/\ln(1/\kappa) ]$

a) $\kappa = 0.20\quad \beta = 0.30$ (Two solutions; $\eta = 0.41$ and $\eta = 0.70$) 
b) $\kappa = 0.20\quad \beta = 0.35$ (No real solutions exist; $\beta > \beta_{\text{Maximum}} = 0.341$)  
c) $\kappa = 0.30\quad \beta = 0.25$ (Two solutions; $\eta = 0.57$ and $\eta = 0.66$)  
d) $\kappa = 0.30\quad \beta = 0.30$ (No real solutions exist; $\beta > \beta_{\text{Maximum}} = 0.254$)  
e) $\kappa = 0.40\quad \beta = 0.15$ (Two solutions; $\eta = 0.55$ and $\eta = 0.81$)  
f) $\kappa = 0.40\quad \beta = 0.20$ (No real solutions exist; $\beta > \beta_{\text{Maximum}} = 0.184$)  

As radius ratio $\kappa$ increases, the maximum value of $\beta$ (see Eq. 2.4-15, BSL) decreases.

**Problem**

Use the Newton-Raphson numerical root-finding technique to calculate the required radius ratio, $\kappa$, such that the volumetric rate of flow on the shell side of the double-pipe heat exchanger is a specified fraction ($\beta$) of the volumetric flow rate for tube flow. Use double precision and report your answer as $\kappa_{\text{root}}$ that satisfies the following convergence criterion; $G(\kappa_{\text{root}}) < 10^{-6}$, for the three cases given below;

a) $\beta = 0.20$ ($\kappa = 0.405$)  
b) $\beta = 0.50$ ($\kappa = 0.147$)  
c) $\beta = 0.80$ ($0.006 \leq \kappa \leq 0.007$) $[\beta = 1 - \kappa^4 - (1 - \kappa^2)^2/\ln(1/\kappa) ]$

**Spatially averaged properties;**

Volumetric flowrate for one-dimensional axial flow through a straight tube,  

$$Q = S\langle v_z \rangle_{\text{Average}} = \iiint_S v_z(r)dS = \int_0^{2\pi} \int_0^R v_z(r)rdrd\Theta \approx \{\Delta P\}^{\frac{1}{n}}$$

where $S$ is the flow cross-sectional area (i.e., $\pi R^2$) and $n$ is the power-law index for the power-law model. When $n=1$ for Newtonian fluids in the laminar flow regime, the relation between volumetric flow rate $Q$ and dynamic pressure difference $\Delta P$ is known classically as the Hagen-Poiseuille law. $dS$ is a differential surface element normal to the flow direction, which is constructed from a product of two differential lengths in the "no-flow" directions.

**Differential surface and volume elements;**
Rectangular coordinate directions; \( x \quad y \quad z \)
Differential lengths in these directions; \( dx \quad dy \quad dz \)

Cylindrical coordinate directions; \( r \quad \Theta \quad z \)
Differential lengths in these directions; \( dr \quad r \, d\Theta \quad dz \)

Spherical coordinate directions; \( r \quad \Theta \quad \phi \)
Differential lengths in these directions; \( dr \quad r \, d\Theta \quad r \, \sin\Theta \, d\phi \)

Construct \( dV \) via the product of differential lengths in all three coordinate directions.
Construct \( dS \) for the flow cross-sectional area or the differential surface element at a solid-liquid interface from the product of two differential lengths (i.e., choose the correct ones).

There are at least 4 methods to induce fluid flow, excluding capillary or surface tension forces:

1. Impose a pressure gradient---forced convection
2. Take advantage of gravity---forced convection
3. Move a solid surface that contacts the fluid---viscous shear
4. Impose a temperature gradient---free convection

**Lecture#9**

Force-flow relations for Newtonian and non-Newtonian fluids through a straight tube with circular cross-section;

\[
\log Q = \left\{ \frac{1}{n} \right\} \log (\Delta P)
\]

Apply a quasi-steady state model for laminar flow through an exit tube to analyze the unsteady state behaviour associated with draining a cylindrical tank or spherical bulb with a cylindrical exit tube. The exit tube has radius \( R \), length \( L \), and it is oriented at angle \( \Theta \) with respect to gravity. The height of fluid in the tank or bulb above the tube is \( h(t) \). For incompressible Newtonian fluids, the volume rate of flow through the exit tube, with radius \( R \), is given by the Hagen-Poiseuille law;

\[
Q = \frac{\pi R^4 \Delta P}{8\mu L}
\]

where;

\[
\Delta P = P_{\text{inlet}} - P_{\text{outlet}}
\]
At the tube outlet, which is chosen as the “zero of potential energy”, one writes;

\[ P_{\text{outlet}} = p_{\text{ambient}} \]

At the tube inlet;

\[ P_{\text{inlet}} = \{ p_{\text{ambient}} + \rho gh(t) \} + \rho g L \cos \Theta \]

Hence;

\[ \Delta P = \rho g \left[ h(t) + L \cos \Theta \right] \]

The previous relations illustrate an example where dynamic pressure is calculated for the approximate hydrostatic conditions in a very large tank or bulb (i.e., \( p = p_{\text{ambient}} + \rho gh \)), and the resulting dynamic pressure difference between the tube inlet and the tube outlet is employed in a hydrodynamic relation to calculate the volumetric flowrate for incompressible Newtonian fluids through the exit tube.

**A note of caution**---the momentum shell balance approach should not be used to analyze fluid flow problems with curved streamlines because centrifugal and coriolis forces will not be accounted for properly in cylindrical and spherical coordinates. Hence, one should apply the vector force balance in fluid dynamics, in general, by considering all three components of the Equation of Motion for a particular problem in only one coordinate system.

**Couette flow**---analysis of the rheology experiment in Ch306 for Newtonian and non-Newtonian fluids; Torque T vs. angular velocity \( \Omega \)

Concentric cylinder viscometer
- Inner solid cylinder at radius \( \kappa R \) rotates at constant angular velocity \( \Omega_{\text{inner}} \)
- Outer cylinder at radius \( R \) rotates at constant angular velocity \( \Omega_{\text{outer}} \)

Fast rotation of the inner solid cylinder produces 2-dimensional flow (i.e., \( v_\theta \) & \( v_r \))
- Centrifugal forces are important, which are responsible for \( v_r \)
- Streamlines are represented by outward spirals
- The Reynolds number is large

Slow rotation of the inner solid cylinder yields 1-dimensional flow (i.e., only \( v_\theta \))
- Centrifugal forces are negligible
- Streamlines are represented by concentric circles
- The Reynolds number is small (i.e., \(<1\)). This is the creeping flow regime.
- It is acceptable to neglect the entire left side of the Equation of Motion.

Steady state analysis implies that \( v_\theta \neq f(t) \)
The Equation of Continuity yields; \((1/r) \partial v_\Theta/\partial \Theta = 0\). Hence, \(v_\Theta \neq f(\Theta)\)

Analysis of the flow problem far from the ends of the rotating cylinders suggests that end effects are not important. Hence, \(v_\Theta \neq f(z)\), but there is no balance or law which provides this result. If end effects are important in practice, then the actual length of the rotating cylinder is usually increased empirically in the final calculations.

One concludes that \(v_\Theta = f(r)\), which is reasonable because the tangential fluid velocity changes considerably as one moves from the inner cylinder to the outer cylinder.

Comparison between tube flow and Couette flow:

<table>
<thead>
<tr>
<th>Consideration</th>
<th>Tube flow</th>
<th>Couette flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Important velocity component</td>
<td>(v_z(r))</td>
<td>(v_\Theta(r))</td>
</tr>
<tr>
<td>Streamlines</td>
<td>straight</td>
<td>curved</td>
</tr>
<tr>
<td>Steady state analysis</td>
<td>(v_z \neq f(t))</td>
<td>(v_\Theta \neq f(t))</td>
</tr>
<tr>
<td>Symmetry</td>
<td>(v_z \neq f(\Theta))</td>
<td>---</td>
</tr>
<tr>
<td>Neglect end effects</td>
<td>---</td>
<td>(v_\Theta \neq f(z))</td>
</tr>
<tr>
<td>Equation of Continuity</td>
<td>(v_z \neq f(z))</td>
<td>(v_\Theta \neq f(\Theta))</td>
</tr>
</tbody>
</table>

For Newtonian fluids in the Couette viscometer that obey Newton's law of viscosity, the following information is useful to determine the important nonzero components of viscous stress;

\[
v_r = 0 \quad v_z = 0 \quad v_\Theta = f(r) \quad \nabla \cdot \mathbf{v} = 0
\]

The only nonzero components of \(\mathbf{\tau}\) are;

\[
\tau_{r\Theta} = \tau_{\Theta r} = -\mu r \frac{d}{dr}\left(\frac{v_\Theta}{r}\right)
\]

This illustrates an example where Newton's law must be modified for flow systems with curved streamlines so that solid body rotational characteristics within the fluid produce no viscous stress. For example, consider the final result when the inner cylinder at radius \(kR\) rotates at constant angular velocity \(\Omega_{\text{inner}}\), and the outer cylinder at radius \(R\) rotates at constant angular velocity \(\Omega_{\text{outer}}\) in the same direction.

\[
v_\Theta(r) = \frac{1}{1 - k^2}\left\{\left(\Omega_{\text{outer}} - k^2 \Omega_{\text{inner}}\right)r - \left(\Omega_{\text{outer}} - \Omega_{\text{inner}}\right)\frac{k^2 R^2}{r}\right\}
\]
Two limiting cases (i.e., see Problem 3B.1(a) on page#105);

(a) Both cylinders rotate at the same angular velocity; \( \Omega_{\text{inner}} = \Omega_{\text{outer}} = \Omega \)
\[ \nu_\theta = \Omega r \]

(b) There is no inner cylinder; \( \kappa = 0 \)
\[ \nu_\theta = \Omega_{\text{outer}} r \]

In both cases, fluid motion is the same as solid body rotation, where \( \mathbf{v} = \mathbf{\Omega} \times \mathbf{r} \), and there is no viscous stress if the angular velocity is constant. If there is no viscous stress, then there is no torque/angular-velocity relation, and the device does not function as a viscosity-measuring device. In the complete expression for \( \nu_\theta \) above, the term which scales as \( r \) does not contribute to \( \tau_{\rho\theta} \) or the torque. The term which scales as \( 1/r \) is solely responsible for the torque/angular-velocity relation.

**Look at the following problems;**

2B.3 page#63 Laminar flow through a rectangular slot

2B.6 page#64 Laminar flow of a falling film on the outside of a tube
Identify the important non-zero component of viscous stress. Identify the radial position where viscous shear stress \( \tau_{rz} \) is maximum. Then calculate the maximum value of viscous shear stress.

2B.7 page#65 Axial annular flow in the laminar regime induced by translation of the inner cylinder

Identify the important non-zero component of viscous stress. Identify the radial position where viscous shear stress \( \tau_{rz} \) is maximum. Then calculate the maximum value of viscous shear stress.

3B.10 page#108 Radial flow between two parallel disks.
Calculate the functional form of the pressure distribution for radial flow between two parallel disks of circular geometry and sketch your answer.

**Lecture#10 Continuation of the Couette viscometer problem**
Consider all three components of the Equation of Motion in cylindrical coordinates for any type of fluid (i.e., EOM in terms of \( \tau \)), with \( \nu_\theta(r) \). Since the most important component of EOM is the \( \Theta \)-component, let’s consider the two components of secondary importance;
r-component:  \[-\rho \frac{v^2_\theta}{r} = -\frac{\partial p}{\partial r} + \rho g_r = -\frac{\partial P}{\partial r}\]

z-component:  \[0 = -\frac{\partial p}{\partial z} + \rho g_z = -\frac{\partial P}{\partial z}\]

If the z-axis of the concentric cylinder configuration is vertical and independent variable z increases downward, such that \(g_z = g\) and both \(g_r\) and \(g_\theta\) vanish, then;

r-component:  \[\frac{\partial p}{\partial r} = \rho \frac{v^2_\theta}{r}, \text{ due to centrifugal forces}\]

z-component:  \[\frac{\partial p}{\partial z} = \rho g, \text{ due to gravitational forces}\]

Intuitively, one invokes symmetry and writes that \(p \neq f(\theta)\), because it is difficult to envision how one could impose a pressure gradient in the \(\theta\)-direction without the existence of normal viscous stress, like \(\tau_{r\theta}\). Certainly, conventional pumps are not equipped to perform this task. Hence, fluid pressure depends on radial position \(r\), due to centrifugal forces, and axial position \(z\), due to gravity. One obtains the fluid pressure distribution \(p(r,z)\) via "partial integration" of the previous two equations after solving for \(v_\theta(r)\).

Consider the most important component of the Equation of Motion in cylindrical coordinates for, \(v_\theta = f(r)\), \(\tau_{r\theta} = g(r)\), and \(P \neq h(\theta)\). All other quantities are zero.

\[\Theta\text{-component of EOM in terms of } \tau:\quad 0 = -\frac{1}{r^2} \frac{d}{dr} \left(r^2 \tau_{r\theta}\right)\]

The general solution is:

\[\tau_{r\theta} = \frac{C_1}{r^2}\]

\(\Theta\text{-component of EOM in terms of velocity gradients for incompressible Newtonian fluids;}\)
\[ 0 = \mu \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (r v_\theta) \right] \]
\[ \frac{1}{r} \frac{d}{dr} (r v_\theta) = C_2 \]
\[ \frac{d}{dr} (r v_\theta) = C_2 r \]
\[ r v_\theta = \left( \frac{1}{2} \right) C_2 r^2 + C_3 \]
\[ v_\theta (r) = \left( \frac{1}{2} \right) C_2 r + \frac{C_3}{r} \]

Check for consistency between the separate solutions for \( \tau_{r\theta} \) and \( v_\theta \) using NLV;

\[ \tau_{r\theta} (r) = -\mu r \frac{d}{dr} \left( \frac{v_\theta}{r} \right) = \frac{C_1}{r^2} \]
\[ = -\mu r \frac{d}{dr} \left( \left( \frac{1}{2} \right) C_2 + \frac{C_3}{r^2} \right) = \frac{2\mu C_3}{r^2} \]

Therefore, \( C_1 = 2 \mu C_3 \).

**Boundary Conditions** (to calculate \( C_2 \) & \( C_3 \));

- @ \( R_{\text{inner}} \)
  - (a) Stationary inner cylinder \( v_\theta = 0 \)
  - (b) Stationary outer cylinder \( v_\theta = \Omega R_{\text{outer}} \)
  - (c) Both cylinders are rotating \( v_\theta = \Omega_{\text{inner}} R_{\text{inner}} \), \( v_\theta = \Omega_{\text{outer}} R_{\text{outer}} \)

**Problem**
Consider tangential annular flow of an incompressible Newtonian fluid between two rotating cylinders. Obtain an expression for the important velocity component when both cylinders are rotating at the same angular velocity \( \Omega \), but in **opposite** directions.
Review Problems for Exam#1

Problem#1
Is the following functional form of the fluid velocity field valid or invalid for two-dimensional (i.e., $v_x$ and $v_y$) laminar flow in the xy-plane for an incompressible non-Newtonian polymer solution at steady state? Provide quantitative support for your answer.

\[ v_x = f(y) \quad v_y = g(x,y) \quad v_z = 0 \]

Answer:
For two-dimensional flow of an incompressible fluid, the Equation of Continuity must be satisfied in the following form;

\[ \nabla \cdot v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \]

for both Newtonian and non-Newtonian fluids. Hence, if $v_x$ is not a function of $x$, as provided in the problem statement, then the 1st term vanishes, and $v_y$ must not be a function of independent variable $y$. The functional form of $v_y$, provided in the problem statement, contradicts this conclusion. The complete velocity field does not satisfy the Equation of Continuity for incompressible fluids, which implies that the microscopic balance on overall fluid mass is not satisfied. Hence, the functional forms of $v_x$ & $v_y$ are invalid when they are analyzed together.

Problem#2
Consider fully-developed one-dimensional flow of an incompressible Newtonian fluid in the polar direction (i.e., only $v_\Theta$) between two stationary concentric spheres of radius $\kappa R$ & $R$, where $\kappa<1$. The Equation of Continuity reveals that the functional form of the steady state fluid velocity profile is;

\[ v_\Theta(r,\Theta) = \frac{f(r)}{\sin \Theta} \]

and the flow configuration is illustrated on page#106 (Fig. 3B.4) in Transport Phenomena.

(a) Construct an integral expression for the volumetric flowrate $Q$. Be sure to include limits of integration in your final answer.

Answer:
The desired result for $Q$ is obtained by integrating $v_\Theta$ over the differential surface element $dS$, which is constructed from a product of two differential lengths in the "no-
flow” directions (i.e., r & φ). One integrates in the radial direction from κR to R, and in the φ-direction from 0 to 2π. Hence;

\[ Q = \iint_S v_\Theta(r, \Theta) dS = \int_0^{2\pi} \int_{\kappa R}^{R} \left[ \frac{f(r)}{\sin \Theta} \right] r \sin \Theta dr d\phi \]

\[ Q = 2\pi \int_{\kappa R}^{R} rf(r) dr \neq g(\Theta) \]

(b) At which polar angle Θ is the volumetric flowrate Q largest? \[ \varepsilon \leq \Theta \leq \pi - \varepsilon \]

Answer:
Since the answer to part (a) does not depend on polar angle Θ, it should be obvious that the volumetric flowrate is constant for incompressible fluids at steady state, with one inlet stream and one outlet stream.

(c) Identify all of the non-zero scalar components of total momentum flux which act across the solid-liquid interface at r = R (i.e., the spherical shell), and then indicate the coordinate direction in which each stress acts. Do not include any quantities in your final answer which are identically zero.

Answer:
The unit normal vector at the solid-liquid interface, where r=R, is oriented in the radial direction. This implies that pressure stress acts in the radial direction, because it is a normal force, and the 1st subscript on the scalar components of convective and viscous momentum flux is r. All of the possible stresses that act across the solid-liquid interface at r=R are tabulated below;

<table>
<thead>
<tr>
<th>Stress</th>
<th>Coordinate Direction</th>
<th>Non-zero?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure</td>
<td>p</td>
<td>r</td>
<td>yes</td>
</tr>
<tr>
<td>Convective momentum flux</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho v_r v_r )</td>
<td>r</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>( \rho v_r v_\Theta )</td>
<td>( \Theta )</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>( \rho v_r v_\phi )</td>
<td>( \phi )</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>Viscous momentum flux</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tau_{rr} )</td>
<td>r</td>
<td>no</td>
<td>( v_r = 0, \nabla \cdot \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>( \tau_{r\Theta} )</td>
<td>( \Theta )</td>
<td>yes</td>
<td>( \partial / \partial r (v_\Theta / r) \neq 0 )</td>
</tr>
<tr>
<td>( \tau_{r\phi} )</td>
<td>( \phi )</td>
<td>no</td>
<td>( v_r = v_\phi = 0 )</td>
</tr>
</tbody>
</table>
Hence, the only non-zero forces due to total momentum flux acting across the solid-liquid interface at \( r=R \) are fluid pressure \( p \), which acts in the radial direction, and viscous shear stress \( \tau_{\text{visc}} \), which acts in the \( \Theta \)-direction.

**Problem #3**
Consider one-dimensional (i.e., only \( v_z \)) steady state laminar flow of an incompressible Newtonian fluid through a straight channel with rectangular cross section. Independent variables \( x \) & \( y \) are defined in the transverse plane, perpendicular to the flow, and \( z \) increases along the axis of the channel in the primary flow direction. Fluid flow is driven by a dynamic pressure gradient \( \frac{dP}{dz} \) in the primary flow direction.

(a) What important information is obtained from the Equation of Continuity for this incompressible one-dimensional flow problem?

*Answer:*
Since \( v_x = 0 \) and \( v_y = 0 \), the Equation of Continuity reduces to;

\[
\nabla \cdot v = \frac{\partial v_z}{\partial z} = 0
\]

which implies that \( v_z \) is not a function of \( z \).

(b) What is the functional form of the non-zero velocity component? \( v_z = f(?) \).

*Answer:*
Steady state analysis implies that \( v_z \) is not a function of time, and the Equation of Continuity reveals that \( v_z \neq f(z) \). Hence, \( v_z \) is a function of \( x \) & \( y \), and neither of these two independent variables can be eliminated because velocity gradients exist in both the \( x \) and \( y \) directions. If the rectangular-shaped cross section has an extremely large aspect ratio, which corresponds to either a large height or width, then gradients of \( v_z \) in the "long" dimension are negligible with respect to similar gradients in the "short" direction.

(c) Write the 2\(^{nd}\)-order differential equation which must be solved to calculate the velocity profile, \( v_z \). Do not include any terms that are trivially zero in the most important component of the Equation of Motion.

*Answer:*
If \( v_x = 0 \), \( v_y = 0 \), and \( v_z = f(x,y) \), then the \( x \)- and \( y \)-components of the Equation of Motion yield the following information;
x-component: \[ 0 = - \frac{\partial P}{\partial x}, \text{ therefore } P \neq f(x) \]

y-component: \[ 0 = - \frac{\partial P}{\partial y}, \text{ therefore } P \neq f(y) \]

These results imply that dynamic pressure is only a function of z at steady state. Now, the important component of the Equation of Motion in terms of velocity gradients for incompressible Newtonian fluids provides the $2^{nd}$-order partial differential equation which allows one to calculate $v_z(x,y)$:

\[
0 = - \frac{dP}{dz} + \mu \left\{ \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} \right\}
\]

If fluid flow is driven only by a pressure gradient in the z-direction, then $g_z = 0$ and one can replace the dynamic pressure gradient in the previous equation by the fluid pressure gradient, $dp/dz$. Separation of variables yields $dp/dz = - \Delta p/L$.

**Numerical analysis directly related to Problem#3**

The axial velocity profile, driven by an imposed pressure drop, for laminar flow of an incompressible Newtonian fluid through a horizontal channel of rectangular cross section represents a complex flow problem. The dimensionless velocity profile is given below in terms of the appropriate dimensionless independent variables.

\[
v_z^\ddagger = \frac{2}{3} \left( 1 - y^\ddagger^2 \right) + 4 \sum_{n=0}^{n \to \infty} (-1)^{n+1} M_n^3 \cosh^{-1}(M_n A_r) \cosh(M_n A_r x^\ddagger) \cos(M_n y^\ddagger)
\]

\[
0 = 4 - \frac{8}{A_r} \sum_{n=0}^{n \to \infty} [M_n^5 \tanh(M_n A_r)]
\]

where;

\[
x^\ddagger = \frac{x}{a} \quad y^\ddagger = \frac{y}{b} \quad A_r = \frac{a}{b} \quad M_n = (2n + 1) \frac{\pi}{2}
\]

The flow cross section has a length of $2a$ in the x-direction and height of $2b$ in the y-direction, and the rectangular coordinate system is oriented such that the range of the independent variables is;

\[-a \leq x \leq a \]
\[-b \leq y \leq b \]
a) Write the differential equation with all of its boundary conditions that must be solved to obtain the complex velocity profile given above.

b) Obtain a **numerical answer** for the volumetric rate of flow in dimensionless form by using Simpson's double integration rule based on second-order Legendre interpolation polynomials when the aspect ratio $A_r = 3$. Numerical integration should be performed over a grid that contains a minimum of 49 points in each coordinate direction.

**Problem#4**

Consider steady state incompressible creeping (i.e., very slow) flow of a Newtonian fluid in a single-screw extruder. The inner solid cylinder of radius $\kappa R$ in a **concentric-cylinder configuration** translates in the z-direction with a constant linear velocity $V$ and rotates at a constant angular velocity $\Omega$. The outer cylinder of radius $R$ is stationary. This motion of the inner cylinder transports fluid through the single-screw extruder. There is no gradient of dynamic pressure in either the $\Theta$- or $z$-directions. **Hint:** Postulate that $v_\Theta = f(r)$ and $v_z = g(r)$.

(a) Use unit vectors in cylindrical coordinates and write a generic expression for the fluid velocity vector when the screw rotates and translates very slowly in the creeping flow regime, such that centrifugal forces are negligible.

**Answer:**

Flow occurs in two dimensions when the screw rotates slow enough such that $v_r = 0$. The fluid velocity vector is:

$$\mathbf{v} = \delta_\Theta v_\Theta(r) + \delta_z v_z(r)$$

(b) Identify all scalar components of the viscous stress tensor that are non-zero.

**Answer:**

If $v_\Theta = f(r)$ and $v_z = g(r)$, with $\nabla \cdot \mathbf{v} = 0$, then the non-zero scalar components of the viscous stress tensor for incompressible Newtonian fluids are $\tau_{r\Theta} = \tau_{\Theta r}$ and $\tau_{rz} = \tau_{zr}$.

(c) Write a detailed expression for each scalar velocity component that is non-zero if the fluid is incompressible and Newtonian, and centrifugal forces are negligible. Please do not derive these expressions.

**Answer:**

The solution for tangential annular flow in cylindrical coordinates, due to slow rotation of the inner cylinder is obtained from the answer to Problem 3B.1(a) on page#105 in
Transport Phenomena, when the angular velocity of the outer cylinder (i.e., \( \Omega_{\text{outer}} \)) is zero. Hence;

\[
v_{\theta}(r) = \Omega R \left( \frac{K^2}{1 - K^2} \right) \left\{ \frac{R}{r} - \frac{r}{R} \right\}
\]

The solution for axial annular flow due to translation of the inner solid cylinder in the absence of any gradient in dynamic pressure is obtained from the answer to 2B.7(a) on page#65 in Transport Phenomena. Hence;

\[
v_z(r) = V \frac{\ln \left( \frac{r}{R} \right)}{\ln \kappa}
\]
Torque vs. angular velocity relation for concentric cylinder viscometers
Consider the solid-liquid interface at the rotating spindle, where \( r = \kappa R \)
Calculate the differential vector force exerted by the solid on the fluid, which is the opposite of the differential vector force exerted by the fluid on the solid, due to total momentum flux (i.e., \( \rho v v, \tau \) and \( p \)) acting across a differential surface element at \( r = \kappa R \)
Unit normal vector from the solid to the fluid is \( n = \delta_r \), in the positive radial direction
Hence, the 1st subscript on \( \rho v v \) and \( \tau \) must be \( r \), and all forces exerted by the solid cylinder on the fluid across the surface at \( r=\kappa R \) have positive signs.
Identify all forces or stresses due to total momentum flux that act across the same differential surface area, \( \kappa R \, d\Theta \, dz \) (i.e., a product of 2 differential lengths in the \( \Theta \)- and \( z \)-directions in cylindrical coordinates, because the simple surface is defined by a constant value of \( r=\kappa R \), and the unit normal vector is in the radial direction)

<table>
<thead>
<tr>
<th>Stress due to;</th>
<th>Coordinate Direction</th>
<th>Non-zero?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure ( p )</td>
<td>( r )</td>
<td>yes</td>
<td>( p &gt; 0 )</td>
</tr>
</tbody>
</table>

Stress due to; Coordinate Direction Non-zero? Reason

Convective momentum flux
- \( \rho v_r v_r \) \( r \) no \( v_r = 0 \)
- \( \rho v_r v_\Theta \) \( \Theta \) no \( v_r = 0 \)
- \( \rho v_r v_z \) \( z \) no \( v_r = 0 \)

Viscous momentum flux
- \( \tau_{rr} \) \( r \) no \( v_r = 0, \nabla \cdot v = 0 \)
- \( \tau_{r\Theta} \) \( \Theta \) yes \( \partial / \partial r (v_\Theta / r) \neq 0 \)
- \( \tau_{rz} \) \( z \) no \( v_r = v_z = 0 \)

As a vector, the differential force exerted by the inner rotating solid cylinder on the fluid in contact with this spindle at \( r = \kappa R \), is given by a product of the non-zero stresses due to total momentum flux acting across the surface at \( r = \kappa R \), including the appropriate unit vectors, and the differential surface element, \( \kappa R \, d\Theta \, dz \);

\[
dF_{\text{Solid on Fluid}} = \{ \delta_r \, p(r=\kappa R) + \delta_\Theta \, \tau_{r\Theta}(r=\kappa R) \} \, \kappa R \, d\Theta \, dz
\]
This differential force exerted by the solid cylinder on the fluid across the differential surface element at \( r=κR \) generates the following differential torque;

\[
\text{Differential Torque } dT = \left[ \text{Lever Arm} \right] \times dF_{\text{Solid on Fluid}}
\]

where the **Lever Arm** is a position vector from the axis of rotation to the point on the differential surface element where stress is transmitted from solid to fluid. This position vector must be perpendicular to the rotation axis. Hence;

\[
\text{Lever Arm} = \delta_r (κR)
\]

Torque calculations are more complex when the lever arm varies over the macroscopic surface across which stress is transmitted from solid to fluid. For the concentric cylinder viscometer with constant lever arm, one calculates the differential torque as follows;

\[
\text{Differential Torque } dT = \left[ δ_r (κR) \right] \times \{ δ_r p(r=κR) + δ_θ τ_{rθ}(r=κR) \} κR dθ dz
\]

\[
dT = \left[ δ_r x δ_θ \right] (κR)^2 τ_{rθ}(r=κR) dθ dz
\]

\[
dT = δ_z (κR)^2 τ_{rθ}(r=κR) dθ dz
\]

Now, recall the basic information for this problem, which was generated primarily from solution of the θ-component of EOM in terms of velocity gradients for incompressible Newtonian fluids;

\[
v_θ(r) = \frac{1}{2} C_2 r + \frac{C_3}{r}
\]

\[
(NLV)τ_{rθ} = -μr \frac{d}{dr} \left( \frac{v_θ}{r} \right) = \frac{C_1}{r^2}
\]

\[
= -μr \left\{ -\frac{2C_3}{r^3} \right\} = \frac{2μC_3}{r^2}
\]

with \( C_1 = 2μC_3 \). Hence, the product of \( r^2 \) and \( τ_{rθ} \), evaluated at \( r=κR \), can be replaced by \( C_1 \), or \( 2μC_3 \). The macroscopic torque vector is obtained by integrating the expression for differential torque;

\[
\text{Torque} = \int dT = 2μC_3 \int_{θ=0}^{2π} \int_{z=0}^{L_z} δ_z dθ dz
\]
Since the orientation of $\delta_z$ never changes on the lateral surface of the solid cylinder [i.e., $\delta_z = f(\Theta, z)$, $\partial\delta_z / \partial \Theta = 0$, and $\partial\delta_z / \partial z = 0$], it is constant and can be removed from the previous integral expression;

$$Torque = \delta_z 2\mu C_3 \int_{\Theta=0}^{2\pi} d\Theta \int_{z=0}^{L} dz = \delta_z 4\pi \mu L^* C_3$$

where $L^*$ represents a manufacturer's suggested corrected length of the rotating spindle, which empirically accounts for end effects. Calculate $C_3$ via comparison of the general solution for $v_\Theta(r)$ with the answer to Problem 3B.1 on page#105 in *Transport Phenomena*;

$$v_\Theta(r) = \frac{1}{1-\kappa^2} \left\{ \left( \Omega_{outer} - \kappa^2 \Omega_{inner} \right) r - \left( \Omega_{outer} - \Omega_{inner} \right) \frac{\kappa^2 R^2}{r} \right\}$$

When the outer cylinder is stationary and $\Omega_{outer} = 0$, and $\Omega_{inner} = \Omega$;

$$v_\Theta(r) = \Omega R \left\{ \frac{\kappa^2}{1-\kappa^2} \left\{ \frac{R}{r} - \frac{r}{R} \right\} = \frac{1}{2} C_2 r + \frac{C_3}{r} \right\}$$

Hence;

$$C_3 = \Omega R^2 \left\{ \frac{\kappa^2}{1-\kappa^2} \right\}$$

$$Torque = \delta_z 4\pi \mu L^* C_3 = \left( \delta_z \Omega \right) 4\pi \mu R^2 L^* \left\{ \frac{\kappa^2}{1-\kappa^2} \right\}$$

where the angular velocity vector for rotation of the inner solid cylinder is $\left( \delta_z \Omega \right)$. The torque required to overcome viscous shear at the solid-liquid interface and spin the inner cylinder at angular velocity $\Omega$ is co-linear with the angular velocity vector of the solid spindle. Think about the necessary modifications which must be implemented if the fluid does not obey Newton's law of viscosity.

**Problem**

Consider laminar flow of an incompressible Newtonian fluid on the shell side of a double-pipe heat exchanger and calculate the z-component of the vector force exerted by the fluid on the outer surface of the inner pipe. Prove that your answer is correct by evaluating this interfacial force at two strategically chosen values of the radius ratio $\kappa$ and comparing your answers with known results for these special cases.
Now, consider three viscometers described below, where very slow rotation of a solid surface produces 1-dimensional fluid flow in the creeping flow regime. The entire left side of the Equation of Motion can be neglected at very low Reynolds numbers, and the non-zero velocity component depends on two spatial coordinates during steady state operation. If one calculates the fluid velocity at the interface with the rotating solid by invoking solid-body rotation, then it is not necessary to any solve partial differential equations to obtain an expression for the important component of the velocity vector.

**Parallel-disk viscometer**

Newtonian fluids; Torque vs. $\Omega$, Problem 3B.5 on page#106 in *Transport Phenomena*

Begin with a picture of the flow configuration and identify $v_\theta$ as the only nonzero component of the fluid velocity vector. The Equation of Continuity in cylindrical coordinates reveals that:

$$\nabla \cdot v = \frac{1}{r} \frac{\partial v_\theta}{\partial \Theta} = 0$$

Hence, one concludes that $v_\theta \neq f(\Theta)$ based on the overall mass balance. One also arrives at this conclusion, qualitatively, by invoking symmetry in cylindrical coordinates. For a parallel disk viscometer in which the fluid of interest is placed in the narrow gap between a stationary plate and a rotating plate, the tangential velocity $v_\theta$ on the rotating plate is $\Omega r$, via solid body rotation. One obtains this result very easily by taking the vector cross product of the angular velocity vector (i.e., $\delta_r \Omega$) with the variable lever arm (i.e., $\delta_r r$). This reveals the radial dependence of $v_\theta$ at any axial position $z$ between the rotating & stationary plates. If one moves into the fluid in the $z$-direction from the moving plate at constant $r$, and a separation of variables solution to the $\Theta$-component of the Equation of Motion is valid, then the $r$-dependence shouldn’t change. Hence, at steady state;

$$v_\theta = f(r) g(z) = r g(z)$$

$g(z) = \Omega$ on the rotating plate at $z=B$

$g(z) = 0$ on the stationary plate at $z=0$

Notice how the postulated "separation-of-variables" form of the fluid velocity profile, based on solid body rotation at the interface between the fluid and the rotating plate, agrees with the Equation of Continuity. For Newtonian fluids, this one-dimensional flow field in the parallel-disk viscometer generates the following important scalar components of the viscous stress tensor;
\[ \tau_{r\theta} = \tau_{\theta r} = -\mu r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) = 0 \]
\[ \tau_{\theta z} = \tau_{z\theta} = -\mu \frac{\partial v_\theta}{\partial z} = -\mu r \frac{dg}{dz} \]

Based on the postulated separation-of-variables form for \( v_\theta \), which accounts for solid body rotation at the interface between the fluid and the rotating plate, the modified form of Newton’s law of viscosity for flow problems with curved streamlines eliminates any contribution of solid body rotation to viscous stress. Hence, the \( r-\theta \) component of the viscous stress tensor vanishes because the radial dependence of \( v_\theta \) was adopted from solid body rotation of the upper plate. As a general rule, if there is only one important scalar component of the fluid velocity vector which depends on two spatial variables, such as \( v_i(x_k, x_m) \), then the following scalar components of the viscous stress tensor will be nonzero; \( \tau_{ik}, \tau_{ki}, \tau_{im}, \) and \( \tau_{mi} \). However, if the dependence of \( v_i \) on \( x_k \) is based on rigid body rotational characteristics within the fluid, then \( \tau_{ik} = \tau_{ki} = 0 \).

**Force balances for the parallel-disk viscometer.** Since the \( \theta \)-component of the Equation of Motion is most important, a consideration of the \( r \)- and \( z \)-components of the force balance will provide information about the fluid pressure distribution.

**r-component:**  
\[ -\rho \frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r} = -\frac{\partial P}{\partial r} \quad (\text{i.e., } g_r = 0) \]

**z-component:**  
\[ 0 = -\frac{\partial P}{\partial z} = -\frac{\partial P}{\partial z} + \rho g_z \]

Once again, fluid pressure depends on radial position \( r \), due to centrifugal forces, and axial position \( z \), due to gravity. Since there are no restraining walls in the parallel-disk viscometer, it is necessary to operate the device in the creeping flow regime such that (i) centrifugal forces are negligible, and (ii) surface tension provides a restraining barrier to prevent fluid from moving in the radial direction. In the creeping flow regime, one neglects the entire left side of the Equation of Motion which scales as the square of the fluid velocity. Now, fluid pressure depends only on axial position \( z \), due to gravity. The fluid pressure distribution in the parallel-disk viscometer is the same in the creeping flow regime as it is in the hydrostatic situation when there is no flow (i.e., \( p = p_{\text{ambient}} + \rho gh \)). It is physically impossible to induce a pressure gradient in the \( \theta \)-direction if there are no normal viscous stresses like \( \tau_{r\theta} \), and \( g_\theta = 0 \) if the viscometer is placed on a horizontal
surface such that gravity acts only in the z-direction. The \( \Theta \)-component of the Equation of Motion for incompressible Newtonian fluids is:

\[
\text{\( \Theta \)-component:} \quad 0 = \mu \left\{ \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} (rv_\Theta) \right] + \frac{\partial^2 v_\Theta}{\partial z^2} \right\}
\]

The 1\textsuperscript{st}-term on the right side of the previous equation vanishes if \( v_\Theta = r g(z) \). This is equivalent to the fact that \( \tau_{r\Theta} \) is not important, as illustrated above, because the \( r \)-dependence of \( v_\Theta \) emulates solid body rotation which generates no viscous stress. Now, one solves for \( g(z) \) from the \( \Theta \)-component of the force balance as follows;

\[
\text{\( \Theta \)-component:} \quad 0 = \mu \frac{\partial^2 v_\Theta}{\partial z^2} = \mu r \frac{d^2 g}{dz^2}
\]

Hence, the function \( g(z) \) must be linear in \( z \), and the general solution for \( v_\Theta \) is;

\[
v_\Theta(r, \Theta) = r \left( C_1 z + C_2 \right)
\]

Integration constant \( C_2 \) must be zero because the no-slip boundary condition suggests that there is no fluid motion at the stationary solid plate \( @ z=0 \) for all \( r>0 \). The boundary condition at the rotating plate (i.e., \( z=B \)) is consistent with the postulated form of the velocity profile, and yields an expression for \( C_1 \). The final result is;

\[
v_\Theta(r, \Theta) = \Omega r \left( \frac{z}{B} \right)
\]

**Differential force due to total momentum flux which acts across the interface between the fluid and the rotating solid plate.** Identify all forces or stresses due to total momentum flux that act across the differential surface with unit normal vector in the \( z \)-direction. In particular, the unit normal vector from the solid to the fluid is in the negative \( z \)-direction. Consequently, all forces transmitted across this interface from the solid to the fluid must include a negative sign. The differential surface area under investigation is \( dS = r \, dr \, d\Theta \) (i.e., a product of 2 differential lengths in the \( r \)- & \( \Theta \)-directions in cylindrical coordinates, because the simple surface is defined by a constant value of \( z=B \), and the unit normal vector is in the \( z \)-direction). The first subscript on any important scalar components of convective and viscous momentum flux must be \( z \).
Stress due to;

<table>
<thead>
<tr>
<th></th>
<th>Coord. Direction</th>
<th>Non-zero?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure</td>
<td>-p</td>
<td>z</td>
<td>yes</td>
</tr>
</tbody>
</table>

**Convective momentum flux**

- $-\rho v_z v_r$  
  r  
  no  
  $v_z = 0$
- $-\rho v_z v_\Theta$  
  $\Theta$  
  no  
  $v_z = 0$
- $-\rho v_z v_z$  
  z  
  no  
  $v_z = 0$

**Viscous momentum flux**

- $-r_{zr}$  
  r  
  no  
  $v_r = v_z = 0$
- $-r_{z\Theta}$  
  $\Theta$  
  yes  
  $\partial v_\Theta / \partial z \neq 0$
- $-r_{zz}$  
  z  
  no  
  $v_z = 0$, $\nabla \cdot \mathbf{v} = 0$

As a vector, the differential force exerted by the upper rotating plate on the fluid in contact with this surface at $z=B$, is given by a product of the non-zero stresses due to total momentum flux acting across the surface at $z=B$, including the appropriate unit vectors, and the differential surface element, $dS = r \, dr \, d\Theta$;

$$dF_{\text{Solid on Fluid}} = - \{ \delta_\Theta \tau_{z\Theta}(z=B) + \delta_z p(z=B) \} \, r \, dr \, d\Theta$$

This differential force generates the following differential torque;

$$\text{Differential Torque } dT = [ \text{Lever Arm} ] \times dF_{\text{Solid on Fluid}}$$

where the **Lever Arm** is a position vector from the axis of rotation to the point on the differential surface element where stress is transmitted from solid to fluid. Hence;

$$\text{Lever Arm} = \delta_r \, r$$

Notice that the **Lever Arm** is not constant. One calculates the differential torque as follows;

$$\text{Differential Torque } dT = - [ \delta_r \, r ] \times \{ \delta_\Theta \tau_{z\Theta}(z=B) + \delta_z p(z=B) \} \, r \, dr \, d\Theta$$

$$dT = - \{ [ \delta_r \times \delta_\Theta ] r^2 \tau_{z\Theta}(z=B) + [ \delta_r \times \delta_z ] r^2 p(z=B) \} \, dr \, d\Theta$$

It is acceptable to neglect the pressure contribution to the relation between torque and angular velocity, even though the pressure force does not extend through the axis of rotation. Detailed vector calculus analysis of the effect of pressure reveals that its
contribution is zero. As a general rule, forces due to fluid pressure will never contribute to the relation between torque and angular velocity. Hence;

\[
\mathbf{dT} = -\delta_z r^2 \tau_{z\Theta}(z=B) \, dr \, d\Theta
\]

Now, recall the basic information for this problem, which was generated primarily from solution of the \( \Theta \)-component of the Equation of Motion in terms of velocity gradients for incompressible Newtonian fluids;

\[
v_\Theta(r) = \Omega r \left( \frac{z}{B} \right)
\]

\[(NLV)\tau_{z\Theta} = -\mu \frac{\partial v_\Theta}{\partial z} = -\frac{\mu \Omega r}{B}\]

The macroscopic torque \( \mathbf{T} \) is obtained via integration of \( \mathbf{dT} \) over the entire interface between the rotating solid and the fluid;

\[
\text{Torque} = \int dT = \int_{\Theta=0}^{2\pi} d\Theta \int_{r=0}^{R} \left\{ -\delta_z r^2 \tau_{z\Theta}(z=B) \right\} dr = \frac{2\pi \mu \Omega}{B} \int_{r=0}^{R} \delta_z r^3 \, dr
\]

Since the orientation of \( \delta_z \) never changes on the surface of the rotating solid plate at \( z=B \), [i.e., \( \delta_z \neq f(r,\Theta) \), \( \partial \delta_z / \partial r=0 \)], it is constant and can be removed from the previous integral expression;

\[
\text{Torque} = (\delta_z \Omega) \frac{2\pi \mu}{B} \int_{r=0}^{R} r^3 \, dr = (\delta_z \Omega) \frac{\pi \mu R^4}{2B}
\]

Once again, the macroscopic torque vector is co-linear with the angular velocity vector (i.e., \( \delta_z \Omega \)), which can be considered as a general rule.

**Rotating sphere viscometer**

Newtonian fluids; Torque vs. \( \Omega \), see pages#95-96 in *Transport Phenomena*

*Statement of the problem, and evaluation of the fluid velocity at the solid-liquid interface via solid-body rotation.* A solid sphere of radius \( R \) is suspended from a wire and rotates very slowly at constant angular velocity \( \Omega \) about the long axis of the wire in an incompressible Newtonian fluid. The fluid is quiescent far from the sphere. For a rotating sphere viscometer, solid-body rotation at the fluid-solid interface suggests that the tangential fluid velocity \( v_\phi \) on the surface of the sphere is \( \Omega R \sin \Theta \). This result is obtained
by analyzing rigid body rotation of a solid sphere about the z-axis of a Cartesian coordinate system and calculating the velocity vector at the fluid-solid interface by invoking the "no-slip" condition;

\[ \mathbf{v} = [\Omega \times \{ \text{Lever Arm} \}]_{r=R} \]

The angular velocity vector is oriented in the z-direction (i.e., \( \Omega = \Omega \delta_z \)), and the variable lever arm from the axis of rotation (i.e., along the wire) to any point on the surface of the solid sphere is;

\[ \text{Lever Arm} = R \sin \Theta \{ \delta_r \sin \Theta + \delta_\phi \cos \Theta \} \]

where \( \Theta \) is the polar angle measured from the z-axis. Upon taking the cross product, one obtains;

\[ \mathbf{v} = \Omega R \sin \Theta \{ [\delta_z \times \delta_r] \sin \Theta + [\delta_z \times \delta_\phi] \cos \Theta \} \]

Trigonometric relations between unit vectors in rectangular & spherical coordinates yield the following expression for \( \delta_z \) (see p.#828, *Transport Phenomena, 2nd edition*, by RB Bird, WE Stewart & EN Lightfoot). Hence;

\[ \delta_z = \delta_r \cos \Theta - \delta_\phi \sin \Theta \quad \delta_z \times \delta_r = \delta_\phi \sin \Theta \quad \delta_z \times \delta_\phi = \delta_\phi \cos \Theta \]

If the sphere rotates very slowly and centrifugal forces do not induce flow in the radial direction or the \( \Theta \)-direction, then one calculates the fluid velocity at the fluid-solid interface via solid body formalism summarized above. Vector algebra reveals that this problem is described by one-dimensional flow in the \( \phi \)-direction, because;

\[ \mathbf{v} = \Omega R \sin \Theta \{ \delta_\phi \sin^2 \Theta + \delta_\phi \cos^2 \Theta \} = \delta_\phi \Omega R \sin \Theta = \delta_\phi \mathbf{v}_\phi \]

This calculation from solid body rotation reveals the angular dependence of \( \mathbf{v}_\phi \) at any radial position, because if one moves into the fluid at larger \( r \) and constant \( \Theta \), and a separation of variables solution to the \( \phi \)-component of the Equation of Motion is valid, then the \( \sin \Theta \) dependence shouldn't change. Hence;

\[ v_\phi(r,\Theta) = f(r) \cdot g(\Theta) = f(r) \sin \Theta \]

\[ f(r) = \Omega R \quad r = R \]

\[ f(r) \to 0 \text{ as } r \to \infty \]

This functional form of the fluid velocity profile satisfies the Equation of Continuity, because when rotation is slow enough and flow occurs only in one direction, the balance on overall fluid mass for incompressible fluids is;
\[ \nabla \cdot v = \frac{1}{r \sin \Theta} \frac{\partial v_\phi}{\partial \phi} = 0 \]

which stipulates that \( v_\phi \) cannot be a function of the azimuthal angle \( \phi \).

**Creeping flow analysis of the Equation of Motion.** In the creeping flow regime where centrifugal forces are negligible, one sets the entire left side of the Equation of Motion to zero and considers only a balance between viscous, pressure and gravity forces. Since the \( \phi \)-component of the vector force balance is most important, the \( r \)- and \( \Theta \)-components yield the following information about dynamic pressure;

\begin{align*}
\text{r-component:} & \quad 0 = - \frac{\partial P}{\partial r} \quad P \neq f(r) \\
\text{\Theta-component:} & \quad 0 = - \frac{1}{r} \frac{\partial P}{\partial \Theta} \quad P \neq f(\Theta)
\end{align*}

Steady state analysis implies that dynamic pressure is not time-dependent, and intuition suggests that it is almost impossible to induce a pressure gradient in the primary flow direction, unless normal viscous stresses like \( \tau_{\phi\phi} \) are nonzero. Since the \( \phi \)-component of gravity vanishes if the sphere is suspended vertically from a wire, one analyzes the \( \phi \)-component of the Equation of Motion in the creeping flow regime when dynamic pressure is constant, similar to a hydrostatic situation. Fluid flow is induced by (i) rotation of the solid and (ii) viscous shear which is transmitted across the solid-liquid interface. As expected, the \( \phi \)-component of the force balance yields useful information to calculate \( v_\phi \). The only terms which survive in the \( \phi \)-component of the Equation of Motion are;

\[ 0 = \frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{\partial}{\partial \Theta} \left[ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} (v_\phi \sin \Theta) \right] \right\} \]

Now, one calculates \( f(r) \) from the previous equation by letting \( v_\phi = f(r) \sin \Theta \). For example;
\[
\frac{\partial}{\partial \Theta} (v_{\phi} \sin \Theta) = 2f(r) \sin \Theta \cos \Theta \\
\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} (v_{\phi} \sin \Theta) = 2f(r) \cos \Theta \\
\frac{\partial}{\partial \Theta} \left[ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} (v_{\phi} \sin \Theta) \right] = -2f(r) \sin \Theta
\]

The \(\phi\)-component of the Equation of Motion reduces to Euler’s differential equation;

\[
\sin \Theta \left\{ \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) - 2f(r) \right\} = 0
\]

If one adopts a trial solution of the form \(f(r) \approx r^n\), or \(r^n \ln r\) if two values of \(n\) are the same for this 2nd-order ordinary differential equation, then both terms in brackets \{\} are proportional to \(r^n\). Substitution of \(f(r) \approx r^n\) into the previous equation yields the following quadratic polynomial with two roots for the exponent \(n\);

\[
\sin \Theta \left[ n (n + 1) - 2 \right] r^{n-2} = 0
\]

The solution is; \(n = -2, 1\). The \(\phi\)-component of the Equation of Motion is a linear differential equation in the creeping flow regime, because one neglects terms that scale as the square of fluid velocity. Hence, the general solution for the \(\phi\)-component of the fluid velocity vector is obtained by adding both solutions for \(f(r)\) and including the appropriate integration constants;

\[
v_{\phi}(r, \Theta) = \left\{ Ar + \frac{B}{r^2} \right\} \sin \Theta
\]

The solution for \(n = 1\) must be discarded because the fluid is stagnant at large \(r\). Hence, \(A = 0\). The boundary condition at the fluid-solid interface yields \(B = \Omega R^3\). The final creeping flow solution is;

\[
v_{\phi}(r, \Theta) = \frac{\Omega R^3 \sin \Theta}{r^2}
\]

Since \(v_{\phi}\) is a function of \(r\) and \(\Theta\), one predicts that there are four nonzero scalar components of the viscous stress tensor. \(\tau_{rr} = \tau_{r\theta}\) is important because \(v_{\phi}\) depends on \(r\),
and this functional dependence does not conform to solid body rotation. In fact, the \( r\phi \)-component of \( \tau \) is solely responsible for the torque/angular-velocity relation. Since \( v_\phi \) depends on \( \Theta \), at first glance, one expects that \( \tau_{\phi\theta} = \tau_{\theta\phi} \) should yield a nonzero contribution to the state of viscous stress. However, the \( \Theta \)-dependence of \( v_\phi \) was constructed from solid body rotation and detailed calculations of the \( \Theta \phi \)- and \( \phi \Theta \)-components of \( \tau \) indicate that no viscous forces are generated from the \( \sin \Theta \) functional dependence of \( v_\phi \). In the parallel disk viscometer, the \( \theta r \)- and \( \Theta r \)-components of \( \tau \) vanish because no viscous forces result from the linear dependence of \( v_\theta \) on \( r \) due to solid body rotation. In summary, when the important nonzero velocity component for one-dimensional fluid flow depends on two independent spatial variables, and the functional dependence on one of these spatial variables is postulated to match solid body rotation at the fluid-solid interface, the viscous shear stress based on this solid-body-type functional dependence vanishes (i.e., \( r \) for the parallel disk viscometer, \( \sin \Theta \) for the rotating sphere viscometer). Hence, the state of viscous stress in the fluid is simplified because the fluid adopts some, but not all, aspects of solid body rotation.

**Differential vector force due to total momentum flux transmitted across the fluid-solid interface at \( r=R \).** Now, one focuses on the fluid-solid interface and calculates the differential vector force \( d\mathbf{F}_{\text{Solid On Fluid}} \) exerted by the solid on the fluid. Begin by identifying the unit normal vector from the solid to the fluid across the surface at \( r=R; \mathbf{n} = + \delta_r \). The 1st subscript on each scalar component of convective and viscous momentum flux is \( r \), each non-zero component of total momentum flux is preceded by a positive sign, & the differential surface element is \( dS = R^2 \sin \Theta \ d\Theta \ d\phi \). Construct the following table;

<table>
<thead>
<tr>
<th>Stress</th>
<th>Coord. direction</th>
<th>Non-zero?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure</td>
<td>( p )</td>
<td>( r )</td>
<td>yes</td>
</tr>
<tr>
<td>Convective momentum flux</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho v_r v_r )</td>
<td>( r )</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>( \rho v_r v_\theta )</td>
<td>( \Theta )</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>( \rho v_r v_\phi )</td>
<td>( \phi )</td>
<td>no</td>
<td>( v_r = 0 )</td>
</tr>
<tr>
<td>Viscous momentum flux</td>
<td>Coord. direction</td>
<td>Non-zero?</td>
<td>Reason</td>
</tr>
<tr>
<td>( \tau_{rr} )</td>
<td>( r )</td>
<td>no</td>
<td>( v_r = 0, \nabla \cdot \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>( \tau_{r\Theta} )</td>
<td>( \Theta )</td>
<td>no</td>
<td>( v_r = v_\theta = 0 )</td>
</tr>
<tr>
<td>( \tau_{r\phi} )</td>
<td>( \phi )</td>
<td>yes</td>
<td>( \partial / \partial r (v_\phi / r) \neq 0 )</td>
</tr>
</tbody>
</table>
Hence, there is a normal pressure force which acts in the r-direction, and a viscous shear force which acts in the $\phi$-direction. The total differential vector force which acts across the solid-liquid interface at $r=R$ is:

$$dF_{\text{Solid On Fluid}} = \{ \delta_r \ p + \delta_\phi \tau_{r\phi} \}_{r=R} R^2 \sin \Theta \ d\Theta \ d\phi$$

The differential vector torque $dT$ that arises from $dF_{\text{Solid On Fluid}}$ acting across the solid-liquid interface @ $r=R$ is calculated by performing the following cross-product operation:

$$dT = \{ \text{Lever Arm} \} \times dF_{\text{Solid On Fluid}}$$

where the lever arm was calculated on page#32. Hence;

$$dT = R \sin \Theta \ \{ \delta_r \ \sin \Theta + \delta_\theta \ \cos \Theta \} \times \{ \delta_r \ p + \delta_\phi \tau_{r\phi} \}_{r=R} R^2 \sin \Theta \ d\Theta \ d\phi$$

$$= \{ \delta_r \ \tau_{r\phi} \ \cos \Theta - \delta_\theta \ \tau_{r\phi} \ \sin \Theta - \delta_\phi \ p \ \cos \Theta \}_{r=R} R^3 \sin^2 \Theta \ d\Theta \ d\phi$$

As a general rule, neglect the pressure contribution to $dT$ because detailed vector calculus analysis will reveal that it vanishes for any type of viscometer. The quantity of interest reduces to;

$$dT = \{ \delta_r \ \cos \Theta - \delta_\theta \ \sin \Theta \} R^3 \tau_{r\phi}(r=R) \sin^2 \Theta \ d\Theta \ d\phi$$

Now, (i) evaluate the important scalar component of the viscous stress tensor at the solid-liquid interface, (ii) rewrite the three spherical coordinate unit vectors in terms of constant unit vectors in rectangular coordinates (i.e., see page#828 in Transport Phenomena, 2nd edition, by RB Bird, WE Stewart & EN Lightfoot), and (iii) integrate the previous expression to calculate the macroscopic torque/angular-velocity relation from which the Newtonian viscosity $\mu$ can be determined via measurements of torque vs. $\Omega$.

$$\tau_{r\phi}(r=R,\Theta) = -\mu \left\{ r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right\}_{r=R} = 3\mu \Omega \sin \Theta$$

$$\delta_r = \delta_x \ \sin \Theta \ \cos \phi + \delta_y \ \sin \Theta \ \sin \phi + \delta_z \ \cos \Theta$$

$$\delta_\theta = \delta_x \ \cos \Theta \ \cos \phi + \delta_y \ \cos \Theta \ \sin \phi - \delta_z \ \sin \Theta$$

$$\delta_\phi = -\delta_x \ \sin \phi + \delta_y \ \cos \phi$$
If the solution for $n=1$ in the expression for $v_\phi$ were not discarded based on qualitative physical arguments, and integration constant $A \neq 0$, then the form of Newton’s law of viscosity for $\tau_{r_0}$ eliminates any contribution from the term $Ar$ in the torque/angular-velocity relation because $Ar$ emulates solid body rotational characteristics within the fluid. Macroscopic torque $T$ is obtained via integration of $dT$ over the surface of the solid sphere, where, for example, $\Theta$ ranges from 0 to $\pi$, and $\Phi$ ranges from 0 to $2\pi$. There are no contributions to $T$ in the $x$- & $y$-coordinate directions because, in all cases, one integrates either $\sin\Phi$ or $\cos\Phi$ over the complete period of these trigonometric functions. Hence, this provides quantitative justification for the claim that fluid pressure does not contribute to the relation between torque and angular velocity in this particular viscometer. It is only necessary to consider terms in the $z$-direction due to $\delta_r$ and $\delta_\Theta$. These are:

$$ Torque = \int dT = 3\mu \Omega R^3 \int_{\phi=0}^{2\pi} \int_{\Theta=0}^{\pi} (\delta_\phi \cos^2 \Theta + \delta_\theta \sin^2 \Theta) \sin^3 \Theta d\Theta d\Phi $$

$$ = \delta_\phi 6\pi \mu \Omega R^3 \int_{\Theta=0}^{\pi} \sin^3 \Theta d\Theta = 8\pi \mu R^3 (\delta_\phi \Omega) $$

Once again, the macroscopic torque vector is co-linear with the angular velocity vector of the solid sphere, and both of these vectors act in the $z$-direction. These results are universal for all viscometers if the $z$-direction is vertical. The origin of $R^3$ in the final expression for torque vs. angular velocity is (i) one factor of $R$ from the lever arm, and (ii) one factor of $R^2$ from both the differential and macroscopic interfacial surface area between the solid sphere and the fluid. The origin of $\sin^3\Theta$ in the integral expression for torque is (i) one factor of $\sin\Theta$ from the lever arm, (ii) one factor of $\sin\Theta$ from the differential surface area, and (iii) one factor of $\sin\Theta$ when the important scalar shear component of the viscous stress tensor is evaluated at the fluid-solid interface.

**Problem#1**
Determine the exponents $a$, $b$ & $c$ in the following scaling relation for the macroscopic torque in the rotating sphere viscometer;

$$ \text{magnitude of the torque} \approx \mu^a \Omega^b R^c $$

**Answer**
$$ a=1, \ b=1 \ & \ c=3. $$

**Problem#2**
How do your answers for the scaling law exponents $a$ & $b$ relate to the fact that the constitutive relation between viscous stress and velocity gradients is linear via Newton’s law of viscosity?
Answer
The values for a & b in the previous scaling law are a direct consequence of the fact that torque is linearly proportional to viscous shear stress, and Newton's law of viscosity is a linear constitutive relation between viscous stress & viscosity (i.e., a=1) and viscous stress & velocity gradients (i.e., b=1), where the velocity gradient can be approximated by the angular velocity \( \Omega \) of the solid sphere.

Problem#3
Estimate the scaling law exponent b if the fluid were non-Newtonian with power-law index \( n \) in the classic Ostwald-de Waele model as described on page#241 in Transport Phenomena, 2nd edition, by RB Bird, WE Stewart & EN Lightfoot.

Answer
For power-law fluids, viscous stress is proportional to the \( n \)th-power of the shear rate, which represents the magnitude of the rate-of-strain tensor. Since torque scales linearly with viscous shear stress, and shear rate scales linearly with angular velocity, it follows directly that torque scales as the \( n \)th-power of \( \Omega \). Hence, \( b = n \).

Cone-and-plate viscometer
All fluids; Torque vs. \( \Omega \)
See Problems 2B.11 on page#67-68, and 8C.1 on page#261 in Transport Phenomena.

An incompressible Newtonian fluid is placed within the gap between a stationary plate and a solid cone whose apex touches the plate. The radius of the cone and the plate is \( R \), and the conical surface is described by a constant value of polar angle \( \Theta = \Theta_1 \), where \( \Theta_1 \) is approximately 89.5°. Hence, the test fluid resides in a very narrow gap between the cone and the plate. The cone rotates very slowly at constant angular velocity \( \Omega \) about the z-axis, and surface tension is sufficient to retain the fluid within the gap because centrifugal forces are negligible in the creeping flow regime at very slow rotational speeds of the cone. One-dimensional fluid flow occurs at low \( \Omega \), whereas either two- or three-dimensional flow is appropriate at high \( \Omega \). The \( r \)- and \( \phi \)-components of the fluid velocity vector are important in the equatorial plane at \( \Theta = \pi/2 \) when the rotational speed of the cone is high enough such that the Reynolds number is sufficiently larger than unity. For creeping flow, analysis of solid-body rotation on the conical surface allows one to obtain the fluid velocity at the solid-liquid interface (i.e., \( \Theta = \Theta_1 \)) by invoking the "no-slip" condition. Hence;

\[
\mathbf{v} = \left[ \Omega \times \{ \text{Lever Arm} \} \right] @ \Theta = \Theta_1
\]
The angular velocity vector is oriented in the z-direction (i.e., $\mathbf{\Omega} = \mathbf{\Omega}_z$), and the lever arm from the axis of rotation (i.e., z-axis) to any point on the conical surface is:

\[
\text{Lever Arm} = r \sin \Theta_1 \left\{ \delta_r \sin \Theta_1 + \delta_\theta \cos \Theta_1 \right\}
\]

where $\Theta_1$ is the polar angle measured from the z-axis to any point on the surface of the cone. Upon taking the cross product, one obtains:

\[
\mathbf{v} = \mathbf{\Omega} r \sin \Theta_1 \left\{ \left[ \delta_z \times \delta_r \right] \sin \Theta_1 + \left[ \delta_z \times \delta_\theta \right] \cos \Theta_1 \right\}
\]

Trigonometric relations between unit vectors in rectangular and spherical coordinates yield the following expression for $\delta_r$, $\delta_\theta$, and $\delta_\phi$ (see p.#828, Transport Phenomena, 2nd edition, by RB Bird, WE Stewart & EN Lightfoot). Hence:

\[
\begin{align*}
\delta_r &= \delta_x \sin \Theta \cos \phi + \delta_y \sin \Theta \sin \phi + \delta_z \cos \Theta \\
\delta_\theta &= \delta_x \cos \Theta \cos \phi + \delta_y \cos \Theta \sin \phi - \delta_z \sin \Theta \\
\delta_\phi &= \delta_x (-\sin \phi) + \delta_y \cos \phi
\end{align*}
\]

The cross products of interest are:

\[
\begin{align*}
\delta_z \times \delta_x &= \delta_y \\
\delta_z \times \delta_y &= -\delta_x \\
\delta_z \times \delta_z &= 0
\end{align*}
\]

Vector algebra reveals that this problem is described by one-dimensional flow in the $\phi$-direction, because the velocity vector at the fluid/rotating-solid interface is:

\[
\mathbf{v} = \mathbf{\Omega} r \sin \Theta_1 \left\{ \delta_y \sin^2 \Theta_1 \cos \phi - \delta_x \sin \Theta_1 \sin \phi + \delta_y \cos^2 \Theta_1 \cos \phi - \delta_x \cos \Theta_1 \sin \phi \right\}
\]

\[
= \mathbf{\Omega} r \sin \Theta_1 \left\{ -\delta_x \sin \phi + \delta_y \cos \phi \right\} = \delta_\phi \mathbf{\Omega} r \sin \Theta_1 = \delta_\phi \mathbf{v}_\phi
\]

This calculation from solid body rotation reveals that $v_\phi = \mathbf{\Omega} r \sin \Theta_1$. Hence, the radial dependence of $v_\phi$ is a linear function of $r$ at any angle $\Theta$ between the rotating cone @ $\Theta_1$ and the stationary plate @ $\Theta_\pi=\pi/2$, because if one moves into the fluid in the $\theta$-direction from the rotating cone toward the stationary plate at constant radial position $r$, and a separation of variables solution to the $\phi$-component of the Equation of Motion is valid, then the r-dependence shouldn't change. Hence:

\[
v_\phi(r, \Theta) = f(r) \ g(\Theta) = r \ g(\Theta)
\]
\[
g(\Theta) = \Omega \sin \Theta_1 \quad \Theta = \Theta_1 \\
g(\Theta) = 0 \quad \Theta = \pi/2
\]

If one postulates \( v_\phi (r, \Theta) \) to agree with the boundary condition at the interface between the fluid and the rotating cone, then the Equation of Continuity is satisfied for an incompressible fluid in which \( v_r = v_\phi = 0 \) when centrifugal forces are negligible in the creeping flow regime, because;

\[
\nabla \cdot v = \frac{1}{r \sin \Theta} \frac{\partial v_\phi}{\partial \phi} = 0
\]

Hence, the important nonzero component of the fluid velocity vector is not a function of the azimuthal angle \( \phi \) in spherical coordinates.

**Important nonzero components of the viscous stress distribution.** If the fluid is Newtonian, then Newton's law of viscosity allows one to estimate the importance of the six independent scalars that summarize the state of viscous stress. For one-dimensional flow of an incompressible fluid in the \( \phi \)-direction, the important quantities are;

\[
\tau_{\Theta\phi} = \tau_{\phi\Theta} = -\mu \frac{\sin \Theta}{r} \frac{\partial}{\partial \Theta} \left\{ \frac{v_\phi}{\sin \Theta} \right\} = h(\Theta) \\
\tau_{\phi\phi} = \tau_{r\phi} = -\mu r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) = 0
\]

The \( r\phi \)-component of the viscous stress distribution vanishes because the \( r \)-dependence of \( v_\phi \) resembles solid-body rotation (i.e., it is a linear function of \( r \)), for which there is no viscous stress. Mathematically, this concept is included in Newton's law when the streamlines are curved. Hence, it is only necessary to consider the \( \Theta\phi \)- and \( \phi\Theta \)-components of the viscous stress tensor in the Equation of Motion.

**Creeping flow analysis of the Equation of Motion.** The force balances in terms of \( \tau \) reveal that the important component of the viscous stress tensor is essentially constant within the gap between the rotating cone and the stationary plate. Since the \( \phi \)-component of the Equation of Motion provides the most important information about \( v_\phi \) and \( \tau_{\phi\phi} \), the other two components of the force balance are considered first;
\[ r\text{-component: } -\rho \frac{v_\phi^2}{r} = - \frac{\partial P}{\partial r} \]

If the z-axis about which the cone rotates is vertical, then the fluid experiences a centrifugal force that is horizontally outward. Hence, the \( r \)-component of the centrifugal force is \( \rho(v_\phi)^2 \sin \Theta \) divided by the radius of curvature at position \( r \) and \( \Theta \), which is \( r\{\sin \Theta}\}. 

This centrifugal force appears on the left side of the \( r \)-component of the Equation of Motion in spherical coordinates. In the creeping flow regime, dynamic pressure is not a function of radial coordinate \( r \) because centrifugal forces are negligible. Surface tension acts in the radial direction and provides the restraining wall that opposes radial fluid motion. However, surface tension is operative only at the boundary of the flow problem. At most, one should account for surface tension in the boundary condition at \( r=R \).

\[ \Theta\text{-component: } -\rho \frac{v_\phi^2 \cot \Theta}{r} = - \frac{1}{r} \frac{\partial P}{\partial \Theta} \]

If the fluid experiences a centrifugal force which is horizontally outward, then the \( \Theta \)-component of this force is \( \rho(v_\phi)^2 \cos \Theta \) divided by the radius of curvature (i.e., \( r \sin \Theta \)), as given by the term on the left side of the previous equation. In the creeping flow regime, dynamic pressure is not a function of polar angle \( \Theta \). Since azimuthal angle \( \phi \) is the symmetry variable in spherical coordinates, qualitative arguments suggest that \( P \) is independent of \( \phi \). This is reasonable because it is almost impossible to impose a pressure gradient in the primary flow direction when the streamlines are curved in a curvilinear coordinate system. One exception is axial flow through a helical cooling tube. For steady state analysis, where \( P \) does not exhibit time dependence, dynamic pressure is constant and the flow is not driven by gravity or a pressure gradient. This is typical for viscometers which contain a rotating solid surface. The \( \phi \)-component force balance yields the following information;

\[ \phi\text{-component: } 0 = - \frac{1}{r \sin \Theta} \frac{\partial}{\partial \Theta} \left\{ \tau_{\Theta \phi} \sin \Theta \right\} - \frac{\tau_{\Theta \phi}}{r} \cot \Theta = - \frac{1}{r} \frac{\partial \tau_{\Theta \phi}}{\partial \Theta} - 2 \frac{\tau_{\Theta \phi}}{r} \cot \Theta \]

Since the viscous stress tensor is symmetric and \( \tau_{\Theta \phi} \) is only a function of polar angle \( \Theta \), separation of variables allows one to obtain the viscous stress distribution as follows;
\[\int \frac{d\tau_{\Theta\phi}}{\tau_{\Theta\phi}} = -2 \int \frac{\cos \Theta}{\sin \Theta} d\Theta \]
\[\ln \tau_{\Theta\phi} = -2 \ln \{\sin \Theta\} + C_1\]
\[\tau_{\Theta\phi}(\Theta) = \frac{C_2}{\sin^2 \Theta}\]

Since the angle between the rotating cone and the stationary plate is approximately 0.5 degree (i.e., from 89.5° to 90°), fluids experience essentially constant shear stress in the cone-and-plate viscometer, which makes this device attractive for both Newtonian and non-Newtonian fluids. Torque is proportional to \(\tau_{\Theta\phi}\), and shear rate or velocity gradient is given by the linear velocity of the cone \(\Omega r \sin \Theta_1\) divided by the spacing between the cone and the plate, \(r(\pi/2 - \Theta_1)\), via an arc-length calculation at radial position \(r\). Hence, the velocity gradient is \(\Omega \sin \Theta_1 / (\pi/2 - \Theta_1)\), and \(\sin \Theta_1\) is approximately equal to unity because the angle of the cone is very close to 90°. Now, one calculates the viscosity of any fluid via the ratio of \(\tau_{\Theta\phi}\) and \(\Omega / (\pi/2 - \Theta_1)\), where the analysis below reveals that the magnitude of the torque transmitted by the fluid to the stationary plate is:

\[\text{Magnitude of the Torque}(\Theta = \pi/2) = (2/3) \pi R^3 \tau_{\Theta\phi}(\Theta = \pi/2)\]

One obtains this expression for the magnitude of the torque rather quickly because the variable lever arm is \(r \sin \Theta \approx r\) and the force transmitted across the fluid-solid interface at \(\Theta = \pi/2\) is given by the product of the viscous shear stress \(\tau_{\Theta\phi} = C_2\) and the differential surface element \(r dr d\phi\). Upon integration over the entire surface of the stationary solid plate;

\[\text{Torque} = \int_0^{2\pi} \int_0^R [\delta_r r] \left[\delta_\phi \tau_{\Theta\phi}(\Theta = \pi/2)\right] r dr d\phi = -\delta_\Theta \tau_{\Theta\phi}(\Theta = \pi/2) \frac{2}{3} \pi R^3\]

one relates viscous shear stress to the magnitude of the torque. It should be emphasized that \(-\delta_\Theta\) everywhere on the surface of the stationary plate corresponds to \(\delta_z\), which is not a function of spatial coordinates \(r\) and \(\phi\) in the previous integral expression.

\textit{Analysis of the important nonzero component of the fluid velocity vector.} If the fluid is Newtonian and one accounts for the slight dependence of viscous shear stress on polar angle \(\Theta\), then the following equations are combined to calculate \(v_\phi\);
\[ v_\phi(r, \Theta) = rg(\Theta) \]

\[ \tau_{\Theta\Phi} = -\mu \frac{\sin \Theta}{r} \frac{\partial}{\partial \Theta} \left( \frac{v_\phi}{\sin \Theta} \right) = -\mu \sin \Theta \frac{d}{d\Theta} \left( \frac{g(\Theta)}{\sin \Theta} \right) = -\frac{C_2}{\sin^2 \Theta} \]

Since \( g(\Theta) \) vanishes on the surface of the stationary plate via the "no-slip" condition, one integrates the previous equation from \( \pi/2 \) to \( \Theta \). The result is;

\[ \int_{\pi/2}^{\Theta} \frac{\cos(\Theta)}{a[m-1] \sin^{m-1}(x)} dx + \frac{m-2}{m-1} \int_{\pi/2}^{\Theta} \frac{dx}{\sin^{m-2} x} = C_2 \frac{\mu}{\sin \Theta} \]

The following integral theorems with \( m \neq 1 \) are useful to complete the development and calculate the fluid velocity profile;

\[ \int \frac{dx}{\sin^m x} = -\frac{\cos(ax)}{a[m-1] \sin^{m-1}(ax)} + \frac{m-2}{m-1} = \int \frac{dx}{\sin^{m-2} x} \]

When \( m=3 \) and \( a=1 \), one obtains;

\[ \int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{dx}{\sin x} \]

Finally;

\[ \int \frac{dx}{\sin x} = \int \csc(x) \frac{\csc x - \cot x}{\csc x - \cot x} dx = \ln \{\csc x - \cot x\} \]

Since the integral of \( 1/\sin^3 x \) vanishes when \( x=\pi/2 \), it is only necessary to evaluate the result at \( x=\Theta \). Hence;

\[ v_\phi(r, \Theta) = rg(\Theta) = -\frac{C_2}{\mu} r \sin \Theta \int_{\pi/2}^{\Theta} \frac{dx}{\sin^3 x} = \frac{C_2r}{2\mu} \left\{ \cot \Theta - \sin \Theta \ln(\csc \Theta - \cot \Theta) \right\} \]

\[ = \frac{C_2}{2\mu} \left\{ \cot \Theta - \frac{1}{2} \sin \Theta \ln \left( \frac{1 - \cos \Theta}{1 + \cos \Theta} \right) \right\} \]
Integration constant $C_2$ is evaluated via the "no-slip" condition at the interface between the fluid and the rotating cone, where $\Theta=\Theta_1$;

$$v_\phi(r, \Theta_1) = \Omega r \sin \Theta_1 = rg(\Theta_1)$$

$$g(\Theta_1) = \Omega \sin \Theta_1 = \frac{C_2}{2\mu} TrigFunction(\Theta_1)$$

$$TrigFunction(\Theta) = \cot \Theta - \frac{1}{2} \sin \Theta \ln \left( \frac{1-\cos \Theta}{1+\cos \Theta} \right)$$

The final result for the fluid velocity profile can be expressed as;

$$v_\phi(r, \Theta_1) = \Omega r \sin \Theta_1 \frac{TrigFunction(\Theta)}{TrigFunction(\Theta_1)}$$

This profile is essentially a linear function of polar angle $\Theta$ when the conical surface is described by values of $\Theta_1$ between $47^\circ$ and $89.9^\circ$.

**Problem#1**

Consider the following flux terms that appear either in the Equation of Motion or in the Equation of Continuity:

(a) \[ \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} \left( \rho v_\theta \sin \{\theta\} \right) \]

(b) \[ \mu \frac{\partial}{\partial r} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right\} \]

(i) Be specific and identify the type of transport and the mechanism in each case.

*Answer:*

- (a) Convective mass flux in the EOC; spherical coordinates
- (b) Normal viscous stress in the EOM; cylindrical coordinates

(ii) Use differential elements of length and identify the surface across which each flux acts.

*Answer:*

- (a) Perpendicular to the $\Theta$-direction; $dS = rsin\Theta dr d\phi$
- (b) Perpendicular to the $r$-direction; $dS = rd\Theta dz$

(iii) Identify the coordinate direction in which each flux acts.

*Answer:*

- (a) Convective mass flux in the $\Theta$-direction; spherical coordinates
(iv) Identify the unit normal vector to the surface across which each flux acts. 

**Answer:**
- (a) Normal vector in the $\theta$-direction; spherical coordinates
- (b) Normal vector in the $r$-direction; cylindrical coordinates

(v) Obtain an analytical solution for the function $\omega(r, \theta)$, as described by the following fluid dynamics equations that describe the flow field in the cone-and-plate viscometer:

\[
\frac{dz(\theta)}{d\theta} = -2z \left\{ \frac{\cos(\theta)}{\sin(\theta)} \right\} 
\]

\[
z(\theta) = -\mu \left[ \frac{\sin(\theta)}{r} \right] \frac{\partial}{\partial \theta} \left\{ \frac{\omega(r, \theta)}{\sin(\theta)} \right\} 
\]

\[
\omega(r, \theta) = rf(\theta)
\]

B.C. \[ \theta = \frac{\pi}{2} \quad f = 0 \]

\[ \theta = \theta_1 \quad f = \Omega \sin(\theta_1) \]

(v) Identify the variable $z(\theta)$ in terms of fluid dynamics nomenclature

**Answer:** $z(\theta) = \tau_{\psi\psi}$ (i.e., the $\theta_\psi$-component of the viscous stress tensor)


**Answer:**
- [1] is the $\phi$-component of EOM in terms of $\tau$; spherical coordinates
- [2] is Newton’s law of viscosity for the $\theta_\phi$-component of $\tau$

(viii) Graph the analytical solution to the problem described in (v) above as follows;

\[
\frac{\omega(r, \theta)}{\Omega R \sin(\theta_1)} \quad \text{vs.} \quad \theta \quad \text{at three (3) constant values of } \frac{r}{R} = 0.2, 0.5, 0.9; \text{ and six (6) different values of the parameter } \theta_1 = 60^\circ, 75^\circ, 85^\circ, 88^\circ, 89^\circ, 89.5^\circ. \text{ The range of the independent variable } \theta \text{ is; } \theta_1 \leq \theta \leq 90^\circ. \text{ When } \theta_1 = 60^\circ, 75^\circ, \text{ scale the horizontal axis from } 60^\circ \text{ to } 90^\circ. \text{ When } \theta_1 = 85^\circ, 88^\circ, 89^\circ, 89.5^\circ, \text{ scale the horizontal axis from } 85^\circ \text{ to } 90^\circ. \text{ You should generate six (6) graphs, one for each}
value of $\Theta_1$. Three solutions should be plotted on each graph, corresponding to the three values of $r/R = 0.2, 0.5, 0.9$

**Answer:** The velocity profile is a linear function of polar angle $\Theta$ for cone angles $\Theta_1$ that are larger than $\approx 50$ degrees. Miraculously, the complex trigonometric function derived above for $v_\phi$ yields a straight line. When the cone angle $\Theta_1$ is less than 54 degrees, deviations of the velocity profile from linearity are greatest near the surface of the cone. For example:

<table>
<thead>
<tr>
<th>$\Theta_1$ (degrees)</th>
<th>% deviations from linearity</th>
</tr>
</thead>
<tbody>
<tr>
<td>54</td>
<td>0</td>
</tr>
<tr>
<td>31</td>
<td>10</td>
</tr>
<tr>
<td>29</td>
<td>13</td>
</tr>
<tr>
<td>28</td>
<td>15</td>
</tr>
<tr>
<td>27</td>
<td>17</td>
</tr>
<tr>
<td>9</td>
<td>74</td>
</tr>
</tbody>
</table>

When the profile for $v_\phi(r, \Theta)$ is a linear function of polar angle $\Theta$ from the surface of the rotating cone to the stationary plate, one postulates the following expression that conforms to the boundary conditions at both solid surfaces:

$$v_\phi(r, \Theta) = \Omega r \sin \Theta_1 \left( \frac{\pi/2 - \Theta}{\pi/2 - \Theta_1} \right)$$

Consistent with this linear velocity profile when the cone angle is $\approx 89^\circ$, one equates the magnitude of the $\Theta \phi$-component of the velocity gradient tensor, which is not a symmetric 2$^{nd}$-rank tensor, with the shear rate in the cone-and-plate viscometer. For example:

$$(\nabla v)_{\Theta \phi} = \frac{1}{r} \frac{\partial v_\phi}{\partial \Theta} = \Omega \sin \Theta_1 \frac{d}{d\Theta} \left( \frac{\pi/2 - \Theta}{\pi/2 - \Theta_1} \right) \approx -\frac{\Omega}{\pi/2 - \Theta_1}$$

As mentioned above, one calculates the viscosity of any fluid in the cone-and-plate viscometer via division of the viscous shear stress $\tau_{\Theta \phi}$ at the surface of the stationary plate, obtained experimentally from the measured torque, by the shear rate which is given by the magnitude of the previous calculation.
(ix) Compare your graphical results of \( \frac{\omega(r,\theta)}{\Omega R \sin(\theta_1)} \) vs. \( \theta \) for a Newtonian fluid within the gap of a cone-and-plate viscometer with three (3) other viscometer problems;

(a) steady state velocity profile \( v_x(y) \) for a fluid contained between two parallel plates; the lower plate is stationary and the upper plate moves unidirectionally in the \( x \)-direction with a constant velocity \( V \),

(b) velocity profile \( v_\theta(r) \) for a Newtonian fluid in the Couette (i.e., concentric cylinder) viscometer,

(c) velocity profile \( v_\theta(r,z) \) for a Newtonian fluid in the parallel-disk viscometer.

One of the four fluid flow examples mentioned above does not conform to the other three profiles. Identify the non-conformer.

**Problem #2**
Solve the following ordinary differential equation for the function \( y(x) \), subject to the boundary condition; \( y = 15 \) when \( x = \pi/2 \) radians.

\[
\frac{dy}{dx} = \frac{4y}{\tan x}
\]

**Problem #3**
Solve the following ordinary differential equation for \( y \) as a function of \( x \), subject to the boundary condition; \( y = 10 \) when \( x = 0 \) radians.

\[
\frac{dy}{dx} = 3y(\tan x)
\]

**Problem #4**
Write an integral expression for the average fluid velocity in the cone-and-plate viscometer. Be sure to include the limits of integration.

*Answer:* Since fluid flow occurs in the \( \phi \)-direction, the differential surface element \( dS \) is constructed from a product of two differential lengths in the \( r \)- and \( \theta \)-
Hence, \( dS = r \, dr \, d\theta \). \( v_\phi \) is averaged over the flow cross-section as follows:

\[
\langle v_\phi \rangle_{\text{Average}} = \frac{\iint v_\phi(r, \theta) \, dS}{\iint dS} = \frac{\frac{\pi R^2}{2} \int_0^{\pi/2} g(\theta) \, d\theta}{\frac{\pi R^2}{2} \int_0^{\pi/2} r \, dr \, d\theta} = \frac{\frac{1}{3} R^3 \int_0^{\pi/2} g(\theta) \, d\theta}{\frac{1}{2} R^2 (\pi/2 - \Theta)} = \frac{2R}{3(\pi/2 - \Theta)} \int_0^{\pi/2} g(\theta) \, d\theta
\]

**Differential vector force transmitted by the fluid to the stationary plate.** To simplify the calculation of the macroscopic torque/angular-velocity relation, one considers the force transmitted by the fluid to the stationary solid plate at \( \Theta = \pi/2 \). Calculations are simplified on the stationary plate because the normal vector always points in the z-direction, and the variable lever arm is completely in the r-direction. In spherical coordinates, the unit normal vector from the fluid to the solid is oriented in the positive \( \Theta \)-direction. Hence, the 1\textsuperscript{st} subscript on the scalar components of convective and viscous momentum flux which act across this surface is \( \Theta \), pressure forces act in the \( \Theta \)-direction, and all scalar components of total momentum flux which act across this surface contain a positive sign. The differential surface element is constructed from a product of two differential lengths in the \( r \)- and \( \phi \)-directions. Hence, \( dS = r \sin \Theta \, dr \, d\phi \), which reduces to \( r \, dr \, d\phi \) on the surface of the stationary plate, where \( \Theta = \pi/2 \). Consider the following table;

<table>
<thead>
<tr>
<th>Stress</th>
<th>Coord. direction</th>
<th>Non-zero?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure ( p )</td>
<td>( \Theta )</td>
<td>yes</td>
<td>( p &gt; 0 )</td>
</tr>
<tr>
<td><strong>Convective momentum flux</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho v_\theta v_r )</td>
<td>( r )</td>
<td>no</td>
<td>( v_\theta = 0 )</td>
</tr>
<tr>
<td>( \rho v_\theta v_\theta )</td>
<td>( \Theta )</td>
<td>no</td>
<td>( v_\theta = 0 )</td>
</tr>
<tr>
<td>( \rho v_\theta v_\phi )</td>
<td>( \phi )</td>
<td>no</td>
<td>( v_\theta = 0 )</td>
</tr>
<tr>
<td><strong>Viscous momentum flux</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tau_{\theta r} )</td>
<td>( r )</td>
<td>no</td>
<td>( v_r = 0, v_\theta = 0 )</td>
</tr>
<tr>
<td>( \tau_{\theta \theta} )</td>
<td>( \Theta )</td>
<td>no</td>
<td>( v_\theta = 0, \nabla \cdot \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>( \tau_{\theta \phi} )</td>
<td>( \phi )</td>
<td>yes</td>
<td>( \partial / \partial \Theta (v_\phi / \sin \Theta) \neq 0 )</td>
</tr>
</tbody>
</table>

Hence, the normal pressure force acts in the \( \Theta \)-direction, and viscous shear acts in the \( \phi \)-direction. The differential vector force which acts across the solid-liquid interface at \( \Theta = \pi/2 \) is;
\[
\text{dF}_{\text{Fluid On Solid}} = \{ \delta_r \rho + \delta_\phi \tau_{\theta \phi} \}_{\Theta = \pi/2} r \, dr \, d\phi
\]

**Macroscopic relation between torque and angular velocity.** The differential vector torque \( dT \) that arises from \( \text{dF}_{\text{Fluid On Solid}} \) acting across the solid-liquid interface @ \( \Theta = \pi/2 \) is calculated as follows;

\[
dT = \{ \text{Lever Arm} \} \times \text{dF}_{\text{Fluid On Solid}}
\]

\[
= \{ \delta_r \, r \} \times \{ \delta_\theta \rho + \delta_\phi \tau_{\theta \phi} \}_{\Theta = \pi/2} r \, dr \, d\phi
\]

Performing the cross-product operation, neglecting the pressure contribution to the torque, and integrating the result over the entire surface of the stationary plate yields;

\[
\text{Torque} = \int dT = \int \int_{0}^{2\pi} \int_{0}^{R} \left[ \delta_r \times \delta_\phi \right] \tau_{\theta \phi} (\Theta = \pi/2) r^2 \, dr \, d\phi
\]

\[
= -\delta_\theta \tau_{\theta \phi} (\Theta = \pi/2) \int d\phi \int_{0}^{R} r^2 \, dr = \delta_\theta \, 2R^2 \tau_{\theta \phi} (\Theta = \pi/2)
\]

where (i) \( \delta_r \times \delta_\phi = -\delta_\theta = +\delta_z \) on the stationary plate, (ii) the viscous shear stress is given by \( C_2 \) when \( \Theta = \pi/2 \), and (iii) and the constant \( C_2 \) is calculated from the "no-slip" boundary condition on the surface of the rotating cone;

\[
\tau_{\theta \phi} (\Theta = \pi/2) = C_2 = \frac{2\mu \Omega \sin \Theta_1}{\text{TrigFunction}(\Theta_1)}
\]

The final result is;

\[
\text{Torque} = \delta_z \, \frac{4}{3} \pi R^3 \mu \Omega \sin \Theta_1 \, \text{TrigFunction}(\Theta_1)
\]

This relation allows one to calculate the viscosity of a Newtonian fluid via measurement of torque and angular velocity. If the fluid does not obey Newton's law of viscosity, then one calculates the viscous shear stress from the magnitude of the macroscopic torque as follows;

\[
\tau_{\theta \phi} (\Theta = \pi/2) = \frac{3\{\text{Torque}\}}{2\pi R^3}
\]

The fluid viscosity is obtained via division of the previous equation by \( \Omega / \{\pi/2 - \Theta_1\} \), where \( \Omega \) is the angular velocity of the cone. Non-Newtonian fluids exhibit a shear-rate-
dependent viscosity. However, when the gap between the cone and plate is very narrow (i.e., $\Theta_1 \approx 89.5^\circ$), the fluid experiences essentially one viscous shear stress, and the velocity profile is a linear function of polar angle $\Theta$, with a constant shear rate given by $\Omega/\{\pi/2 - \Theta_1\}$. The effect of shear rate on the viscosity of a non-Newtonian fluid is obtained, in practice, by incrementing $\Omega$ and repeating the experiment in the cone-and-plate viscometer several times. Then, one constructs a graph of:

$$\frac{3\{\text{Torque}\}(\pi/2 - \Theta_1)}{2\pi \Omega R^3} \text{ vs. } \frac{\Omega}{(\pi/2 - \Theta_1)}$$

### Dimensionless Momentum Transfer Correlations

Forces exerted by moving fluids on stationary solid surfaces via macroscopic dimensionless momentum transfer correlations between the friction factor and the Reynolds number. The magnitude of the dynamic force exerted across a fluid-solid interface represents the focal point for these correlations. For one-dimensional tube flow of an incompressible Newtonian fluid, with $v_z(r)$, $\tau_{rz}(r)$ and $P(z)$, the differential vector force exerted by the fluid on the wall at $r=R$, due to total momentum flux, is:

$$\text{d}F_{\text{Fluid on Solid}} = \{ \delta_r \ p + \delta_z \ \tau_{rz} \}_{r=R} \ R \ \text{d}\Theta \ \text{dz}$$

where $p$ is fluid pressure and $P$ is dynamic pressure. This can be verified rather easily, because the unit normal vector from the fluid to the stationary solid wall is $n = \delta_r$, which implies that the first subscript on all of the important nonzero components of total momentum flux is $r$ and pressure forces across this surface act in the $r$-direction. Now, the component of $\text{d}F_{\text{Fluid on Solid}}$ in the primary flow direction is obtained by performing the scalar product operation of $\text{d}F_{\text{Fluid on Solid}}$ with $\delta_z$, which yields $\tau_{rz} \ \text{d}S$. Hence;

$$\delta_z \cdot \text{d}F_{\text{Fluid on Solid}} = \{ \text{d}F_{\text{Fluid on Solid}} \}_{z\text{-component}} = \tau_{rz}(r=R) \ R \ \text{d}\Theta \ \text{dz}$$

For laminar or turbulent flow of incompressible fluids, the $rz$-component of the viscous stress tensor, evaluated at the tube wall, is;

$$\tau_{rz}(r=R) = \frac{R \Delta P}{2L}$$

Integration of the differential expression for the $z$-component force exerted by the fluid on the stationary wall yields;
By definition, the magnitude of the dynamic vector force exerted by fluids on stationary solid surfaces for any flow problem is given by:

\[
| \{ F_{\text{Fluid on Solid}} \}_z | = \left( \frac{1}{2} \right) \rho V^2 \text{(Shear Area)} f
\]

where \( V \) represents an average velocity. The previous equation can be considered as an operational definition of the friction factor \( f \). If one employs previous results for spatially averaged properties to calculate \( V \) (or \( <v_z> \)) for laminar flow of incompressible Newtonian fluids, then the previous two equations yield \( f = 16/Re \), where the shear area, or lateral surface area of the tube, is \( 2\pi RL \), the Reynolds number is defined by \( Re = \rho <v_z>2R/\mu \), and the correlation is valid when \( Re \) is below 2100.

A word of caution is important, here, if one employs dimensionless momentum transfer correlations between \( f \) and \( Re \) that were developed elsewhere. The following information must be available;

1. How is the Shear Area defined?
2. What is the characteristic length in the definition of \( Re \)?
3. How is the Reynolds number defined?
4. Over what range of \( Re \) is the correlation valid?

**Generalized interpretation of \( f \) vs. \( Re \); what is the physical significance of these correlations?** When the mass flowrate and the Reynolds number increase, the friction factor typically decreases, except for high Reynolds number flow around submerged objects. However, the dynamic force transmitted across a fluid-solid interface increases at higher Reynolds numbers in all flow regimes. The generalized correlations are:

\[
\left| \{ F_{\text{Fluid \Rightarrow Solid}} \}_{\text{PrimaryFlowDirection}} \right| = \left( \frac{1}{2} \right) \rho V^2 \text{(ShearArea)} f
\]

\[
f = \frac{C_1}{Re^a}; \quad Re = \frac{\rho VL}{\mu}
\]

where \( L \) is the characteristic length in the definition of \( Re \), and the exponent "a" is the negative slope of \( f \) vs. \( Re \) on log-log coordinates. The dependence of the dynamic force in the primary flow direction on density, viscosity and average fluid velocity is;
\[
\left\{ F_{\text{Fluid} \Rightarrow \text{Solid}} \right\}_{\text{Primary Flow Direction}} = \mu^a \rho^{1-a} V^{2-a}
\]

Intuitively, this interfacial force should increase for fluids with (i) higher viscosity in the creeping or laminar flow regime, (ii) higher density in the turbulent regime, and (iii) higher flowrates in general. Hence, the acceptable range of the exponent "a" is:

\[
0 \leq a \leq 1
\]

Consider the following correlations for two-dimensional flow of incompressible Newtonian fluids around solid spheres, where the shear area is \(\pi R^2\) (not \(4\pi R^2\)) and the characteristic length in the definition of the Reynolds number is the sphere diameter.

(i) \(f = 24 / \text{Re} \quad \text{Re} \leq 0.5\)

(ii) \(f \approx 18.5 / \text{Re}^{0.6} \quad 2 < \text{Re} < 500\)

(iii) \(f \approx 0.44 \quad 500 < \text{Re} < 2 \times 10^5\)

Correlation (i) in the creeping flow regime represents an exact solution to the steady state Equation of Motion when its entire left-hand side has been neglected because forces due to convective momentum flux are negligible. The latter two experimental correlations [i.e., (ii) and (iii)] have been obtained at higher Reynolds numbers where forces due to convective momentum flux are important, and an exact solution of the \(r\)- and \(\theta\)-components of the Equation of Motion, together with the Equation of Continuity is much more difficult, if not impossible. In the creeping flow regime where \(a=1\):

\[
\left| \left\{ F_{\text{Fluid on Solid}} \right\}_{\text{Primary Flow Direction}} \right| \approx \mu V
\]

At much higher flowrates in the turbulent flow regime, where the exponent "a" approaches zero:

\[
\left| \left\{ F_{\text{Fluid on Solid}} \right\}_{\text{Primary Flow Direction}} \right| \approx \rho V^2
\]

Turbulent flow past a solid sphere is appropriate to describe the flight of a baseball or golfball, for example, and these correlations reveal that the hydrodynamic drag force scales as \(V^2\). Furthermore, density, not viscosity, is the most important fluid property that should be considered. The dimples on a golf ball reduce hydrodynamic drag in the highly turbulent flow regime by approximately a factor of two because the fluid streamlines moving past this rough surface follow the contour of the ball for a longer
distance beyond the stagnation point relative to a smooth surface. This phenomenon has been demonstrated by attaching sandpaper to the front of a bowling ball, at its stagnation point, and plunging the ball into a container of liquid. The fluid streamlines are more efficient at following the contour of the rough bowling ball because the turbulent eddies that are generated by the rough surface allow momentum to be transferred from the free stream into the boundary layer adjacent to the fluid-solid interface. Hence, fluid particles attempting to traverse a streamline on the backside of the submerged object have a higher probability of penetrating regions of higher pressure as their tangential velocity component decreases in magnitude. If these fluid particles are not successful in traversing a streamline on the backside of a submerged object, then they are redirected toward the free stream and a chaotic low-pressure wake develops behind the object. This phenomenon is described as “boundary layer separation”, and the consequence is that the submerged object experiences significant hydrodynamic drag. “Streamlined objects”, whose surface contours match that of a dolphin, for example, are designed to reduce hydrodynamic drag by minimizing boundary layer separation.

**Practical examples of hydrodynamic drag.** Consider the following situations where (i) air or water flow around submerged streamlined objects has been designed to reduce hydrodynamic drag forces, or (ii) hydrodynamic forces are primarily responsible for the direction of flight of an object.

1) Aerobars on a bicycle

2) Bladed bicycle spokes, that look like linguine, which “slice” the air instead of cylindrical spokes that “chop” the air as wheels rotate

3) $900 Disk wheels on a bicycle that shield spokes from "cutting" the wind---disk wheels are extremely unstable while riding in a strong crosswind

4) Air flow transverse to tear-drop bicycle tubes instead of cylindrical tubes

5) New designs for bicycle helmets that include an extended tail---Greg LeMond won the 1989 Tour de France by 8 seconds over Laurent Fignon after more than 100 hours and 2031 miles of racing in three weeks!!! LeMond rode the final 15-mile time trial from Versailles to Paris in 26 minutes 57 seconds (34 miles per hour)

6) Recumbent bicycles that are equipped with wind shields (the world hour record in a human-powered vehicle is 47 miles in 1 hour at sea level on one of these bikes)

7) Bicycle racers forming a paceline where the lead rider does most of the work
8) Canadian geese flying in a V-pattern as they travel from City Park to a golf course south of Fort Collins. It is not known whether it’s one goose, one vote, or some strange version of the electoral college, but however they decide, the goose that starts out at the point of the “V-pattern” is most likely the strongest flier. As the lead goose fatigues, it moves back and is replaced by a bird that continues to fly strong. There’s a tremendous force of wind resistance that the lead bird is breaking, which is extremely power-demanding. Goose-flying formations are actually lessons in physics and fluid dynamics. As a bird flies, vortices of air spiral off its wings, creating slipstreams. When one bird flies directly behind another bird, there is some resistance from this slipstream of air. However, the second bird experiences a suction effect when it flies behind the lead bird’s wingtips. Studies estimate that geese flying in a “V-pattern” can continue to fly approximately 70% farther than one goose could fly alone.

9) Polymer solutions used to coat ship hulls for reduced drag in the ocean---the overall objective is to recapture America’s cup

10) Take-off gear on airplane wings to minimize power requirements for departures

11) Designs for Indy 500 racing cars that travel faster than 200 miles per hour---drinking a quart of milk in the winner’s circle never hurts anyone

12) The air foil on the roof of a semi-cab prevents wind from crashing into the blunt front surface of its trailer on Interstate 25

13) Dimpled golf balls that travel much farther than smooth ones, particularly at the Mount Massive golf course near Leadville (elevation ≈ 9800 feet above sea level). In July 1969, Edwin "Buzz" Aldrin followed Neill Armstrong down the steps of the Eagle, which landed the previous day in the Sea of Tranquility, and hit a golf ball out of sight in the "thin" lunar atmosphere, coupled with a gravitational field that is 6-fold weaker than the one on Earth.

14) Hot wax applied to the base of Nordic skating skis to enhance "slip" at the wall

15) Downhill skiers testing various "tuck" positions in a wind tunnel trying to reduce their time (in seconds) in the first or second decimal place on the giant slalom course at the 2002 scandal-plagued Olympics in Salt Lake City

16) Swimmers who shave their head for the freestyle event in quest of a gold medal
17) Skyscraper designs in large cities to minimize the effects of gale force winds

18) Roofing designs to prevent property destruction in Boulder during wind storms

19) Dandy Don Meredith and Frank Gifford say that it's easier to kick long-distance field goals during Monday night football games at Mile High stadium in Denver relative to any other professional football stadium in the country. Jason Elam tied Tom Dempsey's record with a 63-yard field goal at Invesco Field.

20) If the placekick holder (i.e., most likely, a 2nd- or 3rd-string quarterback) positions the football with the laces facing right, then field goal attempts will be "wide right" because of the hydrodynamic force imbalance. Remember when Scott Norwood missed a last-second field goal attempt in the 1991 Super Bowl and the New York Giants defeated the Buffalo Bills, 20-19? On more than one occasion, the Florida State Seminoles field goal specialist was "wide right" in the final minutes of the game against Miami, and the Hurricanes won the game and the national championship of collegiate football.

21) If a pitcher throws a curve ball, then it breaks more near sea level at Yankee Stadium in the Bronx (i.e., a borough of New York City) than it does at Coors Field in Denver. That might explain why the Colorado Rockies don't generate much offense during "away" games.

22) Relative to baseball games played in humid conditions, there aren't many home runs hit in domed stadiums where water vapor is removed from the air via "air conditioning". Hank Aaron played many baseball games in muggy Atlanta and hit 755 lifetime homers. However, the Seattle Mariners set a new record for most home runs hit by an entire team during the 162-game season in 1997, and half of their games were played in the Kingdome where there is a 20 mile-per-hour wind blowing from home plate toward the outfield. Hey, humid air is lighter. I can ignore it if you only say it once per week. However, you have now mentioned it twice this week and I feel compelled to correct your error. Contrary to what you are telling your readers, humid air is actually lighter than dry air. That's right, with all other things being equal, particularly altitude, curveballs "break" less and batted balls travel farther in humid air relative to dry air. If you don't believe me (and why should you?), then contact someone in the Department of Atmospheric Science at Colorado State University.
23) Major league baseball scores of games played at Coors Field resemble football scores in defensive-minded struggles. Look what happened at the All-Star game in 1998! (i.e., the commissioner let the players finish the game and the American League won by the score of 13-8).

24) At the highest airport on the planet in La Paz, Bolivia, planes must achieve very fast take-off velocities to create the required lift because, at 13,000 feet above sea level, the density of air is reduced considerably relative to that at sea level.

25) **Speed-skating revolution.** When someone asked Michael Jordan, “Is it in the shoes?”, they had the correct question but the wrong sport. Long-track speed skaters definitely will break records at the winter Olympic games, thanks not to their abilities but to a new skate that shaves off a second per lap. The noisy new “klap” (Dutch for “slap”) skate allows a longer and smoother stride. Since their debut at international events, they have helped tie or break more than 16 world records. Every medal contender wears them at international speed skating events. **Fast-forward a few years.** At first glance, they seem insignificant; tiny strips of rubber roughly the size of a seam, hardly noticeable on a speed skater’s skin-tight suit. However, these racing stripes (if that’s the correct word for them, the concept is so new that no one is sure what their official name should be) are the latest weapon in the technological tussle to increase skater’s speeds. Amazingly, the stripes actually transform the “klap” skate into a secondary issue. It used to be that one would simply wake up in the morning and skate. Now, there’s all of this tinkering going on. Theory dictates that the stripes provide an aerodynamic edge by reducing hydrodynamic drag (i.e., wind resistance) when skaters reach speeds of 40 mph. Does it work? The Dutch think so because they tested the concept in wind tunnels and surprised the competition at Olympic events by obtaining permission to attach a few squiggly stripes to each leg of their racing suits, running from knee to ankle, and another stripe was attached to their hood.

Identify two more practical examples of drag reduction, or situations where hydrodynamic drag has a major effect on the flight of a submerged object moving through air or water.

**Use of f vs. Re to analyze pressure drop vs. flowrate in tubes.** Laminar flow of an incompressible Newtonian fluid through a straight tube with radius R and length L corresponds to \( f = 16/Re \) and \( \log Q \approx \log \Delta P \), where \( f \) is the friction factor, Re is the Reynolds number based on the tube diameter, Q is the volumetric flowrate and
P is dynamic pressure. Determine the scaling law exponent $\alpha$ for turbulent flow of an incompressible Newtonian fluid through the same tube;

$$\log Q \approx \alpha \log \{\Delta P\}$$

where $f = 0.0791/Re^{1/4}$. **Hint:** The $z$-component of the dynamic force exerted by the fluid on the wall at $r=R$ is $\pi R^2 \Delta P$.

**Answer**
Prior to solving this problem, it is instructive to consider the underlying fundamentals related to the hint provided above. In terms of forces or stresses due to total momentum flux that act across the solid-liquid interface, the differential vector force exerted by the fluid on the tube wall is;

$$dF_{\text{Fluid on Solid}} = \{ \delta_r (\tau_{rr} + p) + \delta_\theta \tau_{r\theta} + \delta_z \tau_{rz} \} r=R \ R d\Theta \ dz$$

The $z$-component of $dF_{\text{Fluid on Solid}}$ is obvious from the previous expression. Rigorously, it is obtained via the following scalar dot product operation;

$$\delta_z \cdot dF_{\text{Fluid on Solid}} = \{ \tau_{rz}(r=R) \} R d\Theta \ dz$$

Integration of the previous equation over the complete lateral surface (i.e., $0 \leq \Theta \leq 2\pi$, $0 \leq z \leq L$) for incompressible Newtonian fluids yields;

$$\left\{ F_{\text{Fluid to Solid}} \right\}_{z\text{-component}} = \int \delta_z \cdot dF_{\text{Fluid to Solid}} = \int_0^{2\pi} d\Theta \int_0^L \left\{ -\mu \left( \frac{dv_z}{dr} \right)_{r=R} \right\} Rdz$$

For one-dimensional flow in the $z$-direction, where $v_z(r)$ is not a function of spatial coordinates within the lateral surface, the final expression for the macroscopic dynamic force simplifies considerably because $\tau_{rz}$ is also independent of the lateral surface coordinates. Hence;

$$\left\{ F_{\text{Fluid on Solid}} \right\}_{z\text{-component}} = \{ -\mu \ (dv_z/dr)_{r=R} \} 2\pi R L = \pi R^2 \Delta P$$

This result is verified rather easily for laminar flow in terms of the microscopic fluid velocity gradient at the tube wall. For steady state one-dimensional flow through a straight tube in any regime, a combination of the macroscopic mass and momentum balances yields the same result, as given by the previous equation. The solution to this problem begins by employing the macroscopic momentum transfer correlation, which
includes the definition of the friction factor, to evaluate the z-component of the dynamic force exerted by the fluid on the tube wall, with shear area given by $2\pi RL$. For example;

$$\{ \mathbf{F}_{\text{Fluid on Solid}} \}_{\text{z-component}} = \pi R^2 \Delta P = (1/2) \rho <v_z>^2 \{ 2 \pi R L \} f$$

Now, use the dimensionless correlation for $f$ vs. Re, where the Reynolds number is defined in terms of the tube diameter;

$$\text{Re} = \rho <v_z> (2R) / \mu$$

In terms of the scaling law for dynamic force;

$$\{ \mathbf{F}_{\text{Fluid on Solid}} \}_{\text{z-component}} = \pi R^2 \Delta P \approx \mu^a \rho^{(1-a)} <v_z>^{(2-a)}$$

where $a = 1/4$ in the turbulent flow regime. For tube flow, average velocity $<v_z>$ and volumetric flowrate $Q$ are related by the cross-sectional area for flow (i.e., $\pi R^2$). Hence, the dynamic pressure drop $\Delta P$ scales as $Q$ taken to the $(2-a)$ power. In other words;

$$\Delta P \approx Q^{(2-a)}$$

Therefore, the scaling law exponent which relates $Q$ to $\Delta P$ is, $\alpha = 1/(2-a) = 4/7$. The complete result for laminar or turbulent flow of an incompressible Newtonian fluid through a straight tube of radius $R$ and length $L$ is;

$$Q^{2-a} = 2^a \pi^{2-a} R^{5-a} \Delta P / C_1 \rho^{1-a} \mu^a L$$

when the dimensionless momentum transfer correlation is $f = C_1/\text{Re}^a$. In the laminar flow regime, where $C_1 = 16$ and $a = 1$, the previous equation reduces to the classic Hagen-Poiseuille law;

$$Q = \frac{\pi R^4 \Delta P}{8\mu L}$$

The solution to this problem reveals that $Q$ and $\Delta P$ do not follow a linear relation for turbulent flow of an incompressible Newtonian fluid through a tube. Hence, if one has data for $Q$ vs. $\Delta P$ that correspond to flow of an incompressible Newtonian fluid through a straight tube with radius $R$ and length $L$, then the exponent "a" in the experimental $f$ vs. Re correlation can be obtained from the slope of a log-log graph of $Q$ vs. $\Delta P$, provided
that the data do not span more than one flow regime. From the viewpoint of linear least squares analysis for the pressure-drop/flowrate relation in straight tubes, the following procedure should be employed to calculate the experimental scaling law exponent;

(i) polynomial model; \[ y(x) = a_0 + a_1x \]
(ii) independent variable; \[ x_i = \log Q \]
(iii) dependent variable; \[ y_i = \log \Delta P \]
(iv) intercept, or zeroth-order coefficient;
\[
a_0 = \log \left( \frac{C_1 \rho_{\text{sphere}}^{1-a} \mu L}{2^a \pi^{2-a} R^{5-a}} \right)
\]
(v) slope, or 1st-order coefficient; \[ a_1 = 2 - a \]

**Analysis of terminal velocities for submerged objects.** A solid sphere of radius \( R_{\text{sphere}} \) and density \( \rho_{\text{sphere}} \) falls through an incompressible Newtonian fluid that is quiescent far from the sphere. The viscosity and density of the fluid are \( \mu_{\text{fluid}} \) and \( \rho_{\text{fluid}} \) respectively. The Reynolds number is 50, based on the physical properties of the fluid, the diameter of the sphere and its terminal velocity. The following scaling law characterizes the terminal velocity of the sphere in terms of geometric parameters and physical properties of the fluid and solid;

\[
\log v_{\text{terminal}} \approx \alpha \log R_{\text{sphere}} + \beta \log \{ \rho_{\text{sphere}} - \rho_{\text{fluid}} \} + \gamma \log \mu_{\text{fluid}} + \delta \log \rho_{\text{fluid}}
\]

(i) Calculate the scaling law parameters \( \alpha, \beta, \gamma \) and \( \delta \) in the previous equation. Four numerical answers are required, here.

**Answer**
Since there is no longer any acceleration when submerged objects achieve terminal velocity, the sum of all forces acting on the object must be zero. Hence, there is a balance between buoyancy, gravity, and hydrodynamic drag. The gravity force acts downward, and the buoyant and drag forces act in the opposite direction. In general, the hydrodynamic drag force acts (i) in the opposite direction of the motion of the object when the fluid is stationary, (ii) in the same direction as the motion of the fluid when the object is stationary, or (iii) in the direction of the relative motion of the fluid with respect to the object when neither one is stationary. Each force is calculated as follows;
Gravitational force: 
\( (4/3) \pi R_{\text{sphere}}^3 \rho_{\text{solid}} g \)

Buoyant force: 
\( (4/3) \pi R_{\text{sphere}}^3 \rho_{\text{fluid}} g \)

Hydrodynamic drag force: 
\( (1/2) \rho_{\text{fluid}} v_{\text{terminal}}^2 (\pi R_{\text{sphere}}^2) f \)

For flow around spheres in any regime, the dimensionless momentum transfer correlation adopts the following form;

\[ f = \frac{C_1}{Re^a} \quad Re = \frac{\rho_{\text{fluid}} v_{\text{terminal}} (2 R_{\text{sphere}})}{\mu_{\text{fluid}}} \]

Now, the hydrodynamic drag force can be expressed explicitly in terms of physical properties of the fluid and solid;

\[ \text{Hydrodynamic Drag Force} = \frac{\pi C_1}{2^{1+a}} \mu_{\text{fluid}}^a \rho_{\text{fluid}}^{1-a} \left( R_{\text{sphere}} v_{\text{terminal}} \right)^{2-a} \]

Rearrangement of the above-mentioned force balance yields the following solution for \( v_{\text{terminal}} \);

\[ v_{\text{terminal}}^{2-a} = \frac{2^{3+a} R_{\text{sphere}}^{1+a} \left( \rho_{\text{solid}} - \rho_{\text{fluid}} \right) g}{3C_1 \mu_{\text{fluid}}^a \rho_{\text{fluid}}^{1-a}} \]

Therefore, the scaling law parameters are;

\[ \alpha = \frac{1+a}{2-a}; \beta = \frac{1}{2-a}; \gamma = \frac{-a}{2-a}; \delta = \frac{-1-a}{2-a} \]

where \( a = 3/5 \) in the intermediate (i.e., laminar) flow regime. With reference to a creeping flow falling sphere viscometer, one measures the terminal velocity of a solid sphere that falls slowly through an incompressible Newtonian fluid. In the creeping flow regime, the dimensionless momentum transfer correlation for solid spheres is \( f = 24/\text{Re} \), which corresponds to \( C_1 = 24 \) and \( a = 1 \). Hence;

\[ v_{\text{terminal}} = \frac{2 R_{\text{sphere}}^2 \left( \rho_{\text{solid}} - \rho_{\text{fluid}} \right) g}{9 \mu_{\text{fluid}}} \]
One estimates the fluid viscosity by rearranging the previous equation. This prediction is accurate if the Reynolds number is smaller than 0.5. For consistency, one measures the terminal velocity of a solid sphere that falls through an incompressible Newtonian fluid, rearranges the previous equation to calculate the fluid viscosity, and uses the physical properties of the fluid, together with \( v_{\text{terminal}} \), to demonstrate that the Reynolds number corresponds to creeping flow. See page#61 in the 2\textsuperscript{nd} edition of *Transport Phenomena* by RB Bird, WE Stewart & EN Lightfoot.

(ii) A different sphere of the same density with radius \( 2R_{\text{sphere}} \) falls through the same incompressible Newtonian fluid. Now, the Reynolds number is greater than 50, but less than 500, because the diameter of the sphere has increased by a factor of 2. Does the terminal velocity of the sphere increase, decrease, or remain unchanged?

*Answer*
Since the scaling law exponent \( \alpha > 0 \), one achieves larger terminal velocity if the size of the sphere increases.

(iii) By how much, or by what factor, does \( v_{\text{terminal}} \) change in part (ii)? For example, if the terminal velocity of the sphere remains unchanged, then it changes by a factor of one.

*Answer*
The scaling law in part (a) can provide both qualitative and quantitative results. If the sphere radius increases by a factor of 2, then \( v_{\text{terminal}} \) increases by \( \left\{2\right\}^{(1+\alpha)/(2-\alpha)} \), which corresponds to \( \left\{2\right\}^{(8/7)} \) in the intermediate flow regime where \( \alpha = 3/5 \).

(iv) How does the scaling law for terminal velocity change if a non-deformable bubble of radius \( R_{\text{bubble}} \) rises with constant velocity through the same incompressible Newtonian fluid in the same flow regime (i.e., \( 50 \leq \text{Re} \leq 500 \))?

*Answer*
First, one must replace \( R_{\text{sphere}} \) by \( R_{\text{bubble}} \), but this is a minor change. Secondly, and most importantly, the hydrodynamic drag force acts downward when bubbles rise. Now, the upward buoyant force is counterbalanced by gravity and hydrodynamic drag. Consequently, one must replace \( (\rho_{\text{solid}} - \rho_{\text{fluid}}) \) by \( (\rho_{\text{fluid}} - \rho_{\text{bubble}}) \) in the scaling law for \( v_{\text{terminal}} \), as presented in part (i).
Linear least squares analysis of terminal velocities for spheres of different radii in various flow regimes

Algebraic rearrangement of the physical model for terminal velocity in the previous section yields:

\[
\log(v_{\text{terminal}}) = \frac{1}{2 - a} \log \left( \frac{2^{3+a} g (\rho_{\text{solid}} - \rho_{\text{fluid}})}{3C_1 \mu_{\text{fluid}}^a \rho_{\text{fluid}}^{1-a}} \right) + \frac{1 + a}{2 - a} \log(R_{\text{sphere}})
\]

\[
= a_0 + a_1 \log(R_{\text{sphere}})
\]

\[
a_0 = \frac{1}{2 - a} \log \left( \frac{2^{3+a} g (\rho_{\text{solid}} - \rho_{\text{fluid}})}{3C_1 \mu_{\text{fluid}}^a \rho_{\text{fluid}}^{1-a}} \right)
\]

\[
a_1 = \frac{1 + a}{2 - a}
\]

**Creeping flow:**
Re < 0.5, C_1 = 24, a = 1;

\[
a_0 = \log \left( \frac{2 g (\rho_{\text{solid}} - \rho_{\text{fluid}})}{9 \mu_{\text{fluid}}} \right)
\]

\[
a_1 = 2
\]

**Laminar flow:**
2 ≤ Re ≤ 500, C_1 ≈ 18.5, a = 0.6;

\[
a_0 = \frac{5}{7} \log \left( \frac{2^{3.6} g (\rho_{\text{solid}} - \rho_{\text{fluid}})}{3(18.5) \mu_{\text{fluid}}^{0.6} \rho_{\text{fluid}}^{0.4}} \right)
\]

\[
a_1 = \frac{8}{7}
\]

**Turbulent flow:**
500 ≤ Re ≤ 2 \times 10^5, C_1 ≈ 0.44, a = 0;

\[
a_0 = \frac{1}{2} \log \left( \frac{8 g (\rho_{\text{solid}} - \rho_{\text{fluid}})}{3(0.44) \rho_{\text{fluid}}} \right)
\]

\[
a_1 = \frac{1}{2}
\]
Terminal velocity of glass spheres (turbulent flow)

This program analyzes buoyant forces, gravity forces, and hydrodynamic drag forces on glass spheres that fall through an incompressible Newtonian fluid in the turbulent flow regime. The fluid is carbon tetrachloride at 20° C with known density and viscosity. The objective is to achieve a terminal velocity of 65 centimeters per second. The steady state force balance on glass spheres that don't accelerate is solved for the sphere diameter that will accomplish this task. It is necessary to calculate the Reynolds number and verify that fluid flow is actually turbulent and corresponds to the Newton's Law regime.

**density of carbon tetrachloride at 20° C, g/cm³**
\[ \rho_{\text{fluid}} = 1.59 \]

**density of glass spheres, g/cm³**
\[ \rho_{\text{solid}} = 2.62 \]

**terminal velocity that is required, cm/sec**
\[ v_{\text{terminal}} = 65 \]

**viscosity of CCl₄, gram per cm per second, Poise at 20° C**
\[ \mu_{\text{fluid}} = 0.00958 \]

**gravitational acceleration constant, cm/sec²**
\[ \text{gravity} = 980.665 \]

**volume of the glass sphere**
\[ \text{Volume}_{\text{Sphere}} = \frac{4}{3} \pi \left( \frac{D_{\text{Sphere}}}{2} \right)^3 \]

**projected area of the sphere, as seen by the approaching fluid**
\[ \text{Area}_{\text{Projection}} = \pi \left( \frac{D_{\text{Sphere}}}{2} \right)^2 \]
\[ \frac{\text{Volume}_{\text{Sphere}}}{\text{Area}_{\text{Projection}}} = \frac{2}{3} D_{\text{Sphere}} \]

**Steady state force balance that accounts for gravity, buoyancy, and hydrodynamic drag**
\[ (\rho_{\text{solid}} - \rho_{\text{fluid}}) \text{gravity} \{ \text{Volume}_{\text{Sphere}}/\text{Area}_{\text{Projection}} \} = \frac{1}{2} \rho_{\text{fluid}} \frac{v_{\text{terminal}}^2}{2} \{ \text{friction factor} \} \]

**check the Reynolds number to verify the flow regime**
\[ \text{Reynolds number} = \frac{\rho_{\text{fluid}} v_{\text{terminal}} D_{\text{Sphere}}}{\mu_{\text{fluid}}} \]
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**Hint:** this flow problem corresponds to the turbulent flow regime; $500 < \text{Re} < 2 \times 10^5$  
friction factor $= 0.44$

No numerical solution exists when $f = 24/\text{Re}$ and Re is restricted to be less than 0.5  
No numerical solution exists when $f = 18.5/\text{Re}^{3/5}$ and $2 < \text{Re} < 500$

Hence, there are no solutions to this problem in the creeping and laminar flow regimes.

**Solution**

- $\text{Area}_{\text{Projection}} = 3.78e+0$ [cm$^2$]
- $D_{\text{Sphere}} = 2.19e+0$ [cm]
- friction factor $= 4.40e-1$ [dimensionless]
- gravity $= 9.81e+2$ [cm/sec$^2$]
- $\mu_{\text{fluid}} = 9.58e-3$ [g/cm-sec]
- Reynolds number $= 2.37e+4$ [dimensionless]
- $\rho_{\text{fluid}} = 1.59e+0$ [g/cm$^3$]
- $\rho_{\text{solid}} = 2.62e+0$ [g/cm$^3$]
- $\text{Volume}_{\text{Sphere}}/\text{Area}_{\text{Projection}} = 1.46e+0$ [Volume/Area, cm]
- $\text{Volume}_{\text{Sphere}} = 5.54e+0$ [cm$^3$]
- $v_{\text{terminal}} = 6.50e+1$ [cm/sec]

**Applications of hydrodynamic drag forces via $f$ vs. $\text{Re}$ to calibrate a rotameter when the test fluid is different from the calibration fluid.** A rotameter consists of a vertical conical tube that contains a float of higher density than that of the fluid passing through the meter. The tube diameter is not constant, but it increases linearly as the float moves to higher positions in the conical tube. This feature allows the rotameter to measure a wide range of mass flow rates. When the rotameter is calibrated for a particular fluid, it is very straightforward to measure mass flow rates for that fluid in terms of the height of the float under steady state conditions. You are given a rotameter calibration curve for water which illustrates that mass flow rate is linearly proportional to float height. However, experiments on a distillation column require that you must measure the mass flow rates of alcohols using the rotameter that was calibrated for water. Devise a strategy and use that strategy to modify the rotameter calibration curve for water so that one can measure the mass flow rate of an alcohol using the same rotameter.

Your final answer should include strategies when a log-log plot of friction factor vs. Reynolds number for flow through a conical tube that contains a submerged object (i.e., the float);
(i) is a straight line with a slope of -1.0
(ii) is a straight line with a slope of -0.5
(iii) is a straight line with zero slope

Answer
Results from the previous section provide a generalized expression for the terminal velocity of solids or bubbles in stationary fluids. The same results describe the average fluid velocity in the vicinity of a stationary submerged object, like the rotameter float. Of course, the shear area and volume of a solid sphere or gas bubble are well defined in terms of the radius of the submerged object. The corresponding shear area and volume of the rotameter float can be measured, but they can't be expressed in terms of one simple geometric parameter. Fortunately, these quantities don't change when a different fluid passes through the rotameter. The strategy below, which focuses on the following scaling law for the average fluid velocity in the vicinity of the float;

\[
\left< v_{\text{fluid}} \right>^{2-a} \approx \frac{\left( \rho_{\text{solid}} - \rho_{\text{fluid}} \right)}{\mu_{\text{fluid}}^{a} \rho_{\text{fluid}}^{1-a}}
\]

reveals that the shear area, float volume, and gravitational acceleration constant do not affect the rotameter correction factor. One obtains the corresponding mass flow rate from the previous scaling law via multiplication by the fluid density and the cross-sectional area. Since the rotameter correction factor compares mass flow rates for two different fluids when the float height is the same, the flow cross section does not appear in the final result because it remains constant. Hence, it is only necessary to multiply \( \left< v_{\text{fluid}} \right> \) by \( \rho_{\text{fluid}} \). Therefore, when the float is at the same position, the mass flow rate of any fluid through the same rotameter scales as;

\[
\rho_{\text{fluid}} \left< v_{\text{fluid}} \right> \approx \left\{ \rho_{\text{fluid}} \left( \rho_{\text{solid}} - \rho_{\text{fluid}} \right) \right\}^{1/(2-a)} \frac{\mu_{\text{fluid}}^{a/(2-a)}}{\rho_{\text{fluid}}}
\]

The quantity on the right side of the previous equation must be evaluated for the test fluid and the calibration fluid. This ratio (i.e., test fluid/calibration fluid) represents the calibration factor which one must multiply by the mass flow rate of the calibration fluid at a given rotameter float height to obtain the mass flow rate of the test fluid when the float is in the same position.

For part (i), \( a = 1 \) and the mass flow rate for each fluid scales as;
\[
\text{MassFlowRate} \approx \rho_{\text{fluid}} \langle v_{\text{fluid}} \rangle \approx \frac{\rho_{\text{fluid}} (\rho_{\text{solid}} - \rho_{\text{fluid}})}{\mu_{\text{fluid}}}
\]

For part (ii), \(a = 0.5\) and the mass flowrate for each fluid scales as;

\[
\text{MassFlowRate} \approx \rho_{\text{fluid}} \langle v_{\text{fluid}} \rangle \approx \left\{ \frac{\rho_{\text{fluid}} (\rho_{\text{solid}} - \rho_{\text{fluid}})}{\mu_{\text{fluid}}^{1/3}} \right\}^{2/3}
\]

For part (iii), \(a = 0\) and the mass flowrate for each fluid scales as;

\[
\text{Mass flowrate} \approx \left\{ \frac{\rho_{\text{fluid}} (\rho_{\text{solid}} - \rho_{\text{fluid}})}{\mu_{\text{fluid}}^{1/3}} \right\}^{1/2}
\]

(iv) At 20\(^\circ\)C, the density of water is 1.00 g/cm\(^3\) and the density of methanol is 0.79 g/cm\(^3\). The float density is 3.95 g/cm\(^3\). Compare the mass flow rates of water and methanol through the same rotameter at 20\(^\circ\)C when the float rests at the same position in the rotameter. In both cases, the dimensionless momentum transport correlation is \(f \approx \) constant in the high Reynolds number regime.

**Answer**

Evaluate the mass flowrate scaling factor for water and methanol via the prescription from part (iii), because \(a = 0\). Then, construct the ratio of these scaling factors to compare the mass flowrates of the two fluids. For example;

(1) Mass flowrate of water \(\approx \left\{ \rho_{\text{water}} (\rho_{\text{float}} - \rho_{\text{water}}) \right\}^{1/2}\)

(2) Mass flowrate of methanol \(\approx \left\{ \rho_{\text{methanol}} (\rho_{\text{float}} - \rho_{\text{methanol}}) \right\}^{1/2}\)

The ratio of (1) to (2) is 1.09, which indicates that the mass flowrate of water is 9\% larger than that of methanol.

(v) How does your comparison of the mass flow rates of water and methanol at 20\(^\circ\)C from part (iv) change if the float density is only 1.35 g/cm\(^3\)?

**Answer**

Use the scaling laws in part (iv) for water and methanol, but reduce the float density from 3.95 g/cm\(^3\) to 1.35 g/cm\(^3\). Now, the ratio of (1) to (2) is 0.89, which indicates that the mass flowrate of water is about 11\% less than that of methanol.
(vi) In the highly turbulent regime, the mass flow rates of water and methanol will be the same @ 20°C when a particular float rests at the same position in the rotameter. What float density is required for this statement to be true?

**Answer**

Equate the scaling laws in part (iv) for water and methanol and solve for \( \rho_{\text{float}} \):

\[
\left\{ \frac{\rho_{\text{water}} (\rho_{\text{float}} - \rho_{\text{water}})}{2} \right\}^{1/2} = \left\{ \frac{\rho_{\text{methanol}} (\rho_{\text{float}} - \rho_{\text{methanol}})}{2} \right\}^{1/2}
\]

\( \rho_{\text{float}} = 1.8 \text{ g/cm}^3 \).
Review Problems for Exam#2

**Problem #1**
Use one or two sentences and describe the phrase "very low Reynolds number hydrodynamics". Do not include any equations in your description.

**Problem #2** Sketch velocity profiles in 4 viscometers
(a) **Concentric cylinder viscometer**
Sketch \( v_\Theta \) vs. \( r \) when the inner cylinder rotates very slowly at angular velocity \( \Omega \), and the outer cylinder is stationary.

(b) **Parallel-plate viscometer**
Sketch \( v_\Theta \) vs. \( z \) when \( r/R = 0.5 \). The lower plate at \( z=0 \) is stationary and the upper plate at \( z=B \) rotates very slowly at angular velocity \( \Omega \).

(c) **Rotating sphere viscometer**
Sketch \( v_\phi \) vs. \( r \) at \( \Theta = \pi/2 \). The sphere rotates very slowly at angular velocity \( \Omega \) and the fluid is stationary far from the sphere.

(d) **Cone-and-plate viscometer**
Sketch \( v_\phi \) vs. \( \Theta \) at \( r/R = 0.5 \), when the angle of the cone \( \Theta_1 \) is \( 80^0, 85^0 \) & \( 89^0 \). Put all three curves on one set of axes and indicate the value of \( \Theta_1 \) that corresponds to each curve. The cone rotates very slowly at angular velocity \( \Omega \), and the plate at \( \Theta = \pi/2 \) is stationary.

(e) The fluid velocity profile in the cone-and-plate viscometer is "similar" to the fluid velocity profile in one of the other three viscometers listed above. Identify this other viscometer and explain briefly why the fluid velocity profiles are "similar".

**Problem #3**
Use information in *Transport Phenomena* on pages#55-56 in the 2\textsuperscript{nd} edition and page#53 in the 1\textsuperscript{st} edition to develop a relation between the friction factor and the Reynolds number for laminar flow of an incompressible Newtonian fluid between two concentric cylinders, where flow in the z-direction \( v_z(r) \) is driven by a gradient in dynamic pressure. The radius ratio of the inner to the outer cylinder is \( \kappa = R_{\text{inner}}/R_{\text{outer}} < 1 \). It is necessary to consider dynamic forces exerted by the fluid on both solid surfaces, but it is not necessary to simplify your algebraic expression to obtain a concise result for \( f \) vs. \( Re \). The characteristic length, or effective diameter, in the definition of the Reynolds number...
for flow on the shell side of the double-pipe heat exchanger is the difference between the outer and inner cylindrical diameters, \(2R_{\text{outer}}(1-\kappa)\).

**Problem#4**
The volumetric flowrate, \(Q = \pi R^2 <v_z>\), of an incompressible Newtonian fluid through a smooth straight tube of radius \(R\) and length \(L\) increases by a factor of 4. This 4-fold increase in \(Q\) causes the Reynolds number to increase from \(5 \times 10^3\) to \(2 \times 10^4\).

(a) Does the \(z\)-component of the dynamic interfacial shear force exerted by the fluid on the stationary wall at \(r=R\) increase, decrease, remain constant, or is it too complex to obtain a quantitative estimate of the change in this interfacial force?

(b) By what factor does the \(z\)-component of the dynamic interfacial shear force from part (a) change? A factor of 1 indicates no change, a factor less than 1 indicates a decrease, and a factor greater than 1 indicates an increase in the interfacial force. A numerical answer is required, here.

**Problem#5**
It should be obvious that the terminal velocity of a bowling ball in air is much larger than the terminal velocity of a popped corn in air. However, in both cases, a steady state force balance on the object that accounts for buoyancy, gravity, and hydrodynamic drag reveals that:

\[
\log \{ v_{\text{terminal}} \} \approx \zeta \log (\rho_{\text{solid}} - \rho_{\text{air}})
\]

where \(\zeta\) is the scaling law exponent, and \(\rho_{\text{solid}}\) corresponds to either the bowling ball or the popped corn.

(a) What is the value of \(\zeta\) if \(Re_{\text{BowlingBall}} \approx 200,000\)?

(b) What is the value of \(\zeta\) if \(Re_{\text{PoppedCorn}} \approx 0.1\)?

**Problem#6**
Consider a baseball that is 3 inches in diameter (i.e., 0.25 ft) moving at a velocity of 100 miles/hr. (i.e., 147 ft/sec) through stagnant air at 20\(^\circ\)C having a kinematic viscosity (or momentum diffusivity) of \(1.6 \times 10^{-4}\) ft\(^2\)/sec off the bat of Roger Maris in the 6\(^{th}\) inning of the 6\(^{th}\) game of the 1964 World Series (for sports trivia fans, the St. Louis Cardinals beat the New York Yankees, 4 games to 3, and right-handed pitcher Bob Gibson, who won games #5 and #7 with 22 strikeouts in those two games, was the MVP of the series). The Reynolds number is \(2.3 \times 10^5\), based on the sphere diameter. Provide a qualitative ranking of the distance that the baseball will travel if the game is played at each of the
following locations. The temperature of the air, wind conditions, initial velocity of the baseball and its angle of inclination are the same in each location.

(a) Texas Rangers stadium in Arlington, Texas near Dallas where Nolan Ryan pitched without air conditioning and sweated profusely,
(b) Minute-Maid Park in Houston, Texas where the domed stadium is air conditioned to remove unwanted moisture,
(c) Leadville Giants minor league stadium which sits at an elevation of 10,150 feet at the base of Mount Elbert,
(d) Coors' Field in Denver at an elevation of 5280 feet (don't account for the fact that this stadium and major league baseball did not exist in Denver during the Roger Maris era).

Problem#7
A crazy cyclist was riding a bicycle in the horrendous westerly winds when Channel 9 weather experts estimated that the intensity of convective momentum flux was "off of the charts". Use arrows to represent vectors and illustrate the motion of a cyclist in a crosswind, where the wind is blowing perpendicular to the path of the cyclist. Then, draw a vector that illustrates the direction in which the fluid (i.e., air) exerts a hydrodynamic drag force on the submerged object (i.e., the cyclist). Assume that the wind is blowing from the west at 30 miles per hour, and that the cyclist is riding north at 15 miles per hour.

These are additional topics in fluid dynamics which were not discussed during the first 7 or 8 weeks of the course, but they could be mentioned in connection with analogous topics in mass transfer

(a) Why is the Reynolds number important?
(b) Why is the Reynolds number calculated for all momentum transport problems?

Dimensional analysis of the equations governing momentum transport
Dynamic similarity and scaling concepts

Macroscopic mass balances
(a) Application of the unsteady state mass balance and the Hagen-Poiseuille law to calculate the time required to drain capillary tubes with cylindrical or spherical bulbs
(b) Parameters that effect the capillary constant in the rheology laboratory experiments

**Macroscopic Mass Balance**

**Important Considerations in the Development and Use of the Unsteady State Macroscopic Mass Balance**

1. Accumulation rate process is balanced by the net rate of input, since there are no sources or sinks of overall fluid mass.

2. Each term in the macroscopic mass balance has dimensions of mass per time.

3. The accumulation rate process on the left-hand side of the equation requires a time derivative for unsteady state analysis. Ordinary differential equations must be solved to analyze the transient behaviour of a system.

4. Mass flux provides a convective mechanism in the macroscopic mass balance due to fluid flow across the inlet/outlet fictitious planes that bound the system.

5. The divergence of convective mass flux, via the “del” operator (i.e., whom Mr. & Mrs. Shannon named their only son after), accounts for the net rate of output (i.e., output – input) on the right-hand side of the macroscopic mass balance.

6. One equates rates of input to rates of output at steady state, after the transient response of the system decays to zero. Steady state response of a system is analyzed by solving algebraic equations.

7. If a system initially contains a given amount of mass, then this quantity appears in the integrated form of the unsteady state macroscopic mass balance via the initial condition, most likely at t=0, but not as a rate of input or a rate of output.

8. One should only analyze the transient response of a system during a period of time when there are no abrupt changes in rates of input or rates of output. If the input and/or output streams change smoothly, then it is acceptable to analyze the system over a timescale that includes these smooth changes.
If it is necessary to analyze the transient response of a system over several different time intervals because rates of input and/or output change abruptly, or discontinuously, then one must be sure that the mass of the system is continuous from one time interval to the next. No principles are violated if the slope of total system mass vs. time changes abruptly from the end of one time interval to the beginning of the next interval, in response to an abrupt change in rates of input and/or output.

Total system mass achieves an extremum (i.e., maximum or minimum) when the sum of all inlet mass flowrates equals the sum of all outlet mass flowrates.

For a system that is well mixed in an agitated tank, for example, the overall mass density within the tank and in the exit stream might be time-dependent but it will not depend on spatial coordinates due to the mixing process. Hence, one manipulates the accumulation rate process on the left-hand side of the macroscopic mass balance as follows:

\[
\frac{d}{dt} m_{\text{total}} = \frac{d}{dt} \int \int \int \rho dV = \frac{d}{dt} \{ \rho V \} = \rho \frac{dV}{dt} + V \frac{d\rho}{dt}
\]

The system volume will change with time and, hence, \( \frac{dV}{dt} \) will be nonzero if the sum of all inlet volumetric flow rates is different from the sum of all outlet volumetric flow rates. The time-rate-of-change of total system volume is calculated from the difference between the sum of all inlet volumetric flow rates and the sum of all outlet volumetric flow rates. These statements are valid when the system corresponds to fluid in a tank, not fluid plus “empty space”, for example, as the tank is drained.

The most generalized form of the unsteady state macroscopic mass balance for systems with variable volume, multiple inlet/outlet streams, and no sources or sinks of overall fluid mass is:

\[
\frac{d}{dt} m_{\text{total}} = \rho \frac{dV}{dt} + V \frac{d\rho}{dt} = \sum_{\text{inlet stream } i} \{ \rho Q \}_i - \sum_{\text{outlet stream } j} \{ \rho Q \}_j
\]

where \( \rho \) is fluid density and \( Q \) represents volumetric flow rate with respect to the appropriate fictitious inlet or outlet plane.
(14) The time constant for the transient response of any system with inlet and outlet streams is the residence time $\tau$. When the sum of all inlet volumetric flow rates is balanced by the sum of all outlet volumetric flow rates and the total system volume remains constant, approximately five residence times (i.e., $5\tau$) are required for the transient response to decay to within 99.7% of the new steady state response.

**Transient and Steady State Analysis of the Dilution of a Salt Solution in a Well-Mixed Vessel**

A storage vessel contains 1000 Litres of salt solution with a mass of $1.05 \times 10^3$ kg, and the initial salt concentration is 50 grams per Litre. At time $t=0$, salt-free water enters the vessel at a rate of 9 Litres per minute. The outlet valve is opened at $t=0$ and salt solution exits the vessel at a rate of 10 Litres per minute. Hence, the volume of fluid in the tank decreases uniformly at a rate of 1 Litre per minute. Perfect mixing homogenizes the concentration of salt in the liquid within the tank. There is also a large excess of solid salt at the bottom of the tank that dissolves into solution at a steady rate of 5 grams per minute throughout the entire operation of the system.

a) Write all of the equations that are required to calculate the salt mass fraction in the fluid phase within the vessel and in the exit stream. The salt mass fraction is defined as $\rho_{\text{Salt}}/\rho_{\text{Solution}}$, where each density $\rho$ is expressed with respect to the solution volume.

b) Use an ODE solver to generate a quantitative graph of the mass fraction of salt in the tank and in the exit stream as a function of time.

c) Calculate the steady state mass fraction of salt in the exit stream.

d) Does the system reach 99.9% of its steady state operating point before the vessel is drained? If your answer to this question is YES, then calculate the time required for the salt mass fraction in the exit stream to reach 99.9% of its steady state value. In other words, find the value of time $t$ that satisfies the following equation.

$$\frac{\omega_{\text{Salt,Initial}} - \omega_{\text{Salt}}(t)}{\omega_{\text{Salt,Initial}} - \omega_{\text{Salt}}(t > 5\tau)} = 0.999$$
where \( \omega_{\text{Salt}} = \frac{\rho_{\text{Salt}}}{\rho_{\text{Solution}}} \) is the salt mass fraction.

(e) Perform steady state analysis of this well-mixed 1000-Litre vessel when \( Q_{\text{in}} = Q_{\text{out}} = Q = 10 \) Litre/minute and the rate at which solid salt dissolves into the fluid within the vessel is given by:

\[
\text{Dissolution Rate [grams/min]} = 2\left\{\rho_{\text{Salt,Interface}} - \rho_{\text{Salt}}(t)\right\}
\]

where \( \rho_{\text{Salt,Interface}} \) [= grams/Litre] represents the equilibrium liquid phase mass density (i.e., solubility) of salt at the solid-liquid interface via thermodynamic considerations. This dissolution rate is time-dependent because \( \rho_{\text{Salt}}(t) \) decreases during the dilution process. There is a large excess of salt that does not dissolve completely into the liquid phase during the time frame of the dilution process. Is the following expression for the steady state salt mass fraction correct?

\[
\omega_{\text{Salt}}(t \to \infty) = \frac{1}{1 + 6\frac{\rho_{H_2O}}{\rho_{\text{Salt,Interface}}}}
\]

The factor of 6 in the denominator of the previous equation is derived from the sum of \( 1 + \frac{Q}{2} \), and 2 is the coefficient of the mass density difference (i.e., driving force) on the right side of the dissolution rate expression.

**Answer to part (e):**

Begin by writing unsteady state mass balances for the overall solution and the salt, separately. Then, set the accumulation terms to zero at steady state and solve two coupled linear algebraic equations for \( \rho_{\text{Solution}} \) and \( \rho_{\text{Salt}} \) when the transient behaviour vanishes. For example, when the inlet and outlet flowrates are equal and the overall solution volume remains constant;

\[
V_{\text{Solution}} \frac{d\rho_{\text{Solution}}}{dt} = Q\left\{\rho_{H_2O} - \rho_{\text{Solution}}\right\} + k_{\text{MTC}}S\left\{\rho_{\text{Salt,Interface}} - \rho_{\text{Salt}}\right\} \Rightarrow 0
\]

\[
V_{\text{Solution}} \frac{d\rho_{\text{Salt}}}{dt} = 0 - Q\rho_{\text{Salt}} + k_{\text{MTC}}S\left\{\rho_{\text{Salt,Interface}} - \rho_{\text{Salt}}\right\} \Rightarrow 0
\]
Obtain an expression for the steady state mass density of salt in solution from the second equation:

$$\rho_{\text{Salt}} = \left\{ \frac{k_{\text{MTC}} S}{Q + k_{\text{MTC}} S} \right\} \rho_{\text{Salt,interface}}$$

Now, use this result to solve for $\rho_{\text{Solution}}$ from the overall mass balance (i.e., the first equation) when $t \Rightarrow \infty$. One obtains;

$$\rho_{\text{Solution}} = \rho_{H_2O} + \frac{k_{\text{MTC}} S}{Q} \left\{ \rho_{\text{Salt,interface}} - \rho_{\text{Salt}} \right\}$$

The ratio of these steady state mass densities yields the required expression for the mass fraction of salt in the well-mixed vessel and in the exit stream. This answer can be verified numerically by using specific parametric values for $V_{\text{Solution}}$, $Q$, $k_{\text{MTC}} S$, and $\rho_{\text{Salt,interface}}$, together with any reasonable initial conditions, solving both unsteady state mass balances, and inspecting the numerical solution to these coupled ODE’s when time $t$ is greater than five residence times

$$\omega_{\text{Salt}}(t \Rightarrow \infty) = \frac{\rho_{\text{Salt}}}{\rho_{\text{Solution}}} = \frac{\rho_{\text{Salt}}}{\rho_{H_2O} + \frac{k_{\text{MTC}} S}{Q} \left\{ \rho_{\text{Salt,interface}} - \rho_{\text{Salt}} \right\}}$$

$$= \frac{1}{\frac{\rho_{H_2O}}{\rho_{\text{Salt}}} + \frac{k_{\text{MTC}} S}{Q} \left\{ \rho_{\text{Salt,interface}} - \rho_{\text{Salt}} \right\}}$$

$$= \frac{1}{\frac{\rho_{H_2O}}{\rho_{\text{Salt,interface}}} \left\{ 1 + \frac{Q}{k_{\text{MTC}} S} \right\} + \frac{k_{\text{MTC}} S}{Q} \left\{ 1 + \frac{Q}{k_{\text{MTC}} S} - 1 \right\}}$$

$$= \frac{1}{1 + \frac{\rho_{H_2O}}{\rho_{\text{Salt,interface}}} \left\{ 1 + \frac{Q}{k_{\text{MTC}} S} \right\}}$$

(f) For typical operating conditions in this vessel-draining problem, why will $\rho_{\text{Salt,interface}}$ be greater than the initial mass density of salt in solution, $\rho_{\text{Salt}}(t=0)$?
Transient analysis of draining an incompressible Newtonian fluid from a spherical bulb with a tilted capillary tube to simulate the performance of capillary viscometers for the determination of momentum diffusivities

This problem combines the unsteady state macroscopic mass balance and the Hagen-Poiseuille law for laminar tube flow, together with the volume of fluid in a partially filled sphere. The overall objectives are to (i) predict the capillary constant “b”, based solely on geometric parameters of the viscometer, and (ii) compare this prediction with experimental values obtained by calibrating a capillary viscometer using fluids with known viscosity and density. The system is defined as fluid within the bulb plus the capillary, and one seeks the time required to drain only the bulb above a capillary that is oriented at angle Θ with respect to gravity. Hence, this is an example of the unsteady state macroscopic mass balance where the fictitious inlet plane “floats” on the upper surface of liquid in the bulb such that the average velocities of the fluid and the surface are equal. Consequently, there is no contribution from convective mass transfer across the inlet plane. Fluid flow across the stationary outlet plane at the exit from the capillary is described by the Hagen-Poiseuille law for incompressible Newtonian fluids. The macroscopic mass balance for an incompressible fluid with time-varying system volume, no inlet contribution, and one stationary outlet plane reduces to;

$$\rho \frac{dV_{system}}{dt} = -\rho Q_{HP} = -\rho \frac{\pi R_{Tube}^4 \Delta P}{8 \mu L}$$

where the capillary has radius $R_{Tube}$ and length $L$, and $P$ represents dynamic pressure. Laminar flow occurs through a cylindrical capillary tube of length $L$, regardless of whether the capillary is vertical or tilted at angle $\Theta$ with respect to gravity. The angle of tilt is considered in the dynamic pressure difference $\Delta P$ from tube inlet to tube outlet. If $h(t)$ describes the height of fluid within the spherical bulb above the capillary at any time $t$, and the “zero of potential energy” is placed arbitrarily at the exit from the capillary, then fluid pressure at the capillary entrance is $p_{ambient} + \rho gh(t)$, based on approximate hydrostatic conditions in the bulb, and dynamic pressure at the capillary entrance is given by the sum of fluid pressure and gravitational potential energy per unit volume of fluid. Since the capillary entrance is at higher elevation than the capillary exit, by a distance $L \cos \Theta$, one evaluates dynamic pressure at the capillary inlet as follows;

$$P_{inlet} = p_{ambient} + \rho gh(t) + \rho g L \cos \Theta$$
There is no contribution from gravitational potential energy to dynamic pressure at the capillary exit because it coincides with the potential energy reference plane. Ambient pressure exists on the upper surface of liquid in the bulb and at the capillary exit. Hence, the dynamic pressure difference $\Delta P = P_{\text{inlet}} - P_{\text{outlet}}$ is given by $\rho g (h + L \cos \Theta)$. One must solve the following time-dependent ODE to relate momentum diffusivity to efflux time:

$$
\frac{dV_{\text{system}}}{dt} = \frac{d}{dt} \left[ V_{\text{PartiallyFilledSphere}} + \pi R_{\text{Tube}}^2 L \right] = \frac{d}{dt} V_{\text{PartiallyFilledSphere}} \quad = - \frac{\pi R_{\text{Tube}}^4 g}{8(\mu/\rho)L} \left\{ h(t) + L \cos \Theta \right\}
$$

Fluid volume within the capillary tube is constant during the analysis of efflux times because one measures the time required to drain the bulb, not the capillary. The next task is to evaluate the volume of fluid in a partially filled sphere of radius $R_{\text{Sphere}}$ when the fluid achieves height $h(t)$. This calculation is performed in cylindrical coordinates by stacking an infinite number of cylinders with infinitesimal thickness $dz$ and radius $\omega(z)$, such that $\omega(z)$ vanishes when $z = 0$ and $z = 2R_{\text{Sphere}}$, but $\omega(z) = R_{\text{Sphere}}$ when the sphere is 50% filled. Let the spherical bulb sit on the origin of an $xyz$-coordinate system such that the center of the sphere is found at a distance $z = R_{\text{Sphere}}$ upward from the origin in the $z$-direction. If the sphere is filled with fluid to height $z$ that can be greater than or less than the sphere radius, then the liquid surface is circular and the following relation allows one to predict the radius $\omega(z)$ of the circular surface of liquid:

$$
\begin{align*}
\left\{ \omega(z) \right\}^2 + (z - R_{\text{Sphere}})^2 &= R_{\text{Sphere}}^2 \\
\left\{ \omega(z) \right\}^2 &= 2zR_{\text{Sphere}} - z^2
\end{align*}
$$

Now, calculate the volume of an infinite number of cylinders with radius $\omega(z)$ and thickness $dz$ stacked upon each other using a differential volume element in cylindrical coordinates. When fluid achieves height $h(t)$ in this partially filled sphere, one evaluates the following triple integral to obtain the liquid volume:

$$
V_{\text{PartiallyFilledSphere}} = \iiint dV = \int_0^{2\pi} d\Theta \int_0^{h(t)} \int_0^{\omega(z)} dr dz = \pi \int_0^{h(t)} \left\{ \omega(z) \right\}^2 dz
$$

$$
= \pi \int_0^{h(t)} \left\{ 2zR_{\text{Sphere}} - z^2 \right\} dz = \pi \left\{ R_{\text{Sphere}} h^2(t) - \frac{1}{3} h^3(t) \right\}
$$

As expected, the liquid volume vanishes when $h=0$, it achieves the normal volume of a sphere [i.e., $(4/3)\pi R_{\text{Sphere}}^3$] when $h=2R_{\text{Sphere}}$, and it achieves 50% of the normal sphere...
volume when $h=R_{\text{Sphere}}$. The time-rate-of-change of system volume in the unsteady state mass balance is obtained via differentiation of the volume of this partially filled sphere with respect to time, because the fluid height $h(t)$ is time-dependent as the sphere drains. One obtains the following result via separation of variables;

\[
\frac{d}{dt} V_{\text{PartiallyFilledSphere}} = \pi \left\{ 2R_{\text{sphere}} h(t) - h^2(t) \right\} \frac{dh}{dt} = -\frac{\pi R_{\text{Tube}}^4 g}{8(\mu / \rho) L} \left\{ h(t) + L \cos \Theta \right\}
\]

with the following integration limits; $h = 2R_{\text{Sphere}}$ initially at $t = 0$, and $h = 0$ at the efflux time required to drain the bulb. The results of this analysis yield the functional dependence of the capillary constant “$b$”;

\[
b = \frac{gR_{\text{Tube}}^4}{8L} \int_0^{t_{\text{efflux}}} \frac{h(2R_{\text{Sphere}} - h)}{(h + L \cos \Theta)} dh
\]

The capillary constant depends on the (i) dimensions of the capillary tube, (ii) orientation of the capillary with respect to gravity, (iii) volume (or radius) of the spherical bulb, and (iv) strength of the gravitational field. The capillary constant does not depend on temperature or the physical properties of the fluid, provided that the fluid is incompressible and Newtonian.

**Detailed evaluation of the capillary constant and comparison with experimental results.** The next task is to evaluate the complex integral expression in the previous equation for the capillary constant. Begin with the following substitution so that the denominator of the integrand can be rewritten in terms of only one variable $\Psi$. Let $\Psi = h + L \cos \Theta$. Integration proceeds as follows;

\[
\int_0^{2R_{\text{Sphere}}} \frac{h(2R_{\text{Sphere}} - h)}{(h + L \cos \Theta)} dh = \int_{L \cos \Theta}^{2R_{\text{Sphere}} + L \cos \Theta} \frac{(\Psi - L \cos \Theta)(2R_{\text{Sphere}} - \Psi + L \cos \Theta)}{\Psi} d\Psi
\]
The integrand reduces to a simple function of $\Psi$ that can be integrated rather easily;

$$\frac{(\Psi - L \cos \Theta)(2R_{sphere} - \Psi + L \cos \Theta)}{\Psi} = 2R_{sphere} - \Psi + 2L \cos \Theta - \frac{L \cos \Theta(2R_{sphere} + L \cos \Theta)}{\Psi}$$

Integration from $L \cos \Theta$ to $2R_{sphere} + L \cos \Theta$ yields the following expression;

$$\int_{L \cos \Theta}^{2R_{sphere} + L \cos \Theta} \left[ 2(R_{sphere} + L \cos \Theta) - \Psi - \frac{L \cos \Theta(2R_{sphere} + L \cos \Theta)}{\Psi} \right] d\Psi$$

$$= 2\left(R_{sphere} + L \cos \Theta\right)\left(2R_{sphere} + L \cos \Theta - \Psi\right) - \frac{1}{2}\left[(2R_{sphere} + L \cos \Theta)^2 - (L \cos \Theta)^2\right]$$

$$- L \cos \Theta\left(2R_{sphere} + L \cos \Theta\right) \ln \frac{2R_{sphere} + L \cos \Theta}{L \cos \Theta}$$

$$= 2R_{sphere}\left(R_{sphere} + L \cos \Theta\right) - L \cos \Theta\left(2R_{sphere} + L \cos \Theta\right) \ln \left[1 + \frac{2R_{sphere}}{L \cos \Theta}\right]$$

Finally, the capillary constant can be written in terms of the gravitational acceleration constant and several geometric parameters that characterize the spherical bulb and the tilted capillary tube;

$$b = \frac{8R_{Tube}^4}{8L} \frac{2R_{sphere}\left(R_{sphere} + L \cos \Theta\right) - L \cos \Theta\left(2R_{sphere} + L \cos \Theta\right) \ln \left[1 + \frac{2R_{sphere}}{L \cos \Theta}\right]}{2R_{sphere}\left(R_{sphere} + L \cos \Theta\right) - L \cos \Theta\left(2R_{sphere} + L \cos \Theta\right) \ln \left[1 + \frac{2R_{sphere}}{L \cos \Theta}\right]}$$

Geometric parameters and capillary constants are summarized below for two different Cannon-Fenske capillary viscometers. If longer efflux times are desirable to minimize errors associated with end effects and experimental reproducibility, then one should use a viscometer with a smaller capillary constant.

<table>
<thead>
<tr>
<th>Geometric Characteristics</th>
<th>Size#100</th>
<th>Size#150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulb volume, assumed to be spherical (mL)</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Bulb radius, $R_{sphere}$ (cm)</td>
<td>1.24</td>
<td>1.24</td>
</tr>
<tr>
<td>Capillary length, L (cm)</td>
<td>7.6</td>
<td>6.7</td>
</tr>
<tr>
<td>Capillary radius, $R_{Tube}$ (cm)</td>
<td>0.041</td>
<td>0.05</td>
</tr>
<tr>
<td>Capillary tilt angle with respect to gravity (degrees)</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Capillary constant, predicted (cm$^2$/sec$^2$)</td>
<td>1.53x10$^{-4}$</td>
<td>3.45x10$^{-4}$</td>
</tr>
<tr>
<td>Capillary constant, experimental (cm$^2$/sec$^2$)</td>
<td>1.5x10$^{-4}$</td>
<td>3.5x10$^{-4}$</td>
</tr>
</tbody>
</table>
Draining Power-law fluids from a right circular cylindrical tank via a tilted capillary tube

The capillary viscometer in the previous section is re-analyzed when an incompressible power-law fluid is drained from a cylindrical tank instead of a spherical bulb. The unsteady state mass balance with no inlet stream and one outlet is analogous to the previous development, except that it is necessary to (i) modify the time-varying system volume and (ii) use a generalized expression for the volumetric flowrate of non-Newtonian fluids through straight tubes with radius $R_{\text{tube}}$ and length $L$ in the laminar regime. The dynamic pressure difference from capillary inlet to capillary outlet in the generalized *Hagen-Poiseuille law* for tilted tubes is identical to that in the previous section if $h(t)$ represents the variable height of fluid in a cylindrical tank. Hence, the starting point for this analysis, to drain the tank but not the capillary tube, is;

$$\frac{dV_{\text{system}}}{dt} = \frac{d}{dt} \left[ V_{\text{PartiallyFilledTank}} + \pi R_{\text{Tube}}^2 L \right] = \pi R_{\text{Tank}}^2 \frac{dh}{dt} = -\frac{n}{1+3n} \pi R_{\text{Tube}}^{3+1/n} \left[ \frac{\rho g}{2mL} \left\{ h(t) + L \cos \Theta \right\} \right]^{1/n}$$

If the initial height of fluid in the tank is $H$ (i.e., $h=H$ at $t=0$), then one defines the *half-time* $t_{1/2}$ and the *efflux time* $t_{\text{efflux}}$ as $h=H/2$ at $t=t_{1/2}$ and $h=0$ at $t=t_{\text{efflux}}$, respectively. The remainder of this analysis compares half-times and efflux times for incompressible Newtonian fluids, when $n=1$ and $m=\mu$. The overall objective is to prove, unequivocally, that the efflux time is greater than twice the half-time for any set of initial conditions and viscometer geometries, including all orientations (i.e., angle $\Theta$) of the exit capillary with respect to gravity. For fluids that obey Newton’s law of viscosity, the previous expression reduces to;

$$R_{\text{Tank}}^2 \frac{dh}{dt} = -\frac{g R_{\text{Tube}}^4}{8(\mu / \rho)L} \left\{ h(t) + L \cos \Theta \right\}$$

This unsteady state macroscopic mass balance for incompressible Newtonian fluids yields a much simpler result for the momentum diffusivity via the half-time or the efflux time, relative to the final expression for $\mu / \rho$ from the previous section when a spherical bulb is drained. From a practical viewpoint, there are two geometric parameters (i.e., $H$ and $R_{\text{Tank}}$) that must be related to the *volume of the bulb* above the capillary tube (i.e., $V_{\text{Bulb}} \approx \pi R_{\text{Tank}}^2 H$). In contrast, when the bulb volume is modeled as a sphere instead of a right circular cylinder, one identifies the sphere radius via $V_{\text{Bulb}} = (4/3)\pi R_{\text{Sphere}}^3$. Hence, even though two parameters (i.e., $H$ and $R_{\text{Tank}}$) are related by one equation (i.e.,
Volume_{Bulb} \approx \pi R_T^2 H, one predicts the momentum diffusivity for this “tank-draining” problem as follows;

$$\mu = \frac{g R_T^4}{\rho} \frac{\int_0^{t_{\text{efflux}}} dt}{8 R_T^2 \int_0^H \frac{dh}{h + L \cos \Theta}} = \frac{g R_T^4}{\rho} \frac{t_{1/2}}{8 R_T^2 \int_0^{H/2} \frac{dh}{h + L \cos \Theta}}$$

Obviously, one can predict momentum diffusivities for incompressible Newtonian fluids via laboratory measurements of efflux times or half-times. The capillary constant based on efflux times is smaller than the capillary constant based on half-times, because the product of the appropriate capillary constant and either the half-time or the efflux time yields the momentum diffusivity which is insensitive to the time required to drain either one-half of the total volume of the tank (or bulb) or the total volume of fluid above the capillary tube. The rather simple relation between half-time and efflux time, based on the previous equation, is;

$$\frac{t_{\text{efflux}}}{t_{1/2}} = \frac{\int_0^H \frac{dh}{h + L \cos \Theta}}{\int_0^{H/2} \frac{dh}{h + L \cos \Theta}} = \frac{\ln \left\{ \frac{H + L \cos \Theta}{0 + L \cos \Theta} \right\}}{\ln \left\{ \frac{H + L \cos \Theta}{\frac{H}{2} + L \cos \Theta} \right\}} > 2$$

Numerical substitutions for the (i) initial height $H$ of fluid in the cylindrical tank, (ii) length $L$ of the capillary tube, and (iii) angle of tilt $\Theta$ with respect to gravity reveal that the ratio of $t_{\text{efflux}}$ to $t_{1/2}$ is always greater than 2. In fact, this ratio (i.e., $t_{\text{efflux}}/t_{1/2}$) becomes significantly greater than 2 when $H$ is larger, $L$ is smaller, and $\Theta$ approaches $\pi/2$. When the capillary tube is horizontal (i.e., $\Theta = \pi/2$), it is important to emphasize that the half-time is finite;

$$t_{1/2} = \frac{8(\mu/\rho)R_T^2}{g R_T^4} \ln \left\{ \frac{H + L \cos \Theta}{\frac{H}{2} + L \cos \Theta} \right\} \Rightarrow \frac{8(\mu/\rho)R_T^2}{g R_T^4} \ln \{2\}$$

but an infinite amount of time is required to drain the total volume of fluid in the tank. These trends can be rationalized in terms of a dynamic pressure difference from capillary inlet to capillary outlet that decreases at longer times because the hydrostatic pressure at the capillary inlet is directly proportional to the instantaneous height of fluid in the reservoir.
**Macroscopic Momentum Balance**

**Important Considerations and Assumptions in the Development and Use of the Macroscopic Momentum Balance; Steady State and Unsteady State Analysis**

1. The system is defined as fluid within control volume V, bounded by solid surfaces and fictitious inlet and outlet planes.

2. The total system mass within V, which is a scalar, is defined by;

   \[ m_{total} = \iiint_V \rho \, dV \]

3. The total momentum within V, which is a vector, is defined by;

   \[ P_{total} = \iiint_V \rho \mathbf{v} \, dV \]

4. The average fluid velocity \(<\mathbf{v}>\) is a scalar in the contributions from convective momentum flux.

5. Pressure and viscous forces exerted by the fluid on all solid surfaces in contact with the fluid are represented by \( \mathbf{F}_{\text{Fluid on Solid}} \), which contains all of the important contributions due to the viscous stress tensor.

6. Normal pressure forces are much more important than normal viscous forces across all of the fictitious inlet and outlet planes that bound fluid within the control volume.

7. The gravitational acceleration vector \( \mathbf{g} \) is essentially constant within the system. Hence;

   \[ \iiint_V \rho \mathbf{g} \, dV = \mathbf{g} \iiint_V \rho \, dV = m_{total} \mathbf{g} \]

   This is the only external body force in the macroscopic momentum balance. Forces due to electric and magnetic fields are not considered.

8. Across the fictitious inlet and outlet planes which bound fluid within the system, one identifies unit normal vectors \( \mathbf{n}_{\text{inlet}} \) and \( \mathbf{n}_{\text{outlet}} \), respectively, which are oriented
in the direction of primary fluid flow, not necessarily pointing outward from the system to the surroundings.

(9) A qualitative statement of the macroscopic momentum balance is given by:

\[ 1 = 2 - 3 + 4 - 5 - 6 + 7 \]

where

1 is the accumulation of fluid momentum within control volume V
2 represents the rate of input due to convective momentum flux across surface S
3 represents the rate of output due to convective momentum flux across surface S
4 represents normal pressure forces acting across the fictitious inlet planes
5 represents normal pressure forces acting across the fictitious outlet planes
6 represents pressure and viscous forces exerted by the fluid on all solid surfaces
7 represents the external body force due to gravity

(10) A quantitative statement of the macroscopic momentum balance, with terms that correspond to those numbered from 1 to 7 above, is given by:

\[
\frac{dP_{\text{total}}}{dt} = \sum_{\text{inlet planes}} \left( \rho v (v - v_{\text{Surface}}) S \right) n_{\text{inlet}} - \sum_{\text{outlet planes}} \left( \rho v (v - v_{\text{Surface}}) S \right) n_{\text{outlet}} \\
+ \sum_{\text{inlet planes}} \left( pS \right) n_{\text{inlet}} - \sum_{\text{outlet planes}} \left( pS \right) n_{\text{outlet}} \\
- F_{\text{Fluid} \rightarrow \text{Solid}} + m_{\text{total}} g
\]

A word of caution is needed about the fact that the macroscopic momentum balance requires absolute pressure, not gauge pressure, because normal forces due to pressure stress are constructed from a product of pressure and surface area across which pressure acts. In particular, for inlet and outlet planes of a control volume that correspond to a reduction in flow cross-section, pressure forces written in terms of gauge pressure will be incorrect. When one evaluates differences in pressure forces across fictitious inlet and outlet planes that have the same area for unidirectional flow, the use of absolute or gauge pressures yields the same difference because the surface areas are the same and the pressure forces are subtracted. However, when the inlet and outlet planes are described by different surface areas, or when the flow is not unidirectional (i.e.,
the U-tube is an example where pressure forces across the inlet and outlet planes are additive), absolute pressure is required to evaluate pressure forces due to the vector nature of the linear momentum balance and the fact that pressure stress must be multiplied by the appropriate surface area.

Unsteady state applications of the macroscopic momentum balance to rocket propulsion

Consider a propulsion vehicle, like a rocket, with solid mass $m_{\text{rocket}}$ and initial liquid fuel mass $m_{\text{fuel,initial}}$ at time $t_0$ that is traveling vertically upward at constant velocity $v_{\text{propulsion}}$. The system is chosen as the solid rocket and its liquid fuel. Unsteady state analysis is required because, even though the system does not experience any acceleration at constant rocket velocity, the mass of the system decreases at longer time as fuel is burned to provide the required propulsion that maintains constant velocity. Exhaust gases from burned fuel can potentially escape the system at supersonic velocities on the order of 2000 meters/sec through a converging-diverging nozzle if the exit (i.e., back) pressure is low enough. Relative to a stationary frame of reference, the exhaust manifold with flow cross-sectional area $S_{\text{outlet}}$ moves upward at velocity $v_{\text{propulsion}}$. Since there are no inlet planes, and mass is neither depleted nor produced during the combustion reaction, the overall unsteady state macroscopic mass balance yields the following expression for the instantaneous mass of fuel in the system at time $t$ when combustion gases are ejected continuously across the converging-diverging nozzle at constant volumetric flowrate $Q_{\text{exhaust}} = v_{\text{exhaust}} S_{\text{outlet}}$:

$$\frac{d}{dt} m_{\text{system}} = -\rho_{\text{fuel,exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}}$$

$$m_{\text{system}} = m_{\text{rocket}} + m_{\text{fuel}}(t) = m_{\text{rocket}} + m_{\text{fuel,initial}} - \rho_{\text{fuel,exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}} (t - t_0)$$

where $\rho_{\text{fuel,exhaust}}$ is the mass density of the exhaust gas. The mass of the solid rocket $m_{\text{rocket}}$ remains constant during the analysis if booster stages do not separate from the space capsule. The total momentum of the system, and its time derivative, are calculated to evaluate the accumulation term on the left side of the unsteady state macroscopic momentum balance for systems that lose mass, in the form of liquid or gaseous fuel, at constant velocity;
\[
\begin{align*}
\dot{P}_{\text{total}} &= m_{\text{system}} v_{\text{propulsion}} = \left\{ m_{\text{rocket}} + m_{\text{fuel}} \right\} v_{\text{propulsion}} \\
\frac{d\dot{P}_{\text{total}}}{dt} &= v_{\text{propulsion}} \frac{dm_{\text{fuel}}}{dt} = -\rho_{\text{fuel, exhaust}} \left( v_{\text{exhaust}} + v_{\text{propulsion}} \right) S_{\text{outlet}} v_{\text{propulsion}}
\end{align*}
\]

There are no inlet planes across which mass enters the system, and there is only one fictitious outlet plane that provides an escape route for burned exhaust gases that could exceed “Mach 1”. Relative to a stationary frame of reference, the nozzle in the fictitious outlet plane moves upward at the velocity of the system, \(v_{\text{propulsion}}\), whereas the exit stream moves downward at velocity \(v_{\text{exhaust}}\). Consequently, the relative velocity of the exit stream is enhanced with respect to motion of the fictitious outlet plane in the opposite direction. The unsteady state macroscopic momentum balance reduces to;

\[
\frac{d\dot{P}_{\text{total}}}{dt} = -\rho_{\text{fuel, exhaust}} \left( v_{\text{exhaust}} + v_{\text{propulsion}} \right) S_{\text{outlet}} v_{\text{propulsion}} = -\rho_{\text{fuel, exhaust}} v_{\text{exhaust}} \left( v_{\text{exhaust}} + v_{\text{propulsion}} \right) S_{\text{outlet}} n_{\text{outlet}} - p_{\text{exhaust}} S_{\text{outlet}} n_{\text{outlet}} - F_{\text{Fluid -> Solid}} + m_{\text{system}} g
\]

On the right side of the previous equation, the interfacial frictional force exerted by fluid on all of the solid surfaces within the system and at its boundaries is typically neglected relative to the thruster force due to convective momentum flux, the “back-pressure” force (i.e., \(p_{\text{exhaust}} S_{\text{outlet}}\)), and the gravitational force. This approximation allows one to design propulsion vehicles under “ideal” conditions when hydrodynamic drag forces can be neglected. For interplanetary travel, space vehicles that escape the Earth’s atmosphere and gravitation field do not experience forces due to back-pressure, gravity, or hydrodynamic drag, and no thruster force is required to maintain effortless constant-velocity motion. In realistic situations closer to home, the final expressions below for the required initial mass of liquid fuel and the exhaust gas flowrate represent lower limits to obtain a given propulsion velocity. The gravitational force (i.e., last term on the right side of the previous equation) contains variable system mass because fuel is burned continuously to provide propulsion. Since the mass of liquid fuel decreases linearly with time to maintain constant exhaust mass flowrate of the gaseous combustion products, \(m_{\text{system}}\) in the gravitational force term is approximated by the average mass of the solid rocket and liquid fuel during the time frame of operation (i.e., \(t-t_0\)) of the system. Hence;
\[ m_{\text{system}} \Rightarrow \left\{ m_{\text{system}} \right\}_{\text{average}} \]

\[ = m_{\text{rocket}} + \frac{1}{2} \left( m_{\text{fuel, initial}} - \rho_{\text{fuel, exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}} (t - t_0) \right) \]

\[ = m_{\text{rocket}} + m_{\text{fuel, initial}} - \frac{1}{2} \rho_{\text{fuel, exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}} (t - t_0) \]

For the specific example where the propulsion velocity is vertically upward (i.e., in the positive z-direction) and exhaust gases are ejected vertically downward in the same direction as the gravitation acceleration vector, the z-component of the macroscopic momentum balance for constant-velocity motion in the absence of frictional forces reduces to:

\[ -2 \rho_{\text{fuel, exhaust}} Q_{\text{exhaust}} v_{\text{propulsion}} - \rho_{\text{fuel, exhaust}} v_{\text{propulsion}}^2 S_{\text{outlet}} = \rho_{\text{fuel, exhaust}} v_{\text{exhaust}}^2 S_{\text{outlet}} + p_{\text{exhaust}} S_{\text{outlet}} \left\{ m_{\text{rocket}} + m_{\text{fuel, initial}} - \frac{1}{2} \rho_{\text{fuel, exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}} (t - t_0) \right\} g \]

because the unit normal vector at the outlet plane, oriented in the primary direction of exhaust gas flow, points in the negative z-direction. This z-component force balance provides a relation between several design variables for rocket propulsion. If one identifies the mass flow rate of exhaust gases \( \omega_{\text{exhaust}} \), with respect to a stationary reference frame, as;

\[ \omega_{\text{exhaust}} = \rho_{\text{fuel, exhaust}} Q_{\text{exhaust}} = \rho_{\text{fuel, exhaust}} v_{\text{exhaust}} S_{\text{outlet}} \]

then the z-component of the unsteady state macroscopic momentum balance yields the following ideal rocket propulsion design equation;

\[ \frac{1}{\rho_{\text{fuel, exhaust}} S_{\text{outlet}}} \omega_{\text{exhaust}}^2 + \left\{ 2 v_{\text{propulsion}} + \frac{1}{2} g (t - t_0) \right\} \omega_{\text{exhaust}} + p_{\text{exhaust}} S_{\text{outlet}} + \rho_{\text{fuel, exhaust}} v_{\text{propulsion}} S_{\text{outlet}} \left\{ v_{\text{propulsion}} + \frac{1}{2} g (t - t_0) \right\} - \left\{ m_{\text{rocket}} + m_{\text{fuel, initial}} \right\} g = 0 \]

If the system operates for time, \( t-t_0 \), and ejects exhaust gases continuously at constant volumetric flowrate \( Q_{\text{exhaust}} \), then the initial mass of liquid fuel must be greater than;

\[ m_{\text{fuel, initial}} \geq \rho_{\text{fuel, exhaust}} (v_{\text{exhaust}} + v_{\text{propulsion}}) S_{\text{outlet}} (t - t_0) \]
Solution of the nonlinear equation for the mass flowrate of exhaust gases $\omega_{\text{exhaust}}$ requires the positive square root in the quadratic formula. When the initial mass of liquid fuel is greater than its minimum value, as required for continuous rocket propulsion, one obtains the following expression for $Q_{\text{exhaust}}$ to maintain constant-velocity motion during the time frame of system operation, $t-t_0$, if frictional forces are negligible:

$$Q_{\text{exhaust}} = \frac{S_{\text{outlet}}}{2} \left\{ \begin{array}{c} 2v_{\text{propulsion}} + \frac{1}{2}g(t-t_0) \\ 2v_{\text{propulsion}} + \frac{1}{2}g(t-t_0)^2 \\ 4 \rho_{\text{fuel,exhaust}} S_{\text{outlet}} v_{\text{propulsion}} S_{\text{outlet}} \left[ v_{\text{propulsion}} + \frac{1}{2}g(t-t_0) \right] \end{array} \right\}$$

The system of equations searches for positive values of $v_{\text{exhaust}}$, $Q_{\text{exhaust}}$, and $\omega_{\text{exhaust}}$. This occurs when the following inequality is satisfied:

$$(m_{\text{rocket}} + m_{\text{fuel,initial}})g > p_{\text{exhaust}} S_{\text{outlet}} + \rho_{\text{fuel,exhaust}} v_{\text{propulsion}} S_{\text{outlet}} \left[ v_{\text{propulsion}} + \frac{1}{2}g(t-t_0) \right]$$

**Problem**

Design propulsion vehicles with a solid mass of 1250 kg that move vertically upward from sea level at constant velocity until they achieve an altitude of 3000 meters. Liquid fuel is vaporized and burned at 500K. The gaseous combustion products (i.e., primarily CO$_2$ and H$_2$O), with an average molecular weight of 35 daltons, are ejected at 1 atmosphere total pressure (i.e., 101 kiloPascals = 1.01 x $10^5$ Newtons per square meter) through a converging-diverging nozzle that has a diameter of 1.5 feet at the exhaust manifold. It is acceptable to use the ideal gas law with $R_{\text{gas}} = 0.082$ Litre-atm/mol-K to estimate the exit gas molar density with dimensions of *gram-moles per Litre*. Since there are $10^3$ grams per kilogram, and $10^3$ Litres per cubic meter, mass densities in grams per Litre are equivalent to those with dimensions of kg/m$^3$. Use the MKS system of units to correlate the following quantities in tabular and graphical form for propulsion velocities between 10 meters per second and 40 meters per second, in increments of 1 meter per second.

(a) minimum initial mass of liquid fuel (kg)
(b) average velocity of exhaust gases through the exit manifold (m/s)
(c) volumetric flowrate of exhaust gases through the exit manifold (m³/s)
(d) mass flowrate of exhaust gases through the exit manifold (kg/s)

Effect of Propulsion Velocity on Exhaust Velocity

Effect of Propulsion Velocity on Initial Mass of Liquid Fuel
Steady state applications of the macroscopic momentum balance

(11) Steady state analysis of the macroscopic momentum balance is accomplished by neglecting the left hand side of the previous equation. Hence, the sum of all forces acting on the system must vanish. This is reasonable if the system does not (i) accelerate, (ii) gain mass, or (iii) lose mass, because each of these processes gives rise to a time rate of change of total system momentum.

(12) For incompressible fluids with constant density $\rho$, stationary inlet and outlet planes such that $v_{\text{Surface}}=0$, and fluid pressure which does not change much across the entire inlet plane or the outlet plane (but fluid pressure usually decreases from inlet to outlet), one rearranges the steady state macroscopic momentum balance to calculate the pressure and viscous forces exerted by the fluid in contact with all of the solid surfaces;

$$ E_{\text{Fluid} \rightarrow \text{Solid}} = \sum_{\text{inlet planes}} \rho \langle v^2 S \rangle n_{\text{inlet}} - \sum_{\text{outlet planes}} \rho \langle v^2 S \rangle n_{\text{outlet}} + \sum_{\text{inlet planes}} pS_n - \sum_{\text{outlet planes}} pS_n + m_{\text{total}} g $$

(13) Consider steady state one-dimensional flow through a straight horizontal tube with radius $R$ and length $L$. If there is no change in the flow cross-sectional area $S$ from inlet to outlet, then $\langle v \rangle_{\text{inlet}} = \langle v \rangle_{\text{outlet}}$ via the steady state macroscopic mass balance for incompressible fluids. Since the fluid does not change directions from inlet to outlet, the unit normal vectors in the primary flow direction across the fictitious inlet and outlet planes are oriented in the positive $z$-direction. Hence, contributions from convective momentum flux across the inlet and outlet planes cancel because (i) the fluid density doesn’t change if the fluid is incompressible, (ii) the cross-sectional area $S$ remains constant, and (iii) the average fluid velocity across the inlet and outlet planes is the same at steady state. The only surviving terms in the $z$-component of the previous steady state macroscopic momentum balance are, with $g_z = 0$;

$$ \{E_{\text{Fluid} \rightarrow \text{Solid}}\}_{z\text{-component}} = (p_{\text{inlet}} - p_{\text{outlet}})S = \pi R^2 \Delta p $$
Analysis of the vector force per unit area exerted by a moving fluid on the stationary wall of a tube at \( r=R \), with unit normal vector in the +r-direction, yields contributions from pressure and viscous forces, only. There is no contribution from convective momentum flux at the fluid-solid interface (i.e., @ \( r=R \)) because the “no-slip” boundary condition states that the fluid velocity vector must vanish at \( r=R \) if the wall is stationary. Furthermore, the tube wall at \( r=R \) is the only solid surface in contact with the fluid. The fictitious inlet and outlet planes are not solid surfaces. Hence, pressure stress acts in the \( r \)-direction and the \( 1^{\text{st}} \) subscript on the important scalar components of the viscous stress tensor is \( r \);

\[
\left\{ \text{VectorForce/Area} \right\}_{\text{Fluid->Solid} @ r=R} = \left\{ \delta_r (p + \tau_{rr}) + \delta_\theta \tau_{r\theta} + \delta_z \tau_{rz} \right\}_{r=R}
\]

It should seem reasonable to equate the \( z \)-component of the vector force per unit area exerted by the fluid on the solid surface from (14), multiplied by the lateral surface area (i.e., \( 2\pi RL \)), and the \( z \)-component of the pressure and viscous forces (actually 100\% due to \( \tau \)) exerted by the fluid in contact with all of the solid surfaces via the steady state macroscopic momentum balance in (13). Hence;

\[
2\pi RL \tau_{rz} (r = R) = \pi R^2 \Delta p
\]

\[
\tau_{rz} (r = R) = \frac{R\Delta p}{2L} = \frac{1}{2} R \left( -\frac{dp}{dz} \right)
\]

Steady state analysis of the \( z \)-component of the macroscopic momentum balance has been performed together with a microscopic analysis of forces exerted by fluids on simple solid surfaces, due to the momentum flux tensors. One obtains a relation, or balance, between viscous shearing forces on the lateral surface of the tube and normal pressure forces that act across the fictitious inlet and outlet planes at \( z=0 \) and \( z=L \). It is important to emphasize that macroscopic analysis provides an evaluation of \( \tau_{rz} \) only at the tube wall (i.e., \( r=R \)). Microscopic analysis of the linear momentum balance, known as the Equation of Motion, yields the viscous shear stress distribution throughout the fluid which is consistent with the previous equation when \( \tau_{rz}(r) \) is evaluated at \( r=R \). Even though there is an unlimited number of functions of radial position \( r \) that yield \( R \) when they are evaluated at the tube wall (i.e., \( r=R \)), the \( z \)-component of the microscopic linear momentum balance (i.e., Equation of Motion) reveals that the viscous shear stress profile is;

\[
\tau_{rz}(r) = \frac{1}{2} R \left( -\frac{dp}{dz} \right)
\]
One-dimensional flow through tilted tubes with no change in flow cross-sectional area. Let’s revisit steady state one-dimensional incompressible flow through the same straight tube of radius $R$ and length $L$, as described in (13) through (15) above. Now, the flow configuration is oriented at angle $\Theta$ (i.e., $0 \leq \Theta \leq \pi$) with respect to vertical, such that gravity plays a role when one balances forces in the primary direction of fluid flow, which is identified as the $z$-direction in cylindrical coordinates. The $z$-axis and the tube axis are aligned such that the tube wall is a simple surface at constant value of the radial coordinate. The unit normal vector at the fluid-solid interface is oriented in the radial direction at all points on the solid surface. Once again, $z$-component normal forces due to convective momentum flux acting across the fictitious inlet and outlet planes cancel because the average fluid velocity is the same (i.e., $\langle v_z \rangle_{\text{inlet}} = \langle v_z \rangle_{\text{outlet}}$) when the flow cross-sectional area remains constant. The steady state macroscopic momentum balance yields the following expression for the $z$-component of the total vector force exerted by the fluid on the tube wall, which is due completely to viscous shear at the fluid-solid interface because pressure forces at the tube wall act solely in the radial direction;

$$\left\{ F_{\text{Fluid} \rightarrow \text{Solid}} \right\}_{z\text{-component}} = (p_{\text{inlet}} - p_{\text{outlet}}) S + m_{\text{total}} g \cos \Theta$$

$$= \pi R^2 \Delta p + \rho_{\text{fluid}} \pi R^2 L g \cos \Theta = \pi R^2 \left\{ \Delta p + \rho_{\text{fluid}} g L \cos \Theta \right\}$$

The effect of gravitational potential energy per unit volume of fluid on the driving force for flow can be added to fluid pressure (i.e., they have the same dimensions) via the definition of a quantity called dynamic pressure $P$;

$$P = p + \rho_{\text{fluid}} g h$$

where $p$ represents fluid pressure, $h$ is a position variable that increases as one moves vertically upward from an arbitrarily chosen zero of potential energy, and $\rho_{\text{fluid}} g h$ is the gravitational potential energy per unit volume of fluid. The horizontal plane that represents the zero of potential energy can be chosen arbitrarily because driving forces for fluid flow are expressed as differences in dynamic pressure, so the absolute value of potential energy at any position within the fluid has no effect on the final solution to the momentum balance. For example, when forced convective flow is driven by a combination of gravity and a decrease in fluid pressure, as described in this section, and the zero of potential energy is placed at the fictitious outlet plane of the tilted tube such that the inlet plane is either $L \cos \Theta$ higher (i.e., $0 \leq \Theta \leq \pi/2$) or lower (i.e., $\pi/2 \leq \Theta \leq \pi$) than
the outlet plane, the previous result in this section for viscous shear at the fluid-solid interface reduces to;

$$\{F_{\text{Fluid} \rightarrow \text{Solid}}\}_{z\text{-component}} = 2\pi RL \tau_{rz}(r = R)$$

$$= \pi R^2 \left\{ [p_{\text{inlet}} + \rho_{\text{fluid}}gL \cos \Theta] - [p_{\text{outlet}} + 0] \right\} = \pi R^2 \Delta P$$

with assistance from statement (14) above to evaluate the $z$-component of the interfacial force exerted by the fluid on the solid surface due to the viscous stress tensor. Rearrangement of this equation allows one to evaluate $\tau_{rz}$ at the tube wall;

$$\tau_{rz}(r = R) = \frac{R \Delta P}{2L} = \frac{1}{2} R \left\{ \frac{\Delta P}{L} \right\} = \frac{1}{2} R \left\{ -\frac{dP}{dz} \right\}$$

This generalized result for straight tilted tubes is essentially the same as the one in statement (15) above for horizontal tubes, except that fluid pressure $p$ must be replaced by dynamic pressure $P$ when gravity assists or hinders fluid flow. Furthermore, it is possible to extrapolate the previous expression for tilted tubes to obtain the viscous shear stress distribution throughout the fluid;

$$\tau_{rz}(r) = \frac{1}{2} r \left\{ -\frac{dP}{dz} \right\}$$

which is applicable for incompressible flow through straight tubes with constant cross-sectional area, oriented at any angle with respect to gravity. For Newtonian fluids with constant viscosity that cannot store elastic energy upon deformation due to viscous stress, the dynamic pressure gradient in the direction of flow is constant, such that $-dP/dz = \Delta P/L$. As a consequence of this analysis of one-dimensional flow through straight tilted tubes, it is acceptable to modify the generalized macroscopic momentum balance for incompressible fluids and account for gravitational forces via dynamic pressure. Hence, one neglects $m_{\text{total}}g$ in the force balance and replaces fluid pressure $p$ by dynamic pressure $P$. The result is;

$$\frac{dP_{\text{total}}}{dt} = \sum_{\text{inlet planes}} \left\langle \rho v(v - v_{\text{Surface}})S \right\rangle n_{\text{inlet}} - \sum_{\text{outlet planes}} \left\langle \rho v(v - v_{\text{Surface}})S \right\rangle n_{\text{outlet}}$$

$$+ \sum_{\text{inlet planes}} \left\langle PS \right\rangle n_{\text{inlet}} - \sum_{\text{outlet planes}} \left\langle PS \right\rangle n_{\text{outlet}} - F_{\text{Fluid} \rightarrow \text{Solid}}$$
Macroscopic Mechanical Energy Balance

Important Concepts in the Development of the Macroscopic Mechanical Energy Balance (i.e., the Bernoulli Equation)

1) The mechanical energy balance, which is a scalar, is developed from first principles via the (i) balance on overall fluid mass, and the (ii) linear momentum balance. The units of each term in the Bernoulli equation are energy per time, which is equivalent to power.

2) The system is defined as the fluid contained within the control volume $V(t)$, which is not necessarily stationary. Moving control volumes are described by time-dependent boundaries.

3) Total kinetic energy within the system is defined in terms of fluid density $\rho$ and the square of the fluid velocity, as follows;

$$ K_{total} = \iiint_{V(t)} \frac{1}{2} \rho v_{fluid}^2 dV $$

4) Total potential energy within the system is defined in terms of fluid density $\rho$ and height $h$, which is measured vertically upward relative to the “zero of potential energy”.

$$ \Phi_{total} = \iiint_{V(t)} \rho gh dV $$

5) Inlet plane $S_1$ and outlet plane $S_2$ represent fictitious surfaces that are not solid. Fluid flow crosses these fictitious planes. The rate at which kinetic energy enters and leaves the system due to convective flux, or bulk fluid flow, is proportional to the third power of the fluid velocity. There is no molecular flux of kinetic energy.

Net rate of convective input of kinetic energy = $\frac{1}{2} \rho_1 \langle v_1^3 \rangle S_1 - \frac{1}{2} \rho_2 \langle v_2^3 \rangle S_2$

6) The rate at which potential energy enters and leaves the system due to convective flux is given by a product of (i) the potential energy per unit mass of fluid, (ii) overall convective mass flux, and (iii) the surface area normal to fluid flow across the inlet or outlet planes. Once again, there is no molecular flux of potential energy. Molecular fluxes exist for (i) viscous transport of momentum via Newton’s law, (ii) conduction of thermal energy via Fourier’s law, (iii) diffusion
of species mass via Fick’s law, and (iii) entropy via concepts based on irreversible thermodynamics. The net rate of convective input of potential energy to the system is given by;

Net rate of convective input of potential energy = \( g z_1 \rho_1 \langle v_1 \rangle S_1 - g z_2 \rho_2 \langle v_2 \rangle S_2 \)

7) The Bernoulli equation contains several work-related terms which are (i) positive when work is done by the surroundings on the fluid, or (ii) negative when work is done by the fluid on the surroundings. For example, the rate at which work is done on the fluid due to pressure forces is given by a product of (i) the isothermal Gibbs free energy of the fluid, (ii) overall convective mass flux, and (iii) the surface area normal to fluid flow across the inlet or outlet planes. This is equivalent to the fact that the rate at which work is performed (i.e., power) is given by the scalar dot product of a normal pressure force and the fluid velocity vector. Hence;

Net rate of work done on the fluid by pressure forces = \( p_1 \langle v_1 \rangle S_1 - p_2 \langle v_2 \rangle S_2 \)

8) The rate at which work is done on the fluid by viscous forces across the fictitious inlet and outlet planes is negligible, relative to the work done by pressure forces, as outlined in (7).

9) The rate at which work is done by the fluid on moving solid surfaces \( S_{\text{moving}} \) due to pressure forces is given by a surface integral of the product of fluid pressure and the normal component of the fluid velocity vector evaluated at the fluid-solid interface. The fluid velocity is the same as the velocity of the moving solid surface at the point of contact between the fluid and the solid. The unit normal vector in the expression below is directed from the control volume that contains the fluid toward the moving solid surface. This term does not appear in the Bernoulli equation if the solid surfaces are stationary.

\[
\left( \frac{dW}{dt} \right)_{\text{moving, pressure}} = \int_S p \left\{ n \cdot v_{\text{fluid}} \right\} @ S_{\text{moving}} dS
\]

10) The rate at which work is done by the fluid on moving solid surfaces \( S_{\text{moving}} \) due to viscous forces is given by a surface integral of a complex dot product involving (i) the unit normal vector directed from the fluid toward the moving solid surface,
(ii) the fluid velocity vector, and (iii) the viscous stress tensor. Once again, this term vanishes at any solid surface that is stationary.

\[
\frac{dW}{dt}_{\text{moving,viscous}} = \iint_{S_{\text{moving}}} \left\{ n \cdot [\tau \cdot v] \right\} @ S_{\text{moving}}
\]

11) The horsepower requirement of a pump, which is part of the system, is given by the following sum; \((dW/dt)_{\text{pump}} = -(dW/dt)_{\text{moving,pressure}} - (dW/dt)_{\text{moving,viscous}},\) with units of energy per time. Horsepower requirements originate from the rate at which work is done on the fluid due to pressure and viscous forces acting across moving solid surfaces that are in contact with the fluid. Pumps in series act like voltage sources in series. They experience the same flow rates or currents, and the total increase in fluid pressure or voltage is additive. A series configuration of pumps is recommended when each one delivers the correct flow rate, but greater increase in fluid pressure is required. Pumps in parallel function like a parallel configuration of voltage sources. They produce the same pressure or voltage increase, and the total flow rate or current is additive. A parallel configuration of pumps is recommended when each one delivers the correct increase in fluid pressure, but higher flow rates are required. Please refer to (18) below for the design of pumping configurations to meet desired specifications.

12) The non-ideal Bernoulli equation contains an irreversible rate of conversion of mechanical energy to thermal energy due to viscous dissipation, \(E_v.\) Other names for this process are (i) friction loss, (ii) viscous heating, (iii) viscous dissipation, and (iv) the degradation of mechanical energy to thermal energy. The second law of thermodynamics for irreversible processes dictates the path, or direction, by which this process occurs. In fact, the degradation of mechanical or kinetic energy to thermal energy always causes the fluid temperature to increase. If the fluid temperature decreased and all of this thermal energy were converted completely to mechanical energy, then one could construct perpetual motion machines of the second kind. These devices are prohibited by the second law of thermodynamics. The irreversible degradation of mechanical energy to thermal energy, written below using vector-tensor notation, is always positive for Newtonian fluids that cannot store elastic energy.

\[
E_v = -\iiint_{V(t)} (\tau : \nabla v) dV
\]
13) Simple forms for the irreversible conversion of mechanical energy to thermal energy in various fluid flow configurations are summarized below;

Flow in straight channels with arbitrary cross-section;

\[ E_v = \{ \rho \langle v_z \rangle S \} \frac{1}{2} \langle v_z \rangle^2 \frac{L_{\text{TubeLength}}}{R_{\text{HydraulicRadius}}} f_{\text{FrictionFactor}} \]

Flow through valves, fittings, and other obstacles with dimensionless friction loss factor \( e_v \);

\[ E_v = \{ \rho \langle v_z \rangle S \} \frac{1}{2} \langle v_z \rangle^2 e_v \]

14) The complete macroscopic mechanical energy balance, based on all of the concepts mentioned above, for a control volume with one inlet stream (i.e., subscript 1) and one outlet stream (i.e., subscript 2) is;

\[ \frac{d}{dt} \{ K_{\text{total}} + \Phi_{\text{total}} \} = \frac{1}{2} \rho_1 \langle v_1^3 \rangle S_1 - \frac{1}{2} \rho_2 \langle v_2^3 \rangle S_2 + g \zeta_1 \rho_1 \langle v_1 \rangle S_1 - g \zeta_2 \rho_2 \langle v_2 \rangle S_2 + p_1 \langle v_1 \rangle S_1 - p_2 \langle v_2 \rangle S_2 + \left( \frac{dW}{dt} \right)_{\text{Pump}} - E_v \]

15) For systems that operate at steady state with no moving parts (i.e., no pump work term) and negligible friction loss, the macroscopic mechanical energy balance reduces to the ideal Bernoulli equation. Fluids that obey these restrictions are classified as ideal, inviscid, irrotational and isentropic. Flow occurs at constant entropy and the fluid experiences no increase in temperature, because the irreversible conversion of mechanical energy to thermal energy is insignificant. The ideal Bernoulli equation can be summarized as follows; the sum of kinetic energy, potential energy, and isothermal Gibbs free energy (i.e., pressure work term) remains constant everywhere within the fluid. The appropriate mechanical energy balance is;
\[
\frac{1}{2} \rho_1 \langle v_1^3 \rangle S_1 + p_1 \langle v_1 \rangle S_1 + g z_1 \rho_1 \langle v_1 \rangle S_1 \\
= \frac{1}{2} \rho_2 \langle v_2^3 \rangle S_2 + g z_2 \rho_2 \langle v_2 \rangle S_2 + p_2 \langle v_2 \rangle S_2
\]

16) Analysis of the steady state ideal Bernoulli equation is performed after division of the previous equation by the total mass flow rate, which can be written as \(\rho_1 \langle v_1 \rangle S_1\) or \(\rho_2 \langle v_2 \rangle S_2\), because both expressions for the total mass flow rate are the same at steady state with one inlet stream and one outlet stream. The starting point for the analysis of inviscid fluids with no moving parts via the ideal Bernoulli equation is:

\[
\frac{1}{2} \frac{\langle v_1^3 \rangle}{\langle v_1 \rangle} + g z_1 + \frac{p_1}{\rho_1} = \frac{1}{2} \frac{\langle v_2^3 \rangle}{\langle v_2 \rangle} + g z_2 + \frac{p_2}{\rho_2}
\]

In terms of the steady state Bernoulli equation with one inlet stream and one outlet stream, where each term has dimensions of energy per mass, and overall mass flow rate is constant, the pressure terms that are typically subtracted correspond to \(pV\)-work across the fictitious inlet and outlet planes, and no surface areas appear because one divides the equation with dimensions of energy per time by total mass flow rate which does not change from inlet to outlet at steady state. Hence, predictions from the steady state Bernoulli equation with one inlet and one outlet do not depend on the use of absolute or gauge pressure, but absolute pressure is required in the momentum balance due to the vector nature of that equation and the fact that pressure stress must be multiplied by the appropriate surface area.

17) For incompressible fluids, in which fluid density \(\rho\) is a very weak function of pressure, the difference between \(gz + p/\rho\) at the inlet plane \(S_1\) and outlet plane \(S_2\) can be written as \((\Delta P)/\rho\), where the dynamic pressure difference \(\Delta P\) is \(P_1\) minus \(P_2\).

18) Application of the steady state non-ideal Bernoulli equation around a pump reveals that the downstream pressure, after fluid passes through the pump, is greater than the upstream pressure. If the inlet and outlet planes are characterized by the same flow cross-sectional areas such that there is no change in kinetic energy, then the rate at which work must be done on the fluid by pressure and viscous forces acting across moving solid surfaces (i.e., impeller)
in contact with fluid inside of the pump is related to the increase in dynamic pressure via the following equation;

\[
\text{Efficiency} \left( \frac{dW}{dt} \right)_{\text{Pump, Required}} = \left( \frac{dW}{dt} \right)_{\text{Pump Delivered to Fluid}} = \langle v \rangle S \{ (p_2 + \rho g z_2) - (p_1 + \rho g z_1) \} \\
= \langle v \rangle S \{ P_2 - P_1 \} = \langle v \rangle S \Delta P \quad \Rightarrow \quad \langle v \rangle S \Delta p = Q \Delta p
\]

All frictional energy losses within the pump are accounted for by the pump efficiency. Six pumps are available with flowrate specifications at a particular horsepower rating [i.e., \((dW/dt)_{\text{Pump, Required}}\)] when each pump operates very close to its optimum efficiency.

<table>
<thead>
<tr>
<th>Pump #</th>
<th>Volumetric flowrate, Q (gallons/minute, gpm)</th>
<th>(\Delta p) = [Efficiency(dW/dt)\text{Pump, Required}]/Q (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>C</td>
<td>20</td>
<td>20</td>
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<tr>
<td>D</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>E</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>F</td>
<td>40</td>
<td>35</td>
</tr>
</tbody>
</table>

(i) If each pump can be chosen only once and it is desired to use them when they operate near optimum efficiency, then design a pumping configuration that delivers 30 gallons/minute with a 100-psi increase in fluid pressure downstream from the pumps. Remember that pumps in series should operate at matched flowrates and pumps in parallel should operate at matched \(\Delta p\).

Answer: 
Put D (30 gpm; 50 psi) and E (30 gpm; 30 psi) in series with a parallel configuration of A (10 gpm; 20 psi) and C (20 gpm; 20 psi).

(ii) If each pump can be chosen only once and it is desired to use them when they operate near optimum efficiency, then design a pumping configuration that delivers 60 gallons/minute with a 50-psi increase in fluid pressure downstream from the pumps. Remember that pumps in series should operate at matched flowrates and pumps in parallel should operate at matched \(\Delta p\).
Answer:
Put E (30 gpm; 30 psi) in series with a parallel configuration of A (10 gpm; 20 psi) and C (20 gpm; 20 psi). Then, put this entire configuration in parallel with D (30 gpm; 50 psi)

Orifice meter analysis via the ideal and non-ideal Bernoulli equation with frictional energy loss

The discharge coefficient for an orifice meter, $C_v$, is defined as the ratio of the actual mass flow rate with frictional energy loss (i.e., $E_v > 0$) to the ideal mass flow rate without friction loss (i.e., $E_v = 0$). If the frictional energy loss $E_v$, with units of energy per unit mass of fluid, for a sharp-edged orifice meter is described by:

$$E_v = \frac{1}{2} \left( \frac{v_{\text{tube}}}{v_v} \right)^2 e_v$$

and the dimensionless friction loss factor $e_v$ in the turbulent regime is approximated by:

$$e_v = \lambda \left( 1 - \beta \right) \left( 1 - \beta^4 \right) \beta^4$$

where $\beta = D_2/D_1$ is defined as the ratio of the orifice diameter $D_2$ to the tube diameter $D_1$, then calculate the discharge coefficient, $C_v$, when $\beta = 0.75$

Answer:
Begin with the steady state nonideal Bernoulli equation and define the fictitious inlet and outlet planes that provide the necessary surfaces which identify the control volume and the system as the fluid between these planes. Inlet plane “1” is upstream from the orifice meter, within the tube, and outlet plane “2” is placed at the throat of the meter where maximum velocity and minimum pressure exist. There are no moving solid surfaces within the control volume, so the pump work term vanishes. All remaining terms in the steady state Bernoulli equation, including frictional energy loss, are divided by the total mass flow rate which remains constant for steady state analysis. The kinetic energy correction factor $\alpha$ is defined as the ratio of the cube of the average velocity relative to the average of the velocity cubed. In other words;
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\[ \alpha = \frac{1}{S_{\text{cross-section}} \int v_z dS} \]

The primary objective is to predict the mass flow rate either in the tube or in the throat of the meter via the following equation;

\[ \frac{\langle v_1 \rangle^2}{2\alpha} + g\bar{z}_1 + \frac{p_1}{\rho_1} = \frac{\langle v_2 \rangle^2}{2\alpha} + g\bar{z}_2 + \frac{p_2}{\rho_2} + E_v \]

The steady state mass balance for incompressible fluids allows one to relate \( \langle v_1 \rangle \) and \( \langle v_2 \rangle \) via flow cross-sectional areas \( S_1 \) in the tube and \( S_2 \) at the contraction. Hence;

\[ \langle v_1 \rangle S_1 = \langle v_2 \rangle S_2 \]

\[ \langle v_1 \rangle = \frac{S_2}{S_1} \langle v_2 \rangle = \beta^2 \langle v_2 \rangle \]

Now, the nonideal Bernoulli equation is solved for average fluid velocity \( \langle v_2 \rangle \) at the meter;

\[ \frac{\langle v_2 \rangle^2}{2\alpha} \left( 1 - \beta^4 + \alpha \beta^4 e_v \right) = \frac{1}{\rho} (p_1 - p_2) + g(z_1 - z_2) \]

\[ \langle v_2 \rangle = \sqrt{\frac{2\alpha \left( \frac{1}{\rho} (p_1 - p_2) + g(z_1 - z_2) \right)}{1 - \beta^4 + \alpha \beta^4 e_v}} = \sqrt{\frac{2\alpha \left( \frac{1}{\rho} (P_1 - P_2) \right)}{1 - \beta^4 + \alpha \beta^4 e_v}} \]

Multiplication of the previous expression for \( \langle v_2 \rangle \) by \( \rho S_2 \) allows one to predict the total mass flow rate for both real (i.e., \( e_v > 0 \)) and ideal (i.e., \( e_v = 0 \)) fluids. The ratio of these mass flow rates yields the coefficient of discharge \( C_v \). It is important to emphasize that both mass flow rates are evaluated at the same dynamic pressure difference \( P_1 - P_2 = \Delta P \) on the far right side of the previous equation for orifice meters that are either horizontal or tilted at any angle with respect to gravity. The appropriate equation for \( C_v \) is;
Two important analyses follow. First, the previous equation is inverted to express the dimensionless friction loss factor \( e_v \) in terms of the coefficient of discharge \( C_v \). The result is:

\[
e_v = \left\{ \frac{1}{C_v^2} - 1 \right\} \frac{1-\beta^4}{\alpha \beta^4}
\]

Hence, experimental determination of the coefficient of discharge via (i) measurement of the real mass flow rate and (ii) use of the ideal Bernoulli equation to predict the ideal mass flow rate allows one to estimate frictional energy loss for an orifice meter. The dynamic pressure drop across the meter is given by:

\[
\Delta P = \frac{1}{2} \rho \langle v_{\text{tube}} \rangle^2 e_v = \frac{1}{2} \rho \langle v_{\text{tube}} \rangle^2 \left\{ \frac{1}{C_v^2} - 1 \right\} \frac{1-\beta^4}{\alpha \beta^4}
\]

via application of the nonideal Bernoulli equation with no change in flow cross-sectional area upstream and downstream from the meter. The second important calculation in this section employs the dimensionless friction loss model from the previous page to estimate the coefficient of discharge for sharp-edged orifice meters. One obtains the following result:

\[
C_v = \sqrt{\frac{1 - \beta^4}{1 + \alpha \beta^4}} e_v = \sqrt{\frac{1}{1 + \alpha \lambda (1 - \beta)}}
\]

where \( \beta \) is the diameter ratio for the orifice meter, \( \lambda \) is a parameter in the model for the dimensionless friction loss factor, and \( \alpha \) is approximately unity (i.e., 0.945) for turbulent flow.
Mass Flow Rate vs. Pressure Drop in Venturi Meters

Obtain an expression for the actual, not the ideal, mass flow rate through the tilted venturi meter illustrated below, where the primary direction of flow makes an angle $\theta$ with respect to the horizontal. An incompressible fluid with density $\rho_1$ flows down the incline from left to right. The manometer contains mercury with density $\rho_2$. Mercury, with density $\rho_{\text{Hg}}$, rises to height $z_4$ in the left leg of the manometer, and it rises to height $z_3$ in the right leg of the manometer. It is important to realize that the pressure taps at positions #1 and #2 are at elevations $z_1$ and $z_2$, respectively, in the gravitational field, where $z_1 > z_2$.

\[ e_V = \left( \frac{1}{C_v^2 - 1} \right) \frac{1 - \beta^4}{\alpha \beta^4} \]

the strategy involves starting with the steady state ideal Bernoulli equation. One defines the fictitious inlet and outlet planes that provide the necessary surfaces which identify (i) the control volume and (ii) the system as the fluid between these planes. Inlet plane “1” at position $z_1$ is upstream from the orifice meter, within the tube, and outlet plane “2” at position $z_2$ is placed at the throat of the meter where maximum velocity and minimum pressure exist. There are no moving solid surfaces within the control volume, so the pump work term vanishes. All remaining terms in the ideal Bernoulli equation are divided by the total mass flow rate that remains constant for steady state analysis. The primary objective is to predict the mass flow rate either in the tube or in the throat of the meter via the following equation without frictional energy loss;
The steady state mass balance for incompressible fluids allows one to relate \( <v_1> \) and \( <v_2> \) via flow cross-sectional areas \( S_1 \) in the tube and \( S_2 \) at the contraction. Hence;

\[
\frac{\langle v_1 \rangle^2}{2\alpha} + gz_1 + \frac{p_1}{\rho_1} = \frac{\langle v_2 \rangle^2}{2\alpha} + gz_2 + \frac{p_2}{\rho_2}
\]

Now, the steady state ideal Bernoulli equation for incompressible fluids (i.e., \( \rho_1 = \rho_2 = \rho \)) is solved for average fluid velocity \( <v_2> \) at the meter;

\[
\langle v_2 \rangle = \frac{S_2}{S_1} \langle v_2 \rangle = \beta^2 \langle v_2 \rangle
\]

Multiplication of the previous expression for \( <v_2> \) by \( \rho S_2 \) allows one to predict the total mass flow rate for ideal fluids (i.e., \( e_v = 0 \)). Then, multiplication of \( \rho <v> S_2 \) by the coefficient of discharge \( C_v \) yields the actual mass flowrate through the Venturi meter. Hence;

\[
\text{ActualMassFlowrate} = C_v \rho \langle v_2 \rangle S_2 = C_v \rho \frac{\pi}{4} d_{\text{contraction}}^2 \sqrt{\frac{2\alpha \left\{ \left( \frac{\rho_1}{\rho} - \rho_2 \right) + \rho g(z_1 - z_2) \right\}}{\rho \left( 1 - \beta^4 \right)}
\]

The coefficient of discharge \( C_v \) for Venturi meters is approximately 0.92-0.95

The venturi meter is equipped with a very dangerous and outdated mercury manometer to measure the pressure difference between positions #1 and #2 for subsequent calculation of the pressure drop \( \Delta p = p_1 - p_2 \). If the difference between the height of mercury in the two legs of the manometer is the same when the venturi is
tilted at angle \( \Theta \) and when the venturi is horizontal, then in which orientation is the mass flow rate smaller?

**Answer:** Both flow rates are the SAME.

When the meter is horizontal, \( z_1 = z_2 \) and the dynamic pressure difference in the previous equation for the actual mass flowrate reduces to the actual fluid pressure difference. However, upon applying hydrostatics within the manometer for tilted meters, in general, and equating fluid pressure at elevation \( z_4 \) in both legs, one obtains the following relation;

\[
\begin{align*}
p_1 + \rho g(z_1 - z_4) &= p_2 + \rho g(z_2 - z_3) + \rho_{Hg} g(z_3 - z_4) \\
p_1 - p_2 &= \rho g(z_2 - z_3) - \rho g(z_1 - z_4) + \rho_{Hg} g(z_3 - z_4) \\
p_1 - p_2 + \rho g(z_1 - z_2) &= g(z_3 - z_4) \{ \rho_{Hg} - \rho \}
\end{align*}
\]

\[
\text{ActualMassFlowrate} = C_v \rho \frac{\pi}{4} d_{\text{Contraction}}^2 \sqrt{\frac{2ag(z_3 - z_4) \{ \rho_{Hg} - \rho \}}{\rho(1 - \beta^4)}}
\]

where \( \rho \) is the fluid density of interest and \( \rho_{Hg} \) is the density of mercury in the bottom of the manometer. As indicated by the previous equation for the actual mass flowrate in tilted Venturi meters, elevations \( z_1 \) and \( z_2 \) at the inlet and outlet planes do not affect the final answer, so one obtains the same mass flowrate in tilted and horizontal meters when the difference between the height of mercury in both legs of the manometer (i.e., \( z_3 - z_4 \)) is the same. Why does this happen? The hydrostatic contribution to fluid pressure in each leg of the manometer cancels the effect of gravitational potential energy differences at the inlet and outlet planes.

The venturi meter is equipped with 21st-century pressure transducers at positions #1 and #2 to measure the pressure drop \( \Delta p = p_1 - p_2 \). If the pressure transducers measure the same \( \Delta p \) when the venturi is tilted at angle \( \Theta \) and when the venturi is horizontal, then in which orientation is the mass flow rate larger?

**Answer:** The tilted Venturi, in which flow proceeds downhill, exhibits the larger mass flowrate.

Now, hydrostatic calculations within each leg of the manometer do not contribute to fluid pressure at the inlet and outlet planes because the transducers provide a direct measure of fluid pressure. When \( z_1 \neq z_2 \), gravitational potential energy differences at the inlet and
outlet planes cannot be neglected and no cancellation occurs. Hence, if $\Delta p = p_1 - p_2$ is the same for horizontal and tilted meters, then the tilted meter exhibits a larger flowrate when $z_1 > z_2$ for downhill flow, and the horizontal meter exhibits a larger flowrate when $z_1 < z_2$ for uphill flow.

**Application of the fluid flow meter equation for laminar tube flow**

Frictional energy loss, with units of energy per unit mass of fluid, for one-dimensional fluid flow through a straight tube with circular cross-section is; $E_v = (1/2) \langle v_{tube} \rangle^2 e_v$, where the dimensionless friction loss factor for a tube with diameter $D = 2R$ and length $L$ is; $e_v = (4L/D)f$, and the friction factor for laminar flow of an incompressible Newtonian fluid is; $f = 16/Re$ when $Re = \{\rho \langle v_{tube} \rangle D/\mu \} < 2100$. Use the fluid flow meter equation to calculate $\langle v_{tube} \rangle$ when $\beta = 1$. Remember that $\Delta p \neq 0$ when $\beta = 1$. You should obtain a classic result in fluid mechanics.

**Answer:**

Begin with the fluid flow meter equation from the nonideal Bernoulli equation for either $\langle v_{tube} \rangle$ or $\langle v_{meter} \rangle$, because both are the same when the diameter ratio $\beta = 1$, and combine fluid pressure and gravitational potential energy via the definition of dynamic pressure;

$$\langle v \rangle \approx \sqrt{2\alpha \left[ \frac{1}{\rho} \left( p_1 - p_2 \right) + g \left( z_1 - z_2 \right) \right]} = \sqrt{\frac{2\alpha \left( P_1 - P_2 \right)}{\rho \left( 1 - \beta^4 + \alpha \beta^4 e_v \right)}} \rightarrow \frac{2\alpha \Delta P}{\rho \alpha e_v} \quad \beta = 1$$

Now, use the dimensionless friction loss factor expression for straight tubes in the laminar flow regime, where the kinetic energy correction factor $\alpha = 1/2$. One obtains the following result;

$$\langle v \rangle \approx \frac{2\alpha \Delta P}{\rho \alpha e_v} \rightarrow \frac{2\alpha \Delta P}{\rho \left( 4L/D \right) f(Re)} \rightarrow \frac{2\alpha \Delta P}{\rho \left( 4L/16 \mu R \right)} \rightarrow \frac{R^2 \langle v \rangle \Delta P}{8\mu L} \rightarrow \frac{R^2 \Delta P}{8\mu L}; \text{This is the HP Law!!}$$
Viscous flow through a parallel configuration of orifice and venturi meters

Water at ambient temperature flows into a branch point with an inlet volumetric flowrate of $10^3$ cubic centimeters per second (i.e., 1 Litre per second). The flow configuration splits into two segments at the branch point. The first horizontal segment contains 100 cm of smooth 5-cm inner diameter straight tubing and a sharp-edged orifice meter with a diameter ratio of 0.504. The second horizontal segment contains 100 cm of smooth 5-cm inner diameter straight tubing and a venturi meter with a diameter ratio of 0.316. The coefficients of discharge are 0.62 for the orifice meter and 0.90 for the venturi. Calculate the fraction of the inlet stream (i.e., $Q_{inlet} = 10^3 \text{ cm}^3/\text{sec.}$) that flows through each branch.

The incorrect approach to solve this problem
The steady state macroscopic mass balance at the junction point, with one inlet stream and two outlet streams for incompressible fluids, reveals that the inlet volumetric flow rate of $10^3 \text{ cm}^3/\text{sec}$ must be balanced by the sum of volumetric flowrates through the first branch that contains the orifice meter and the second branch that contains the venturi. The fluid flow meter equation provides a route to calculate volumetric flowrates, because $z_1 = z_2$ for horizontal configurations, and the coefficient of discharge $C_V$ and the diameter ratio $\beta$ are provided for each meter. Hence;

\[
Q_{inlet} = 10^3 \text{ cm}^3/\text{sec} = Q_{orifice} + Q_{venturi}
\]

\[
Q_{orifice} = C_{V,orifice} \frac{\pi}{4} d_{orifice}^2 \sqrt{\frac{2\alpha_{orifice}(p_1 - p_2)}{\rho(1 - \beta_{orifice}^4)}}; Q_{venturi} = C_{V,venturi} \frac{\pi}{4} d_{venturi}^2 \sqrt{\frac{2\alpha_{venturi}(p_1 - p_2)}{\rho(1 - \beta_{venturi}^4)}}
\]

Since the meters are arranged in parallel, the pressure drop across each one (i.e., $p_1 - p_2$) should be the same. A second relation between volumetric flowrates through both meters is obtained by rearranging the flow meter equations and equating $p_1 - p_2$. It is assumed that the flow regime is the same in each branch and that the kinetic energy correction factor $\alpha$ will not appear in the final result. If, for example, laminar flow occurs through the orifice meter and turbulent flow occurs through the venturi, then $\alpha_{orifice} = 1/2$, $\alpha_{venturi} = 0.945$, and cancellation does not occur. Upon equating pressure drops, one obtains;

\[
p_1 - p_2 = \left[\frac{16\rho}{2\alpha_{orifice}\pi^2 d_{Tube\ orifice}^4}\right] \frac{Q_{orifice}^2}{C_{V,orifice}^2} \left[\frac{1}{\beta_{orifice}^4} - 1\right] = \left[\frac{16\rho}{2\alpha_{venturi}\pi^2 d_{Tube\ venturi}^4}\right] \frac{Q_{venturi}^2}{C_{V,venturi}^2} \left[\frac{1}{\beta_{venturi}^4} - 1\right]
\]
Why is the previous expression incorrect?
The pressure drop in the fluid flow meter equation corresponds to the difference between upstream pressure and fluid pressure at the contraction, where minimum pressure exists. When the fluid subsequently experiences a gradual (i.e., venturi) or sudden (i.e., orifice) expansion and exits the meter, the streamlines recover from the obstacle (i.e., meter) in the line of flow and fluid pressure increases to some extent, but it doesn’t increase beyond the upstream pressure. The permanent pressure drop across each meter, defined by the difference between upstream pressure and downstream pressure, is significantly less than $p_1 - p_2$ in the fluid flow meter equation. Branches in parallel experience the same overall, or permanent, pressure drop. Hence, it is not correct to equate $p_1 - p_2$ for both meters because $p_1$ corresponds to the upstream pressure, but $p_2$ represents minimum fluid pressure at the vena contracta. Furthermore, it is necessary to add the pressure drop across smooth straight tubing to the permanent pressure drop across each meter before one should equate the overall pressure drop across each branch in parallel.

The preferred approach to solve this problem
Frictional energy loss across any obstacle in the line of flow can be correlated using dimensionless friction loss factors $e_v$ and the square of the average fluid velocity upstream from the obstacle, but within the appropriate branch. This methodology was adopted earlier in the discussion of the Macroscopic Mechanical Energy Balance to obtain a relation between dimensionless friction loss factors and coefficients of discharge $C_v$ for orifice and venturi meters. The desired relation between $C_v$ and $e_v$ for any meter with diameter ratio $\beta$ is;

$$e_v = \left\{ \frac{1}{C_v^2} - 1 \right\} \frac{1 - \beta^4}{\alpha \beta^4}$$

where the kinetic energy correction factor $\alpha$ is flow-regime-specific. Application of the non-ideal Bernoulli equation to a control volume in which the fictitious inlet plane lies upstream from the meter and the fictitious outlet plane is at least eight tube diameters downstream from the meter allows one to estimate the dynamic pressure difference across the obstacle. For horizontal configurations, the dynamic pressure difference is identical to the permanent fluid pressure drop because there is no change in potential energy from inlet to outlet. Furthermore, there are no moving solid surfaces within the control volume, and there is no change in kinetic energy from inlet to outlet if the upstream tube diameter is the same as the downstream tube diameter. Hence;
\[ P_{\text{upstream}} - P_{\text{downstream}} \Rightarrow p_{\text{upstream}} - p_{\text{downstream}} = \frac{1}{2} \rho (v_{\text{Tube}})^2 e_v \]

If the control volume within each branch of the parallel configuration contains smooth straight tubing in addition to the appropriate meter, then application of the non-ideal Bernoulli equation yields the following result;

\[ P_{\text{upstream}} - P_{\text{downstream}} \Rightarrow p_{\text{upstream}} - p_{\text{downstream}} = \frac{1}{2} \rho (v_{\text{Tube}})^2 e_v \left[ \frac{1}{C_v^2} \left( \frac{1}{\alpha} \frac{1-\beta^4}{\alpha^4} + 4 \frac{L}{d_{\text{Tube}}} f(\text{Re}_{\text{Tube}}) \right) \right] \]

It is reasonable to add permanent pressure drops across obstacles in series, including smooth straight tubing in which \( f = \frac{16}{\text{Re}} \) for laminar flow and \( f \approx \frac{0.08}{\text{Re}^{0.25}} \) for turbulent flow. The correct strategy to analyze partitioned flow in a parallel configuration of orifice and venturi meters is;

1. Apply the steady state overall macroscopic mass balance at the junction point and relate the inlet volumetric flowrate to the sum of flowrates through both meters (i.e., \( Q_{\text{inlet}} = Q_{\text{orifice}} + Q_{\text{venturi}} \)).

2. Express the average tube velocity in each branch in terms of the corresponding volumetric flowrate and cross-sectional area.

3. Assume turbulent flow to evaluate the kinetic energy correction factor (i.e., \( \alpha \approx 0.945 \)) and identify the appropriate friction factor correlation. It will be necessary to verify the flow regime by inspecting the Reynolds number when the volumetric flowrates are determined.

4. Use the appropriate discharge coefficient and geometric parameters for each meter and equate the permanent pressure drop across each branch, as given by the previous expression.

5. This problem is completely defined, with no degrees of freedom, and it can be solved implicitly by non-linear algebraic equation solvers. A summary of the appropriate equations and numerical results is provided on the following page.
Introduction to Transport Phenomena

Ch406/Detailed Lecture Topics
Frictional Energy Loss and Pump Horsepower Requirements

Calculate the horsepower requirement of a pump to transport water at 68°F from reservoir A to tank B at the same elevation in the gravitational field. Reservoir A is at ambient pressure (i.e., 1 atmosphere) and tank B is at an absolute pressure of 3 atmospheres. The flow configuration contains 100 feet of hydraulically smooth 2-inch inner diameter tubing with four 90° rounded elbow, one globe valve fully opened, and one sharp-edged orifice meter that has a diameter ratio of 0.5 and the following coefficient of discharge (i.e., \( C_v \approx 0.62 \)). The volumetric flowrate is 100 gallons per minute, and the pump efficiency is 30%. The physical properties of water at 68°F and some important conversion factors are provided for your calculations.

Data:
- \( \mu = 6.72 \times 10^{-4} \text{ lb-mass/foot-second} \) (viscosity of water @ 68°F)
- \( \rho = 62.4 \text{ lb-mass per cubic foot} \) (density of water @ 68°F)
- \( g = 32.174 \text{ feet/second}^2 \) (gravitational acceleration constant)
- 1 gallon = 0.13368 cubic feet
- 1 cubic foot = 28.316 Litres
- 1 atmosphere = 68105.9 \( \text{lb-mass/foot-second}^2 \) = 14.7 \( \text{lb-force/inch}^2 \)
- 1 \( \text{lb-force} = 32.174 \text{ lb-mass foot/second}^2 \)
- 1 BTU/second = 778.223 foot-\( \text{lb-force} \)/second
- 1 Horsepower = 0.7068 BTU per second = 550 ft-\( \text{lb-force} \) per second

This sequence of calculations analyzes the power requirements of a pump via the nonideal Bernoulli equation which includes frictional energy losses through various obstacles, straight tubing, and pump inefficiencies.

Enter the fluid density

density = 62.4 \{ \text{lb-mass per cubic foot for water @ 68°F} \}

Enter the fluid viscosity

viscosity = 6.72 \times 10^{-4} \{ \text{lb-mass per foot per second for water @ 68°F} \}

Enter the required volumetric flowrate

VolumeFlow = 100 \{ \text{gallons per minute} \}

Convert the volumetric flowrate from gallons per minute to cubic feet per second

1 gallon = 0.13368 cubic feet

cfs = VolumeFlow \times 0.13368/60

Enter the inner diameter of the tube through which fluid flows, feet
diameter = 2/12 {2-inch inner diameter tubing}

Fluid velocity in feet per second
Velocity = 4*cfs/(pi*diameter**2)

Reynolds number, turbulent flow
Re# = density*Velocity*diameter/viscosity

Kinetic energy correction factor for turbulent flow
alpha = 0.945

f vs. Re correlation for hydraulically smooth tubes, turbulent flow
friction = 0.0791/Re#**0.25

Length of straight tubing in feet
Length = 100

Friction loss factor for contraction from an infinite reservoir
SuddenContraction = 0.45

Friction loss factor for expansion into an infinite reservoir, turbulent flow
SuddenExpansion = 1/alpha

Enter the number of rounded 90-degree elbows, e_v = 0.75
Elbows90 = 4

Enter the number of gate valves, completely opened, e_v = 0.17
GateValves = 0

Enter the number of globe valves, completely opened, e_v = 8.0
GlobeValves = 1

Enter the number of 180-degree return bends, e_v = 1.5
ReturnBends180 = 0

Enter the number of orifice or venturi meters, either 1 or 0
Meter = 1

Enter the diameter ratio for the orifice/venturi meter, smaller/larger
DiameterRatio = 0.5
Enter the coefficient of discharge for the orifice/venturi meter, highly turbulent flow
\[ C_V = 0.62 \]

Friction loss in the meter
\[ \text{MeterLoss} = \frac{1}{(C_V**2)-1} \times (1-DiameterRatio**4)/(\alpha*(DiameterRatio**4)) \]

\[ \text{EvStraightTube} = 0.5\times\text{Velocity}**2 \times (4\times\text{Length} \times \text{friction/diameter}) \text{ \{ft}^2/\text{sec}^2 \}
\]

\[ \text{EvObstacles} = 0.5\times\text{Velocity}**2 \times (\text{SuddenContraction} + \text{SuddenExpansion} + 0.75\times\text{Elbows90} + 0.17\times\text{GateValves} + 8.0\times\text{GlobeValves} + 1.5\times\text{ReturnBends180} + \text{Meter} \times \text{MeterLoss}) \]

Total frictional energy loss, ft\(^2\)/sec\(^2\)
\[ \text{EvHat} = \text{EvStraightTube} + \text{EvObstacles} \]

Fluid pressure in reservoir#1, prior to the pump, atmospheres
\[ p_1 = 1 \]

\[ \text{lb-mass/ft-sec}^2, \text{1 atmosphere} = 68105.9 \text{ lb-mass/ft-sec}^2 \]
\[ p_1\text{Convert} = p_1 \times 68105.9 \]

Fluid pressure in reservoir#2, downstream from the pump, atmospheres
\[ p_2 = 3 \]

\[ \text{lb-mass/ft-sec}^2, \text{1 atmosphere} = 14.7 \text{ lb-force/square-inch} \]
\[ p_2\text{Convert} = p_2 \times 68105.9 \]

Height of reservoir#1 relative to the zero of potential energy, feet
\[ z_1 = -250 \]

Height of reservoir#2 relative to the zero of potential energy, feet
\[ z_2 = -250 \text{ \{pressurizing liquid fuel tanks in Death Valley, California\}} \]

Gravitational acceleration constant, feet per square second
\[ \text{gravity} = 32.174 \]

Dynamic pressure in reservoir#1, lb-mass/ft-sec\(^2\)
\[ \text{scriptP1} = p_1\text{Convert} + \text{density} \times \text{gravity} \times z_1 \]
Dynamic pressure in reservoir #2, lb-mass/ft-sec^2
\[ \text{scriptP}_2 = \text{p}_2\text{Convert} + \text{density}\times\text{gravity}\times z_2 \]

Power delivered by the pump to the fluid in units of foot lb-force/second
There are 32.174 lb-mass foot per square second in 1 lb-force
\[ \text{WpumpDelivered} = \text{density}\times\text{cfs}\times((\text{scriptP}_2 - \text{scriptP}_1)/\text{density} + \text{EvHat})/\text{gravity} \]

The pump is 30% efficient
\[ \text{PumpEfficiency} = 0.30 \]

Required pump power, ft lb-force/second
\[ \text{WpumpRequired} = \text{WpumpDelivered}/\text{PumpEfficiency} \]

Calculate the horsepower delivered to the fluid and required by the inefficient pump
The following conversions are useful;
1 BTU per second = 778.223 ft lb-force per second
1 Horsepower = 0.7068 BTU per second = 550 ft lb-force per second
\[ \text{HorsepowerDelivered} = \frac{\text{WpumpDelivered}}{778.223\times0.7068} \]
\[ \text{HorsepowerRequired} = \frac{\text{WpumpRequired}}{778.223\times0.7068} \]

\[ \alpha = 9.45e-1 \text{ [KE correction factor]} \]
\[ \text{CV} = 5.20e-1 \text{ [Discharge Coefficient]} \]
\[ \text{diameter} = 1.67e-1 \text{ [feet]} \]
\[ \text{Elbows\#} = 4.00e+0 \text{ [\# of Elbows]} \]
\[ \text{EvObstacles\#} = 1.98e+3 \text{ [ft}^2/\text{sec}^2] \]
\[ \text{Friction} = 3.97e-5 \text{ [Friction Factor]} \]
\[ \text{Gate\# of\ valves} = 1.00e+0 \text{ [\# of valves]} \]
\[ \text{HorsepowerDelivered} = 3.66e+0 \text{ [hp]} \]
\[ \text{Length} = 1.00e+2 \text{ [feet, Straight Tubing]} \]
\[ \text{Meter\# of\ flow\ meters} = 2.54e+1 \text{ [Meter, e-sub-v]} \]
\[ \text{p1\ Convert} = 5.01e+4 \text{ [lb-mass/ft-sec}^2] \]
\[ \text{p2\ Convert} = 2.04e+5 \text{ [lb-mass/ft-sec}^2] \]
\[ \text{Re}^* = 1.50e+5 \text{ [Reynolds Star]} \]
\[ \text{scriptP}_1 = 1.18e+5 \text{ [lb-mass/ft-sec}^2] \]
\[ \text{Sudden Contraction} = 4.50e-1 \text{ [e-sub-v, friction loss]} \]
\[ \text{Velocity} = 1.02e+1 \text{ [feet/second]} \]
\[ \text{Volume\ Flow} = 1.00e+2 \text{ [gallons/minute]} \]
\[ \text{Wpump\ Required} = 5.71e+3 \text{ [ft lb-force/sec]} \]
\[ z_2 = 2.50e+1 \text{ [feet, height]} \]
Mass Transfer in Non-reactive Systems

Begin with the mass transfer equation (MTE) for species i; \( \#1 = \#2 + \#3 + \#4 \), which is a scalar equation with dimensions of moles per volume per time, for isothermal problems. The system is defined as the fluid within the control volume (CV). Since the control volume is differentially thick in all coordinate directions, this mass balance yields a partial differential equation. Consider each mass transfer rate process, as described below;

#1; Accumulation rate process
Accumulation of the moles of species i within a stationary control volume; \( \partial C_i / \partial t \)
This term appears on the left side of the mass transfer equation.

#2; Convective mass transfer
Net rate at which the moles of species i enters a stationary control volume due to convective mass flux acting across all of the surfaces that surround fluid within the CV;

\[- \nabla \cdot C_i \mathbf{v} = - \{ C_i \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla C_i \} \]

where \( \mathbf{v} \) represents the mass-average velocity vector from fluid dynamics. The net rate of input is given by the negative of the divergence (i.e., convergence) of convective mass flux. For an incompressible fluid, the Equation of Continuity reveals that \( \nabla \cdot \mathbf{v} = 0 \). Hence, the contribution from convective mass transfer on the right side of the mass transfer equation for incompressible liquids reduces to;

\[- \nabla \cdot C_i \mathbf{v} = - \mathbf{v} \cdot \nabla C_i \]

#3; Molecular mass transfer via Fick’s 1st law of diffusion
Net rate at which the moles of species i enters a stationary control volume (CV) due to diffusional mass flux acting across all of the surfaces that surround fluid within the CV, where the diffusional flux is given by Fick’s 1st law (i.e., \( - D_{i,\text{Mixture}} \nabla C_i \));

\[- \nabla \cdot ( - D_{i,\text{Mixture}} \nabla C_i ) = D_{i,\text{Mixture}} \nabla \cdot \nabla C_i + \nabla C_i \cdot \nabla D_{i,\text{Mixture}} \]

The net rate of input is given by the negative of the divergence (i.e., convergence) of diffusional mass flux. For non-reactive or chemically reactive mixtures with constant physical properties, like density \( \rho \) and mass diffusivity \( D_{i,\text{Mixture}} \), only the first term on the right side of the previous equation survives. Hence, the term that characterizes molecular mass transfer via diffusion on the right side of the mass transfer equation is represented by;
\[ \text{D}_{i,\text{Mixture}} \nabla \cdot \nabla C_i = \text{D}_{i,\text{Mixture}} \nabla^2 C_i \]

where \( \nabla^2 C_i \) corresponds to the Laplacian of the molar density of species \( i \). Diffusion occurs, at most, in three coordinate directions. Partial differential equations must be solved for problems that are described by two- or three-dimensional diffusion. When one-dimensional diffusion is sufficient to describe a mass transfer problem, the governing equation (i.e., MTE) corresponds to an ODE. One obtains the following forms for the Laplacian of a scalar (i.e., molar density of species \( i \)), which represents one-dimensional diffusion in three important coordinate systems;

- **Rectangular coordinates (z-direction);**
  \[ \nabla^2 C_i = \frac{d^2 C_i}{dz^2} \]

- **Cylindrical coordinates (r-direction);**
  \[ \nabla^2 C_i = \frac{1}{r} \frac{d}{dr} \left( r \frac{dC_i}{dr} \right) \]

- **Spherical coordinates (r-direction);**
  \[ \nabla^2 C_i = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_i}{dr} \right) \]

The additional factors of \( r \) in cylindrical coordinates and \( r^2 \) in spherical coordinates represent curvature correction factors because the surface area normal to radial mass flux scales as either \( r \) or \( r^2 \), respectively.

### Rates of production due to multiple chemical reactions
If species \( i \) participates in several chemical reactions, and its stoichiometric coefficient in the \( j \)th reaction is \( \upsilon_{ij} \), then one must consider each reaction to account for the rate of production of the moles of species \( i \). The final result, which includes information from chemical kinetics to construct the appropriate rate law for the \( j \)th chemical reaction, is;

\[ \Sigma_j \upsilon_{ij} R_j \]

where \( R_j \) represents the intrinsic rate of the \( j \)th reaction. For simple \( n \)th-order chemical kinetics, where the rate law is only a function of the molar density of one species;

\[ R_j = \left( k_{n,j} \right) C_i^n \]

where \( n_j \) is the overall order of the \( j \)th reaction with \( n \)th-order kinetic rate constant \( k_n \).
The **Mass Transfer Equation** for species $i$ that will be analyzed with various combinations of mass transfer rate processes in an incompressible fluid is:

$$\frac{\partial C_i}{\partial t} = -\mathbf{v} \cdot \nabla C_i + D_{i,\text{Mixture}} \nabla^2 C_i + \sum_j v_{ij} R_j$$

Usually, the 1st-term on the right side of the previous equation appears on the left side with a positive sign because its dimensional scaling factor is the same as that for the accumulation rate process. Basic information at the microscopic continuum level is obtained by solving the previous equation (i.e., MTE) for the molar density of species $i$ in a mixture.

**Steady state diffusion in a stagnant medium with no chemical reaction.** Molecular mass transfer via one-dimensional diffusion is considered for these simple steady state problems, which correspond to the film theory of interphase mass transfer with a constant film thickness that contains all of the concentration gradients. The molar density of species $i$ must satisfy the following equation:

$$D_{i,\text{Mixture}} \nabla^2 C_i = 0$$

This is Laplace’s equation, which stipulates that the Laplacian of molar density must vanish. One obtains the following generalized molar density profiles for species $i$ in three important coordinate systems, as dictated by the following steady state mass balances;

**Rectangular coordinates (x,y,z)**

3-dimensional diffusion (see page#832 in *Transport Phenomena, 2nd-edition*);

$$\nabla^2 C_i = \frac{\partial^2 C_i}{\partial x^2} + \frac{\partial^2 C_i}{\partial y^2} + \frac{\partial^2 C_i}{\partial z^2} = 0$$

1-dimensional diffusion in the z-direction, $C_i(z)$;

$$\frac{d^2 C_i}{dz^2} = 0$$

1-dimensional solution; $C_i(z) = a_0 + a_1 z$
Cylindrical coordinates \((r, \Theta, z)\)

3-dimensional diffusion (see page#834 in Transport Phenomena, 2nd-edition);

\[
\nabla^2 C_i = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 C_i}{\partial \Theta^2} + \frac{\partial^2 C_i}{\partial z^2} = 0
\]

1-dimensional diffusion in the radial direction, \(C_i(r)\);

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dC_i}{dr} \right) = 0
\]

1-dimensional solution; \(C_i(r) = a_2 + a_3 \ln r\)

Spherical coordinates \((r, \Theta, \phi)\)

3-dimensional diffusion (see page#836 in Transport Phenomena, 2nd-edition);

\[
\nabla^2 C_i = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial C_i}{\partial r} \right) + \frac{1}{r^2 \sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial C_i}{\partial \Theta} \right) + \frac{1}{r^2 \sin^2 \Theta} \frac{\partial^2 C_i}{\partial \phi^2} = 0
\]

1-dimensional diffusion in the radial direction, \(C_i(r)\);

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_i}{dr} \right) = 0
\]

1-dimensional solution; \(C_i(r) = a_4 + a_5 / r\)

Analogy with steady state conductive heat transfer in a stagnant medium with no viscous dissipation, no radiation, and no work terms due to compression or expansion of the system. The thermal energy balance is rather complex, but when molecular transport of thermal energy in a pure material via conduction is the only rate process that must be considered, the following equation describes the process;

\[
\nabla \cdot \mathbf{q} = 0
\]

Convective transport of thermal energy, reversible compression/expansion work, irreversible conversion of mechanical energy to thermal energy, and radiation are not included in the previous balance. The molecular flux of thermal energy, given by \(\mathbf{q}\), is related to temperature gradients via Fourier's law of heat conduction in pure materials;
\[ q = - k_{TC} \nabla T \]

with thermal conductivity \( k_{TC} \) of an isotropic medium. The steady state thermal energy balance for conductive heat transfer reduces to;

\[ \nabla \cdot \left( - k_{TC} \nabla T \right) = - k_{TC} \nabla \cdot \nabla T = - k_{TC} \nabla^2 T = 0 \]

Once again, this is Laplace's equation, and the Laplacian of temperature must vanish for steady state heat conduction in a pure material. If one replaces \( C_i \) by \( T \), then molar density profiles in three important coordinate systems can be adopted from the previous section as the solution to Laplace's equation for the corresponding temperature profiles. Hence, the one-dimensional and three-dimensional steady state heat conduction equations are summarized below;

**Rectangular coordinates \((x,y,z)\)**

3-dimensional conduction (see page#832 in *Transport Phenomena, 2nd-edition*);

\[ \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

1-dimensional conduction in the z-direction, \( T(z) \);

\[ \frac{d^2 T}{dz^2} = 0 \]

1-dimensional solution; \( T(z) = a_0 + a_1 z \)

**Cylindrical coordinates \((r, \Theta, z)\)**

3-dimensional conduction (see page#834 in *Transport Phenomena, 2nd-edition*)

\[ \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \Theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \]

1-dimensional conduction in the radial direction, \( T(r) \);

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0 \]
1-dimensional solution; \( T(r) = a_2 + a_3 \ln r \)

**Spherical coordinates \((r, \Theta, \phi)\)**

3-dimensional conduction (see page#836 in *Transport Phenomena, 2nd-edition*);

\[
\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial T}{\partial \Theta} \right) + \frac{1}{r^2 \sin^2 \Theta} \frac{\partial^2 T}{\partial \phi^2} = 0
\]

1-dimensional conduction in the radial direction, \( T(r) \);

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0
\]

1-dimensional solution; \( T(r) = a_4 + \frac{a_5}{r} \)

The analogous problem in fluid dynamics which is described by *Laplace's equation*. The cross product of the "del" operator with the velocity vector, yields the vorticity vector [i.e., \((1/2) \nabla \times \mathbf{v}\)] with dimensions of inverse time. Fluid flow problems that exhibit rotational characteristics, like all of the viscometers discussed in previous sections of this document, are described by nonzero vorticity. When fluid flow occurs far from a high-shear no-slip solid-liquid interface, or in the vicinity of a zero-shear perfect-slip gas-liquid interface, it is reasonable to invoke no vorticity. Hence;

\[
\nabla \times \mathbf{v} = 0
\]

This is the realm of irrotational, potential, inviscid, isentropic, or ideal flow, where there is no tendency for an object placed within the fluid to undergo any type of rotation. For comparison, if the velocity vector of a rigid solid which experiences solid-body rotation is given by;

\[
\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}
\]

then the vorticity vector corresponds to the angular velocity vector. For 2-dimensional flow in cylindrical coordinates, with \( \mathbf{v}_r(r, \Theta) \) and \( \mathbf{v}_\Theta(r, \Theta) \), the volume-averaged vorticity vector, defined by;
\frac{1}{V} \int_V \left[ \nabla \cdot (\mathbf{v} \times \nabla) \right] dV = \frac{1}{2\pi R^2 L} \int_V \left[ \delta_r \left( \frac{\partial v_z}{r} - \frac{\partial v_\theta}{\partial z} \right) + \delta_\theta \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) + \delta_z \left( \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \right] r dr d\Theta dz

= \frac{\delta_z}{2\pi R^2} \int_0^R \int_0^{2\pi} \left[ \frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] r dr d\Theta

only contains a non-trivial z-component which simplifies considerably to the average angular velocity vector of the fluid. Potential flow in liquids implies that there are no rotational tendencies within the fluid, especially near a boundary. The microscopic description of potential flow requires that the vorticity vector must vanish. The macroscopic description of potential flow requires that there is no large-scale vorticity, which implies that the volume-averaged vorticity vector must vanish. From a mathematical viewpoint based on the microscopic description, the vorticity vector will vanish if one identifies any scalar velocity potential \( \Phi \) (not to be confused with the gravitational energy per unit mass of fluid), such that;

\[ \mathbf{v} = \nabla \Phi \]

because;

\[ \nabla \times \nabla \Phi = 0 \]

via Stokes' theorem if \( \Phi \) is an exact differential. This is true for any multivariable scalar function that is analytic and path-independent, because the order of mixed 2\textsuperscript{nd} partial differentiation can be reversed without affecting the final result. Hence, the requirement of no vorticity at the microscopic level, which is consistent with irrotational flow, suggests that the fluid velocity vector can be expressed as the gradient of a scalar velocity potential. However, the requirement of no vorticity does not provide a unique function for \( \Phi \) because any scalar that is an exact differential will satisfy the previous equation. The unique scalar velocity potential for a particular ideal flow problem is calculated by invoking incompressibility. Hence;

\[ \nabla \cdot \mathbf{v} = \nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0 \]

which is Laplace's equation. Potential flow solutions in n-dimensions (i.e, \( 1 \leq n \leq 3 \)) are obtained by solving one 2\textsuperscript{nd}-order partial differential equation (i.e., Laplace's equation) for \( \Phi \) in terms of \( n \) independent spatial variables. This is one of the most straightforward routes to calculate 3-dimensional flows.
Mass transfer coefficients and Sherwood numbers that are consistent with the steady state film theory of interphase mass transfer. The film theory describes steady state one-dimensional diffusion across an interface and into a stagnant medium. The examples presented below are specific to planar, cylindrical, and spherical interfaces. The overall objective is to develop relations between mass transfer coefficients, molecular transport properties, and film thicknesses. Curvature correction factors are required when radial diffusion occurs across cylindrical and spherical interfaces. Analogies between heat and mass transfer are invoked to obtain the corresponding heat transfer coefficients. General molar density and temperature profiles have been obtained in three important coordinate systems from the solution of Laplace's equation in one dimension. Two boundary conditions are required to evaluate the integration constants in each case. In terms of the molar density of mobile component A, equilibrium is achieved at the interface (i.e., \( C_{A,\text{equilibrium}} \)), which corresponds to the solubility of A in the phase of interest. As one moves away from the interface in the direction of the unit normal vector and travels through the stagnant film which contains all of the concentration gradients, bulk fluid conditions are achieved (i.e., \( C_{A,\text{bulk}} \)) at the edge of the film. Three separate mass transfer problems are considered below.

**Steady state diffusion and conduction across flat interfaces in rectangular coordinates;**

If \( z \) is the independent variable measured normal to the interface and the stagnant film is described by a constant thickness \( L \), then the boundary conditions are;

\[
\begin{align*}
(1) & \quad \text{At } z = 0, C_A = C_{A,\text{equilibrium}} \\
(2) & \quad \text{At } z = L, C_A = C_{A,\text{bulk}}
\end{align*}
\]

The two integration constants in the linear molar density profile are calculated as follows;

\[
\begin{align*}
(1) & \quad C_{A,\text{equilibrium}} = a_0 \\
(2) & \quad C_{A,\text{bulk}} = a_0 + a_1 L
\end{align*}
\]

Hence, \( a_1 = -\Delta C_A/L \), where \( \Delta C_A = C_{A,\text{equilibrium}} - C_{A,\text{bulk}} \) represents the overall concentration driving force for mobile component A. The basic information for mass transfer at the microscopic continuum level, consistent with steady state one-dimensional diffusion into a stagnant medium, is;

\[
\frac{C_{A,\text{equilibrium}} - C_A(z)}{\Delta C_A} = \frac{z}{L}
\]

The corresponding temperature profile for steady state one-dimensional conduction in an isotropic solid or stagnant fluid is;
where $T_{\text{Surface}}$ is analogous to $C_{A, \text{equilibrium}}$, and $T_{\text{Bulk}}$ is analogous to $C_{A, \text{bulk}}$. If film thickness $L$ is interpreted as a hydrodynamic factor that decreases at higher Reynolds numbers for the phase where $C_A(z)$ and $T(z)$ were determined, then this simple model exhibits some of the characteristics of more complex heat and mass transfer problems, even though convective transport was neglected.

**Effect of curvature for radial diffusion and conduction across cylindrical interfaces;**

Results from the previous section must be modified slightly when radial diffusion occurs across a curved interface, like the lateral surface of a long cylinder. The logarithmic profile accounts for the fact that the surface area normal to diffusional flux in the radial direction is not constant (i.e., it scales linearly with radial position $r$). If a stagnant film of radius $R_{\text{Film}}$ surrounds a cylindrical solid of radius $R_{\text{Solid}}$, then the appropriate "film theory" boundary conditions are:

1. At $r = R_{\text{Solid}}$, $C_A = C_{A, \text{equilibrium}}$
2. At $r = R_{\text{Film}}$, $C_A = C_{A, \text{bulk}}$

The film thickness is $R_{\text{Film}} - R_{\text{Solid}}$, and the integration constants in the molar density profile are calculated from the following equations:

1. $C_{A, \text{equilibrium}} = a_2 + a_3 \ln R_{\text{Solid}}$
2. $C_{A, \text{bulk}} = a_2 + a_3 \ln R_{\text{Film}}$

One obtains:

$$a_3 = \frac{-\Delta C_A}{\ln \left( R_{\text{Film}} / R_{\text{Solid}} \right)}; a_2 = C_{A, \text{bulk}} + \Delta C_A \left\{ \frac{\ln R_{\text{Film}}}{\ln \left( R_{\text{Film}} / R_{\text{Solid}} \right)} \right\}$$

The molar density profile for radial diffusion across a cylindrical interface is given by:

$$\frac{C_A(r) - C_{A, \text{bulk}}}{\Delta C_A} = \frac{\ln \left( R_{\text{Film}} / r \right)}{\ln \left( R_{\text{Film}} / R_{\text{Solid}} \right)}$$
The corresponding temperature profile is;

\[
\frac{T(r) - T_{\text{Bulk}}}{T_{\text{Surface}} - T_{\text{Bulk}}} = \frac{\ln(R_{\text{Film}} / r)}{\ln(R_{\text{Film}} / R_{\text{Solid}})}
\]

**Steady state radial diffusion and conduction in spherical coordinates;**

Now, the surface area normal to diffusional flux in the radial direction scales as the square of radial position \(r\), and this effect of curvature yields a molar density or temperature profile that depends inversely on \(r\). A stagnant film of radius \(R_{\text{Film}}\) surrounds a bubble, liquid droplet, or spherical solid pellet of radius \(R_{\text{Solid}}\). The constant film thickness is \(R_{\text{Film}} - R_{\text{Solid}}\). This is the only coordinate system that will produce reasonable results when the thickness of the stagnant film becomes infinitely large (i.e., when \(R_{\text{Film}} \Rightarrow \infty\)). The appropriate boundary conditions are;

1. At \(r = R_{\text{Solid}}\), \(C_A = C_{A,\text{equilibrium}}\)
2. At \(r = R_{\text{Film}}\), \(C_A = C_{A,\text{bulk}}\)

Integration constants \(a_4\) and \(a_5\) are calculated as follows;

1. \(C_{A,\text{equilibrium}} = a_4 + a_5 / R_{\text{Solid}}\)
2. \(C_{A,\text{bulk}} = a_4 + a_5 / R_{\text{Film}}\)

Hence;

\[
a_5 = \frac{\Delta C_A}{R_{\text{Solid}}} - \frac{1}{R_{\text{Film}}}; a_4 = C_{A,\text{bulk}} - \frac{\Delta C_A}{R_{\text{Film}}} \frac{1}{R_{\text{Solid}}} \frac{1}{R_{\text{Film}}}
\]

The final results at the microscopic continuum level for molecular transport in the radial direction across a spherical interface are;

\[
\frac{C_A(r) - C_{A,\text{bulk}}}{\Delta C_A} = \frac{1}{r} - \frac{1}{R_{\text{Film}}}; \quad \frac{T(r) - T_{\text{Bulk}}}{T_{\text{Surface}} - T_{\text{Bulk}}} = \frac{1}{R_{\text{Solid}}} - \frac{1}{R_{\text{Film}}}
\]

When the stagnant film is infinitely thick, which implies that one must travel an infinite distance away from the spherical interface to achieve bulk conditions, the right side of the previous two equations for \(C_A(r)\) and \(T(r)\) reduces to \(R_{\text{Solid}} / r\).
**General strategy to calculate interphase transfer coefficients;**

(i) Draw a picture of the problem and identify the coordinate system that exploits the symmetry of the macroscopic boundary. Now, the interface under investigation should conform to a “simple surface”.

(ii) Solve a simplified version of the mass transfer equation that includes only the most important mass transfer rate processes, together with the supporting boundary conditions. The objective is to obtain the molar density profile of mobile component A that is transported across the interface. When the mass transfer Peclet number is sufficiently large, it is reasonable to neglect molecular transport (i.e., conduction, diffusion, or viscous stress) in the primary flow direction. However, one should never neglect molecular transport normal to the interface.

(iii) Identify the unit normal vector $n$, which is perpendicular to the interface and points in the direction of interphase transport.

(iv) Construct the scalar "dot" product of $n$ with the molecular flux of species A whose molar density profile within the mass transfer boundary layer was obtained in the previous sections for interfaces with rectangular, cylindrical, or spherical symmetry.

(v) Define the interphase transfer coefficient in terms of the normal component of molecular flux, evaluated at the interface;

$$\{ n \cdot \text{Molecular flux} \} \text{Evaluated at the Interface} = \left[ \text{Transfer Coefficient} \right] \left[ \text{Driving Force} \right]$$

**Rectangular coordinates;**

The flat interface is defined by $z=0$, and the unit normal vector, oriented in the direction of interphase mass transfer, is $n = \delta_z$. The molar density profile for mobile component A, which is consistent with one-dimensional diffusion in the $z$-direction, was developed in a previous section;

$$\frac{C_{A,\text{equilibrium}} - C_A(z)}{\Delta C_A} = \frac{z}{L}$$

The normal component of the diffusional mass flux vector of species A, evaluated at the interface, is;

$$n \cdot \left\{ -D_{A,\text{mixture}} \nabla C_A \right\}_{z=0} = -D_{A,\text{mixture}} \left( \frac{dC_A}{dz} \right)_{z=0} = D_{A,\text{mixture}} \frac{\Delta C_A}{L} = k_C \Delta C_A$$
Hence, the simplest mass transfer coefficient for steady state one-dimensional diffusion across a flat interface into a stagnant fluid with a constant film thickness $L$, is given by;

$$k_C = \frac{D_{A,mixture}}{L}$$

This corresponds to a Sherwood number of unity, when the characteristic length is $L$ in the definition of the Sherwood number.

Unsteady state diffusion with no chemical reaction

(a) Fick's 2nd law
(b) Introduction to boundary layer analysis

Fluid dynamics p.#115
Heat transfer Eq.#11.2-10 on p.#375, also p.#338
Mass transfer Eq.#19.1-18 on p.#585, also p.#613

**Leibnitz rule for differentiating one-dimensional integrals when the limits of integration are not constant.** This theorem is based on the following integral expression, where the integrand and the limits of integration depend on the independent variable. For example, if;

$$\Gamma(t) = \int_{x=u(t)}^{w(t)} f(x,t)dx$$

then, in general;

$$\Gamma(t) = \Gamma\{t,u(t),w(t)\} = \Gamma(t,u,w)$$

The hierarchy is that $\Gamma$ depends directly on $t$, $u$, and $w$; whereas $u$ and $w$ both depend on $t$. Hence, the total differential of $\Gamma$ is;

$$d\Gamma = \left(\frac{\partial \Gamma}{\partial t}\right)_{u,w} dt + \left(\frac{\partial \Gamma}{\partial u}\right)_{t,w} du + \left(\frac{\partial \Gamma}{\partial w}\right)_{t,u} dw$$

Therefore;

$$\frac{d\Gamma}{dt} = \left(\frac{\partial \Gamma}{\partial t}\right)_{u,w} + \left(\frac{\partial \Gamma}{\partial u}\right)_{t,w} \frac{du}{dt} + \left(\frac{\partial \Gamma}{\partial w}\right)_{t,u} \frac{dw}{dt}$$
The 1st term on the right side of the previous equation is obtained by taking the partial derivative of \( f(x,t) \) with respect to \( t \) "inside the integral", because the limits of integration are treated as constants. Now, it is necessary to obtain expressions for \((\partial \Gamma/\partial u)_{t,w}\) and \((\partial \Gamma/\partial w)_{t,u}\). This is achieved by introducing a new function \( \Xi(x,t) \), which represents the indefinite integral of \( f(x,t) \) with respect to \( x \). In other words;

\[
\Xi(x,t) = \int f(x,t) \, dx \\
\Gamma(t,u,w) = \int_{x=u(t)}^{w(t)} f(x,t) \, dx = \Xi(w,t) - \Xi(u,t)
\]

The "fundamental theorem of calculus" states that;

\[
\left( \frac{\partial \Xi}{\partial x} \right)_t = f(x,t)
\]

Therefore, if one replaces \( x \) by either \( u \) or \( w \), then;

\[
\left( \frac{\partial \Xi}{\partial u} \right)_t = f(u,t); \left( \frac{\partial \Xi}{\partial w} \right)_t = f(w,t)
\]

The relation between \( \Gamma \) and \( \Xi \) above yields the following partial derivatives of interest;

\[
\left( \frac{\partial \Gamma}{\partial u} \right)_{t,w} = -\left( \frac{\partial \Xi}{\partial u} \right)_t = -f(u,t) \\
\left( \frac{\partial \Gamma}{\partial w} \right)_{t,u} = +\left( \frac{\partial \Xi}{\partial w} \right)_t = +f(w,t)
\]

The final result is;

\[
\frac{d\Gamma}{dt} = \int_{x=u(t)}^{w(t)} \left\{ \frac{\partial f(x,t)}{\partial t} \right\} dx + f(w,t) \frac{dw}{dt} - f(u,t) \frac{du}{dt}
\]

Convection and diffusion through permeable membranes

(a) Plug flow models
(b) Analysis of blood capillaries
Review for Exam#3

Problem#1
Consider steady state heat conduction across a solid-liquid interface with spherical symmetry. A solid sphere of radius $R_{\text{Solid}}$ is suspended by a metal wire in a stagnant fluid. An electrical current generator maintains the solid sphere at constant temperature, given by $T_{\text{Solid}}$, everywhere throughout the solid. Thermal energy is transported from the solid to the stagnant fluid solely by radial conduction, and the stagnant film which surrounds the solid sphere is infinitely thick, such that $T_{\text{Bulk}}$, which is less than $T_{\text{Solid}}$, is achieved as $r \to \infty$. There is no source of thermal energy within the fluid. Calculate the heat transfer coefficient for this steady state process. Then, modify your result for a very thin stagnant film in the liquid phase surrounding the sphere, such that $R_{\text{Film}} \approx R_{\text{Solid}}$.

Answer:
Begin with the temperature profile that represents the solution to Laplace's equation for steady state conduction exclusively in the radial direction in spherical coordinates;

\[
\frac{T(r) - T_{\text{Bulk}}}{T_{\text{Surface}} - T_{\text{Bulk}}} = \frac{1}{r} - \frac{1}{R_{\text{Film}}}
\]

Use Fourier's law and evaluate the molecular flux of thermal energy, normal to the solid-liquid interface in the radial direction. By definition, this conductive heat flux, evaluated at the interface, is given by the product of a heat transfer coefficient and a temperature driving force;

\[
n \cdot \{ -k_{TC} \nabla T \}_{r=R_{\text{Solid}}} = -k_{TC} \left( \frac{dT}{dr} \right)_{r=R_{\text{Solid}}} = -k_{TC} \left( T_{\text{Surface}} - T_{\text{Bulk}} \right) \left( -\frac{1}{R_{\text{Solid}}} \right) \frac{1}{1} = h_{HTC} \left( T_{\text{Surface}} - T_{\text{Bulk}} \right)
\]

\[
h_{HTC} = \left( \frac{k_{TC}}{R_{\text{Solid}}} \right) \frac{1}{1 - \frac{R_{\text{Solid}}}{R_{\text{Film}}}}
\]

Now, evaluate the result for the heat transfer coefficient when the steady state film thickness (i.e., $R_{\text{Film}} - R_{\text{Solid}}$) is infinitely large (i.e., $R_{\text{Film}} >> R_{\text{Solid}}$). The final result for this “stagnant film theory” problem in spherical coordinates is;
h_{HTC} = k_{TC} / R_{Solid}

which corresponds to a Nusselt number of \( \text{Nu} = 2 \) when the characteristic length is the diameter of the sphere. When the stagnant film is very thin, the previous heat transfer coefficient expression is manipulated by defining \( \varepsilon = R_{Film} / R_{Solid} \), expanding the denominator of the curvature correction factor in spherical coordinates about \( \varepsilon = 1 \), and truncating the Taylor series after the linear term because \( \varepsilon \) is very close to unity;

\[
1 - \frac{R_{Solid}}{R_{Film}} \approx 0 + \frac{\varepsilon - 1}{1!} \frac{d}{d\varepsilon} \left( 1 - \frac{1}{\varepsilon} \right)_{\varepsilon=1} + ... \approx \varepsilon - 1
\]

\[
\text{CurvatureCorrectionFactor} = \frac{1}{1 - \frac{R_{Solid}}{R_{Film}}} \approx \frac{1}{\varepsilon - 1} \approx \frac{R_{Solid}}{R_{Film} - R_{Solid}}
\]

One obtains the following expression for the steady state heat transfer coefficient that is consistent with radial conduction into a stagnant medium when the thermal boundary layer thickness, \( \delta_T = R_{Film} - R_{Solid} \), is very small and localized at the solid-liquid interface;

\[
h_{HTC} = \left( \frac{k_{TC}}{R_{Solid}} \right) \frac{1}{1 - \frac{R_{Solid}}{R_{Film}}} \approx \left( \frac{k_{TC}}{R_{Solid}} \right) \frac{R_{Solid}}{\delta_T} \approx \frac{k_{TC}}{\delta_T}
\]

This is the heat-transfer analog of steady state diffusion across a “locally flat” interface (see pages 132-133 where \( L \) represents the mass transfer boundary layer thickness) because the curvature correction factor in spherical coordinates is negligible for thin boundary layers.

**Problem #2**

(a) A cold solid sphere of radius \( R_{Solid} \) at initial temperature \( T_{Solid} \) is submersed in a large container of liquid. A heater maintains this liquid bath at a temperature which is much hotter than the initial temperature of the solid sphere, and vigorous stirring of the liquid within the container eliminates all temperature gradients in the liquid phase such that \( T = T_{Surface} \) at \( r = R_{Solid} \) throughout the entire analysis. There is no source of thermal energy within the solid sphere. Write the appropriate differential equation that must be solved to calculate the local temperature profile \( T \) within the solid, and don’t include any unnecessary terms.
Answer:
This problem is classified as unsteady state heat conduction in a solid sphere. Since there is no convective transport in solids, and viscous dissipation is nonexistent, one must solve the unsteady state heat conduction equation;

\[
\frac{\partial T}{\partial t} = \alpha \nabla \cdot \nabla T = \alpha \nabla^2 T
\]

where \( \alpha = k_{TC}/(\rho C_p) \) is the thermal diffusivity of the solid. If heat conduction occurs exclusively in the radial direction, then the previous equation requires the contribution in the \( r \)-direction for the divergence of the gradient of a scalar in spherical coordinates. This is equivalent to the contribution in the \( r \)-direction for the Laplacian of a scalar. Hence, one must account for the fact that the surface area normal to the molecular flux of thermal energy in the radial direction is variable, and scales as the square of radial position. The partial differential equation for \( T(r,t) \) is;

\[
\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right)
\]

(b) Include all of the boundary conditions that are required to obtain a unique solution to your thermal energy balance in part (a).

Answer:
One condition is required on independent variable \( t \);

\[
(1) \quad t = 0, \text{ for all } r < R_{\text{Solid}}; \quad T = T_{\text{Solid}} \quad \text{(initial condition)}
\]

Two conditions are required on independent spatial variable \( r \) for heat conduction;

\[
(2) \quad r = R_{\text{Solid}}, \text{ for all } t > 0; \quad T = T_{\text{Surface}} \quad \text{(interfacial condition)}
\]

At short times, this problem can be treated as a boundary layer problem via combination of variables. Hence, if \( \delta_T \) represents the thermal boundary layer thickness measured inward from the fluid-solid interface at \( r = R_{\text{Solid}} \), then;

\[
(3a) \quad r < R_{\text{Solid}} - \delta_T, \text{ for finite } t; \quad T = T_{\text{Solid}} \quad \text{(boundary layer BC)}
\]

However, if one seeks the solution for \( T(r,t) \) from the initial condition until the solid equilibrates with the constant temperature bath, then separation of variables is the
method of choice, and symmetry at the center of the sphere is appropriate. In other words, the molecular flux of thermal energy vanishes at \( r=0 \) due to symmetry, and Fourier's law stipulates that:

\[
(3b) \quad r = 0; \; \frac{dT}{dr} = 0 \quad \text{(symmetry at the center of the sphere)}
\]

Another possible condition is \( T = T_{\text{Surface}} \) at long times for all values of \( 0 \leq r \leq R_{\text{Solid}} \), but this obvious result might be satisfied automatically by the exponentially decreasing function of time in the separation-of-variables solution for \( T(r,t) \).

**Problem#3**
An incompressible Newtonian fluid flows past a rectangular solid plate that is soluble in the liquid. Hence, mobile component A is transported from the solid to the liquid. The interface between the solid and the liquid is locally flat, and laminar flow is appropriate to describe flow of the liquid parallel to the interface. The \( x \)-direction is parallel to the interface, and the \( y \)-direction is perpendicular to the interface.

(a) If the mass transfer Peclet number is very large, then write the simplified version of the steady state microscopic mass transfer equation which must be solved to calculate the molar density profile of species A, \( C_A(x,y) \), within the incompressible liquid phase for this non-reactive problem. Include all three boundary conditions that are required to obtain a unique solution for \( C_A(x,y) \).

**Answer:**
Begin with the most general form of the microscopic mass transfer equation for species A;

\[
\frac{\partial C_A}{\partial t} = -v \cdot \nabla C_A + D_{A,\text{Mixture}} \nabla^2 C_A + \sum_j v_{A_j} R_j
\]

For steady state analysis of non-reactive systems, the MTE reduces to;

\[
v \cdot \nabla C_A = D_{A,\text{Mixture}} \nabla^2 C_A
\]

Convection and diffusion occur, at most, in three coordinate directions. For two-dimensional transport in rectangular coordinates, the MTE for \( C_A(x,y) \) reduces to;

\[
v_x \frac{\partial C_A}{\partial x} + v_y \frac{\partial C_A}{\partial y} = D_{A,\text{Mixture}} \left\{ \frac{\partial^2 C_A}{\partial x^2} + \frac{\partial^2 C_A}{\partial y^2} \right\}
\]
Since the primary direction of fluid flow coincides with the x-direction, and the mass transfer Peclet number is large, it is reasonable to neglect molecular transport by diffusion in the x-direction relative to convective mass transfer in the x-direction. If convective mass transfer occurs in both the x- and y-directions, where flow in the y-direction is of secondary importance, then one obtains the solution for $C_A$ from the following partial differential equation:

$$v_x \frac{\partial C_A}{\partial x} + v_y \frac{\partial C_A}{\partial y} = D_{A,Mixture} \frac{\partial^2 C_A}{\partial y^2}$$

If convective mass transfer occurs exclusively in the x-direction, then the appropriate form of the mass balance for $C_A(x,y)$ is:

$$v_x \frac{\partial C_A}{\partial x} = D_{A,Mixture} \frac{\partial^2 C_A}{\partial y^2}$$

(b) The solution to part (a) for the molar density profile of mobile component A is given by:

$$\frac{C_{A,\text{equilibrium}} - C_A(\eta)}{C_{A,\text{equilibrium}} - C_{A,\text{bulk}}} = P(\eta) = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_0^\eta \exp\left(-\frac{u^3}{3}\right)du$$

$$\eta = \frac{y}{\delta_C(x)}$$

where $\Gamma(4/3)$ is the gamma function evaluated when the argument is $4/3^{rd}$ and $\delta_C(x)$ represents the mass transfer boundary layer thickness that increases at larger values of x along the interface, with $\delta_C=0$ at x=0. Calculate the local mass transfer coefficient, $k_{C,\text{local}}$, and indicate its functional dependence on the appropriate spatial coordinate or coordinates.

**Answer:**

Since there is no convective flux at a high-shear no-slip interface, use Fick’s 1st-law of diffusion and calculate the y-component of diffusional mass flux at $y=0$, for all $x>0$ (i.e., $\eta=0$). The unit normal vector at the interface, oriented in the direction that mass transfer occurs, is $\mathbf{n} = \delta_y$. Hence, one performs the following scalar dot product operation and equates the result to the product of a local mass transfer coefficient, $k_{C,\text{local}}$, and the overall concentration driving force, $C_{A,\text{equilibrium}} - C_{A,\text{bulk}}$: 

$$151$$
\[ n \cdot \left\{ -D_{A, \text{Mixture}} \nabla C_A \right\}_{y=0} = -D_{A, \text{Mixture}} \frac{\partial C_A}{\partial y}_{y=0} = -D_{A, \text{Mixture}} \frac{dC_A}{dP} \frac{1}{\delta_c(x)} = k_{C, \text{local}} (C_{A, \text{equilibrium}} - C_{A, \text{bulk}}) \]

Hence, the local mass transfer coefficient is;

\[ k_{C, \text{local}} (x) = D_{A, \text{Mixture}} \frac{dP}{d\eta}_{\eta=0} \frac{1}{\delta_c(x)} \]

The Leibnitz rule indicates that the derivative of the dimensionless molar density profile with respect to the combined variable \( \eta \), evaluated at the solid-liquid interface for all \( x > 0 \) (i.e., \( \eta = 0 \)), is;

\[ \left( \frac{dP}{d\eta} \right)_{\eta=0} = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \left\{ \exp(-\eta^3) \right\}_{\eta=0} = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \]

The final expression for the local mass transfer coefficient, which depends on spatial coordinate \( x \) measured parallel to the interface, is;

\[ k_{C, \text{local}} (x) = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \frac{D_{A, \text{Mixture}}}{\delta_c(x)} \]

(c) Without using any equations, describe how the local mass transfer coefficient changes as one moves along the solid-liquid interface in the primary flow direction (i.e., increasing \( x \)), when the Reynolds number remains constant. Does \( k_{C, \text{local}} \) increase, decrease, remain constant, or is it too complex to determine how the local mass transfer coefficient changes? Provide a qualitative explanation for your answer in one or two sentences.

**Answer:**

The local mass transfer coefficient **decreases** as one moves along the interface in the direction of flow (i.e., increasing \( x \)) because the resistance to mass transfer increases, due to the fact that the boundary layer thickness increases. The answer to part (b) reveals that \( k_{C, \text{local}} \) depends inversely on the thickness of the mass transfer boundary layer, \( \delta_c \).
Helpful hint: The Leibnitz rule for differentiating a one-dimensional integral, where the integrand and both limits of integration depend on the independent variable, is:

$$
\Gamma(t) = \int_{x=u(t)}^{w(t)} f(x,t) \, dx
$$

$$
\frac{d\Gamma}{dt} = \int_{x=u(t)}^{w(t)} \left\{ \frac{\partial f(x,t)}{\partial t} \right\} \, dx + f(w,t) \frac{dw}{dt} - f(u,t) \frac{du}{dt}
$$

Diffusion and Chemical Reaction Across Spherical Gas-Liquid Interfaces
see Chapter #13 in TPfCRD and Chapter #19 in BSL’s Transport Phenomena.

Radial diffusion and 1st-order chemical reaction in spherical coordinates
Chemical reaction enhancement of mass transfer coefficients
Problem 19B.6, pp. 607-8 (BSL); transient, steady state and quasi-steady state analyses

Convection, Diffusion & Chemical Reaction in Multiphase Reactors
see Chapter #24 in TPfCRD

- Chlorination of benzene in a gas-liquid continuous stirred tank
- Gas phase mass balances with interphase mass transfer
- Liquid phase mass balances with chemical reaction and interphase transfer
- Chemical reaction enhancement of interphase mass transfer coefficients
- Equilibrium at the gas-liquid interface
- Minimal gas phase resistance in the mass transfer boundary layer
- Interfacial area for gas-liquid mass transfer
- Time constants for three important mass transfer rate processes
- Dimensionless mass balances in the gas and liquid phases
- Molecular diffusion in liquids

Coupled Heat and Mass Transfer in Non-Isothermal Liquid Phase
Tubular Reactors with Strongly Exothermic Chemical Reaction
see Chapter #4 in TPfCRD

- Thermal Runaway, Parametric Sensitivity and Multiple Stationary States
  Strategies to control thermal runaway
Plug flow mass balance at high mass transfer Peclet numbers
Thermal energy balance at high heat transfer Peclet numbers
Thermodynamics of multicomponent mixtures
Conductive heat transfer across the lateral surface of the reactor
   Adiabatic reactors
   Constant heat flux across the wall
   Constant outer wall temperature
      Manipulating the outer wall temperature
      Manipulating the surface-to-volume ratio of the reactor
Coupled heat and mass transfer in PFR's with cocurrent cooling
   Manipulating the flowrate of a cocurrent cooling fluid
Parametric sensitivity analysis in non-isothermal tubular reactors
Exothermic/endothermic reactions with cocurrent cooling
Concentric double-pipe configurations that are not insulated
Countercurrent cooling in concentric double-pipe configurations
Multiple stationary states in PFR's with countercurrent cooling
   Examples of multiple stationary states

Review for Exam#4

Problem#1
Consider steady state mass transfer via convection and diffusion through a blood vessel at very large mass transfer Peclet numbers with no chemical reaction.

(a) Obtain an expression for the molar density profile of mobile component A within the blood vessel, $C_A(z)$, as a function of axial coordinate $z$, which increases in the primary flow direction through the permeable capillary. The well-mixed concentration of species A outside of the capillary is approximately constant (i.e., $C_{A,\text{bulk}}$), and $C_A$ at the capillary inlet (i.e., $z=0$) is zero.

(b) Sketch $C_A$ vs. $z$ in the laminar flow regime when the Reynolds number, based on the capillary diameter, is 200 and 1000. Put both curves on one graph, with $C_A$ on the vertical axis and $z$ on the horizontal axis. Label each curve with the appropriate value of the Reynolds number.

(c) If the Reynolds number is 500 in the laminar flow regime, then sketch $C_A$ vs. $z$ at $35^\circ\text{C}$ and $40^\circ\text{C}$. Put both curves on one graph, with $C_A$ on the vertical axis and $z$ on the horizontal axis. Label each curve with the appropriate temperature.
Problem#2
(a) Consider one-dimensional radial diffusion and 1\textsuperscript{st}-order irreversible chemical reaction in the liquid phase external to a gas bubble of radius \( R \). Obtain an expression for the molar density profile of reactant \( A \), \( C_A(r) \), if the steady state film that surrounds the bubble is infinitely thick. \textbf{Hint}: Adopt a solution in terms of exponential functions, not hyperbolic functions.

Answer:
In dimensional notation for the molar density of species \( A \), the spherical coordinate mass transfer equation in the liquid phase with radial diffusion and 1\textsuperscript{st}-order irreversible chemical reaction yields the following 2\textsuperscript{nd}-order ODE;

\[
D_{A,\text{Mixture}} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_A}{dr} \right) = k_1 C_A
\]

The following transformation [i.e., \( C_A = (1/r)\Psi_A \)] allows one to rewrite the previous ODE with variable coefficients as a frequently occurring 2\textsuperscript{nd}-order ODE with constant coefficients;

\[
\frac{dC_A}{dr} = \frac{1}{r} \frac{d\Psi_A}{dr} - \Psi_A
\]

\[
r^2 \frac{dC_A}{dr} = r \frac{d\Psi_A}{dr} - \Psi_A
\]

\[
\frac{d}{dr} \left( r^2 \frac{dC_A}{dr} \right) = r \frac{d^2\Psi_A}{dr^2}
\]

\[
D_{A,\text{Mixture}} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dC_A}{dr} \right) = D_{A,\text{Mixture}} \frac{1}{r} \frac{d^2\Psi_A}{dr^2} = k_1 C_A = k_1 \frac{1}{r} \Psi_A
\]

\[
D_{A,\text{Mixture}} \frac{d^2\Psi_A}{dr^2} = k_1 \Psi_A
\]

If \( \eta = r/R \), then the solution to this ODE for \( \Psi_A(\eta) \) is written as follows in terms of exponential functions;

\[
\Psi_A(\eta) = A \exp(\Lambda_A \eta) + B \exp(-\Lambda_A \eta)
\]

\[
\Lambda_A^2 = \frac{k_1 R^2}{D_{A,\text{Mixture}}}
\]
where $\Lambda_A^2$ is the Damkohler number for species A, which represents a ratio of the rate of chemical reaction to the rate of molecular mass transfer via diffusion. The general solution for the molar density of reactant A is:

$$C_A = \frac{1}{r} \Psi_A(\eta) = \frac{1}{r} \left\{ A \exp(\Lambda_A \eta) + B \exp(-\Lambda_A \eta) \right\}$$

(b) If the boundary conditions are:

(i) At $r = R$ (i.e., $\eta = 1$), $C_A = C_{A,\text{equilibrium}}$
(ii) As $r \to \infty$, $C_A = 0$

then calculate the integration constants in your expression for $C_A(r)$ from part (a) and express your final answer for $C_A(r)$ in terms of the Damkohler number for reactant A.

Answer:
The exponential term with the positive argument in the general solution for $C_A$ increases in unbounded fashion as radial position $r$ (or $\eta$) tends toward infinity. This can be verified via one application of l'Hopital's rule. Hence, one sets integration constant $A$ to zero, as required by boundary condition (ii). Integration constant $B$ is evaluated using boundary condition (i);

$$C_{A,\text{equilibrium}} = \frac{B}{R} \exp(-\Lambda_A)$$

$$C_A(r) = C_{A,\text{equilibrium}} \left( \frac{R}{r} \right) \exp(-\Lambda_A \eta)$$

(c) Obtain an expression for the chemical-reaction-enhanced mass transfer coefficient $k_{C,\text{Liquid}}$ on the liquid side of the gas-liquid interface, which exhibits dependence on the Damkohler number that is slightly different from the results discussed in class.

Answer:
Use Fick's 1st law and evaluate the diffusional flux of reactant A in the radial direction on the liquid side of the gas-liquid interface, at $r = R$. Then, equate this result to the product of a liquid-phase mass transfer coefficient and the concentration difference, or driving force, $C_{A,\text{equilibrium}} = 0$. This generalized procedure yields the following expression for $k_{C,\text{Liquid}}$
\[-D_{A,\text{Mixture}} \left( \frac{dC_A}{dr} \right)_{r=R} = -D_{A,\text{Mixture}} C_{A,\text{equilibrium}} \frac{R}{\exp(-\Lambda_A)} \left\{ \frac{d}{dr} \left[ \frac{1}{r} \exp\left( -\frac{\Lambda_A r}{R} \right) \right] \right\}_{r=R} \]
\[= -D_{A,\text{Mixture}} C_{A,\text{equilibrium}} \frac{R}{\exp(-\Lambda_A)} \left\{ \frac{1}{r^2} \exp\left( -\frac{\Lambda_A r}{R} \right) - \frac{\Lambda_A}{R} \exp\left( -\frac{\Lambda_A r}{R} \right) \right\}_{r=R} \]
\[= \frac{D_{A,\text{Mixture}}}{R} \left[ 1 + \Lambda_A \right] \left( C_{A,\text{equilibrium}} - 0 \right) = k_{C,\text{Liquid}} \left( C_{A,\text{equilibrium}} - 0 \right) \]
\[k_{C,\text{Liquid}} = \frac{D_{A,\text{Mixture}}}{R} \left[ 1 + \Lambda_A \right] \]

The second term in brackets \( [ \] for \( k_{C,\text{Liquid}} \) dominates for diffusion-controlled chemical reactions because the Damköhler number is very large (i.e., \( 1 + \Lambda_A \approx \Lambda_A \)), whereas the first term in brackets \( [ \) is more important for reaction-controlled situations (i.e., \( 1 + \Lambda_A \approx 1 \)) or when no reaction occurs (i.e., \( \Lambda_A \Rightarrow 0 \)). This latter case reduces to steady state diffusion or heat conduction across a spherical interface into a stagnant medium in which the boundary layer thickness is infinitely large and curvature effects cannot be neglected.

**Problem #3 Reactive distillation**

Pure liquid B is flowing at steady state from left to right across a perforated tray in a distillation column and bubbles of gas A rise through the liquid. Gas A is soluble in liquid B, and A reacts irreversibly with B only in the liquid phase. Due to the high concentration of B in the liquid phase, the "method of excess" suggests that the kinetic rate law is pseudo-first-order with respect to the liquid phase molar density of solubilized gas A. The rising motion of the bubbles produces a "well-stirred" liquid mixture of A and B, but the two streams do not leave the tray in equilibrium with each other. At most, equilibrium is established at the spherical gas-liquid interface.

(a) Consider mass transfer rate processes and their corresponding time constants to describe the conditions that must exist if the outlet liquid stream contains a significant fraction of species A, realizing that the inlet stream contains pure liquid B. Do not use any equations.

**Answer:**

*The time constant for interphase mass transfer of species A must be significantly smaller than either of the time constants for (i) chemical reaction in the liquid phase or (ii) convective mass transfer of the liquid phase across the tray in the distillation column (i.e., residence time).*
(b) In the diffusion-limited regime, perform a macroscopic balance on the liquid phase and obtain an algebraic equation that relates the outlet liquid phase molar density of reactant A, $C_{A, outlet}$, to the following quantities:

- $C_{A, equilibrium}$: equilibrium molar density of species A on the liquid side of the gas-liquid interface (i.e., equilibrium solubility of gas A in liquid B, g-mol/cm$^3$)
- $D_{A, Liquid}$: diffusion coefficient of species A in liquid B, (cm$^2$/sec)
- $k_1$: pseudo-first-order kinetic rate constant in the liquid phase, (1/sec)
- $q$: volumetric flow rate of the liquid, (cm$^3$/sec)
- $V_L$: liquid phase volume on the tray, (cm$^3$)
- $\tau$: liquid phase residence time on the tray, $V_L/q$ (sec)
- $a_L$: interfacial area per unit volume of liquid, (1/cm)

Answer:
The liquid phase can be analyzed as a well-mixed CSTR operating at steady state. Hence, one equates rates of input to rates of output for reactant A in its liquid phase mass balance. Since the inlet liquid stream contains pure component B, there is no contribution from convective mass transfer across the inlet plane.

Rate of input due to interphase mass transfer = $\{k_1 D_{A, Liquid}\}^{1/2} [C_{A, equilibrium} - C_{A, outlet}] a_L V_L$ (using a chemical-reaction-enhanced mass transfer coefficient in the diffusion-limited regime, where curvature effects are negligible for thin mass transfer boundary layers)

Rate of output due to convective mass transfer = $q C_{A, outlet}$

Rate of disappearance of reactant A due to 1st-order irreversible reaction = $k_1 C_{A, outlet} V_L$

The steady state liquid-phase CSTR mass balance for reactant A, with dimensions of moles per time, is:

$$\sqrt{k_1 D_{A, Liquid}} (C_{A, equilibrium} - C_{A, outlet}) a_L V_L = q C_{A, outlet} + k_1 C_{A, outlet} V_L$$

$$C_{A, outlet} = C_{A, equilibrium} \frac{a_L \sqrt{k_1 D_{A, Liquid}}}{a_L \sqrt{k_1 D_{A, Liquid}} + k_1 + \frac{1}{\tau}}$$
(c) The Arrhenius activation energy for diffusion of solubilized gas A in liquid B, $E_{act/\text{Diffusion}}$, is much smaller than the Arrhenius activation energy for the chemical reaction, $E_{act/\text{ChemicalReaction}}$. Hence,

$$E_{act/\text{Diffusion}} \ll E_{act/\text{ChemicalReaction}}$$

$$\frac{d}{dT} \ln D_{A,\text{Liquid}} = \frac{E_{act/\text{Diffusion}}}{RT^2} > 0$$

$$\frac{d}{dT} \ln k_1 = \frac{E_{act/\text{ChemicalReaction}}}{RT^2} > 0$$

Describe how a decrease in temperature $T$ will affect the outlet liquid phase molar density of reactant A. Will $C_A$ increase, decrease, remain unchanged, or is it too complex to determine how $C_A$ will change?

**Problem#4**

Use numerical methods (i.e., finite difference calculus) to solve the steady state microscopic mass transfer equation for convective diffusion in heterogeneous catalytic “tube-wall” reactors with circular cross-section in the laminar flow regime for incompressible Newtonian fluids. Chemical reaction at the catalytic surface (i.e., $r=R$) is irreversible and first-order with respect to reactant A. Let the tube radius $R$ be the characteristic length in the definitions of the Damköhler (i.e., $\beta$) and mass transfer Peclet (i.e., $Pe_{MT}$) numbers, and consider the regime where $Pe_{MT}$ is large enough to justify the neglect of axial diffusion.

**Answer:**

The appropriate mass transfer equation is given in Step#5 of Problem 23-7 on page#649 of TPfCRD, and the laminar flow velocity profile is provided in Step#7 on page#650. Hence, the primary objective of this exercise is to calculate the molar density of reactant A, $C_A(r,z)$, from the following partial differential equation and its boundary conditions in cylindrical coordinates, with variable coefficients and chemical reaction at the boundary of the flow configuration;

$$2\langle v_z \rangle_{\text{Average}} \left\{ 1 - \eta^2 \right\} \frac{\partial C_A}{\partial z} = D_A \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial C_A}{\partial r} \right\} = D_A \left\{ \frac{\partial^2 C_A}{\partial r^2} + \frac{1}{r} \frac{\partial C_A}{\partial r} \right\}$$

$$C_A = C_{A,\text{inlet}} @ z = 0, r < R$$

$$\left\{ \frac{\partial C_A}{\partial r} \right\}_{r=0} = 0; -D_A \left\{ \frac{\partial C_A}{\partial r} \right\}_{r=R} = k_{1,\text{Surface}} C_A(r = R, z)$$
The zero-flux boundary condition along the tube axis at \( r=0 \) is a consequence of symmetry, and the *radiation* boundary condition at the catalytic surface (i.e., \( r=R \)) represents a balance between diffusion and chemical reaction. Radial and axial positions are dimensionalized using tube radius \( R \). Hence, \( \eta = r/R \) and \( \zeta = z/R \). Reactant molar density is dimensionalized via the inlet condition, \( \Psi_A(\eta, \zeta) = C_A(r, z)/C_{A,\text{inlet}} \). In terms of the important dimensionless numbers that govern the solution to this problem:

\[
\text{Damkohler\#; } \beta = \frac{k_{1,\text{Surface}} R}{D_{A,\text{ordinary}}},
\]

\[
\text{Peclet\#; } Pe_{MT} = \frac{\langle v_z \rangle_{\text{Average}} R}{D_{A,\text{ordinary}}},
\]

the mass transfer equation and its boundary conditions can be written as follows using dimensionless variables:

\[
2Pe_{MT}\left\{1-\eta^2\right\} \frac{\partial^2 \Psi_A}{\partial \zeta^2} - 2\eta \frac{\partial \Psi_A}{\partial \eta} + \frac{1}{\eta^2} \frac{\partial^2 \Psi_A}{\partial \eta^2} = 0; \quad \Psi_A = 1 \text{ at } \zeta = 0, \eta < 1
\]

\[
\left\{ \frac{\partial \Psi_A}{\partial \eta} \right\}_{\eta=0} = 0; \quad \left\{ \frac{\partial \Psi_A}{\partial \eta} \right\}_{\eta=1} = -\beta \Psi_A (\eta = 1, \zeta)
\]

Problem 23-7 in TPfCRD provides an asymptotically exact mass transfer boundary layer solution for \( C_A(r, z) \) in the inlet region (i.e., \( z>0 \)) for heterogeneous catalytic tubular reactors (see *Step #16* on page #652). A much simpler approach is adopted below to initiate the numerical algorithm by applying the radiation boundary condition at \( z=0 \) and \( r=R \) (or \( \zeta=0 \) and \( \eta=1 \)) to estimate the molar density of reactant A at the wall near the inlet plane. For example:

\[
\left\{ \frac{\partial \Psi_A}{\partial \eta} \right\}_{\eta=1} \approx \frac{\Psi_A(\eta = 1, \zeta = 0) - 1}{\Delta \eta} = -\beta \Psi_A (\eta = 1, \zeta = 0)
\]

\[
\Psi_A (\eta = 1, \zeta = 0) \approx \frac{1}{1 + \beta \Delta \eta}
\]
which exhibits the correct trend, because reactant molar density at the catalytic surface decreases when the rate of reaction is faster and the Damköhler number increases. In an effort to check the validity of the finite-difference solutions to the microscopic mass transfer equation, one poses the following question; Do the microscopic results satisfy the quasi-macroscopic mass balance? Hence, it is necessary to evaluate the bulk molar density of reactant A at each axial position, given by equation (23-19) in TPfCRD, explicitly for tubular reactors. Analogous to equation (23-51) for rectangular ducts (see page#632 in TPfCRD), the dimensionless bulk reactant molar density in tubular reactors is;

\[
C_{A,bulk}(z) = \frac{\int v_z(r)C_A(r,z)rd\Theta}{\pi R^2 \langle v_z \rangle_{Average}} = 4 \int_{\eta=0}^{1} C_A(r,z) \{1-\eta^2\} \eta d\eta
\]

\[
\Psi_{A,bulk}(\zeta) = 4 \int_{\eta=0}^{1} \Psi_A(\eta,\zeta) \{1-\eta^2\} \eta d\eta
\]

Finally, the quasi-macroscopic mass balance for heterogeneous catalytic reactors with first-order irreversible chemical reaction at the boundary, as described on pages 634-636 (TPfCRD), is analyzed completely for uniform catalyst activity on the inner wall of tubes in Problem 23-6 on pages 647-648. Hence, the second equation on page#648 of TPfCRD is dimensionalized as follows;

\[
\langle v_z \rangle_{Average} \pi R^2 \left\{-\frac{dC_{A,bulk}}{dz}\right\} = 2\pi Rk_{1,\text{Surface}} C_A \{r = R, z\}
\]

\[-\frac{d\Psi_{A,bulk}}{d\zeta} = \frac{2\beta}{Pe_{MT}} \Psi_A(\eta = 1, \zeta)\]

These equations are analyzed via the following finite-difference algorithm that can be implemented in conjunction with a linear equation solver. A nonlinear equation solver is required if the chemical kinetics are not first-order.

Important parameters that govern the solution to the convective diffusion mass transfer equation for laminar flow tube-wall reactors

- \(\beta = 150\) Damköhler number; heterogeneous reaction rate wrt diffusion rate
- \(Pe_{MT} = 25\) Mass transfer Peclet number; rate of convection wrt diffusion rate
Numerical grid parameters that determine the total number of grid points and mesh size

\[ N_r = 101 \]  
number of discretized points in the radial direction  
\[ \Delta \eta = 1/(N_r-1) \]  
step size in the radial direction  
\[ \Delta \zeta = 0.001 \]  
step increment in axial position

Establish the dimensionless inlet molar density profile of reactant A

\[ \Psi_A(j,k=0) = 1; \ 1 \leq j \leq N_r-1 \] no conversion in the feed stream at \( \zeta=0 \)  
\[ \Psi_A(N_r,k=0) = 1/(1+\beta \Delta \eta) \] approximate molar density at the wall via the BC at \( \eta=1 \)

Evaluate the dimensionless laminar flow velocity profile at each radial mesh point

\[ \eta(j) = (j-1) \Delta \eta; \ 1 \leq j \leq N_r \]  
\[ v^*_z(j) = 2\{1-[\eta(j)]^2\} \]

Initiate a counter and calculate the dimensionless axial position

\[ k=1 \]  
\[ ***\zeta = k \Delta \zeta \] use a loop and return to this statement each time counter \( k \) is incremented

Symmetry boundary condition at the center of the tube (i.e., \( \eta=0 \))
Second-order-correct forward difference representation for first derivatives, Eq. (23-35)

\[ \frac{1}{2 \Delta \eta} \{ -\Psi_A(3,k) + 4 \Psi_A(2,k) - 3 \Psi_A(1,k) \} = 0 \]

Radiation boundary condition at the catalytic wall (i.e., \( \eta=1 \))
Second-order-correct backward difference representation for first derivatives, Eq. (23-40)

\[ \frac{1}{2 \Delta \eta} \{ 3 \Psi_A(N_r,k) - 4 \Psi_A(N_r-1,k) + \Psi_A(N_r-2,k) \} = -\beta \Psi_A(N_r,k) \]

Implicit finite-difference representation of the convective diffusion mass transfer equation within the tube; \( 1^{st} \) derivative with respect to axial position \( \zeta \) is first-order correct; \( 1^{st} \) and \( 2^{nd} \) spatial derivatives with respect to radial position \( \eta \) are second-order correct, Eq. (23-24)

\[ 2 \leq j \leq N_r-1 \]

\[ Pe_{MT} v^*_z(j) \frac{\Psi_A(j,k) - \Psi_A(j,k-1)}{\Delta \zeta} = \frac{\Psi_A(j+1,k) - 2 \Psi_A(j,k) + \Psi_A(j-1,k)}{(\Delta \eta)^2} + \frac{\Psi_A(j+1,k) - \Psi_A(j-1,k)}{\eta(j)2 \Delta \eta} \]
Calculate the bulk molar density of reactant A via the trapezoidal rule

\[ \Psi_{A,\text{bulk}}(k) = 2\left\{ \frac{\Delta \eta}{2} \right\}^{N_R-1} \sum_{j=2}^{N_R-1} 2\eta(j)v_z^*(j)\Psi_A(j,k) \]

Verify that the finite-difference solution of the microscopic convective diffusion equation also satisfies the quasi-macroscopic mass balance, using the trapezoidal rule

\[ \frac{\Psi_{A,\text{bulk}}(k-1) - \Psi_{A,\text{bulk}}(k)}{\Delta \zeta} = \frac{2\beta}{Pe_{MT}} \Psi_A(N_R,k) \]

Increment the counter, return to the step denoted by 3 asterisks ***, solve the system of linear algebraic equations at the next axial step, calculate the bulk molar density of reactant A and verify that the finite-difference solution also satisfies the quasi-macroscopic mass balance

\[ k = k+1 \]

Go To ***

Calculate the dimensionless tube length \( \zeta = z/R \) that is required to achieve 50% conversion of reactant A to products when the Damköhler number is 150 and the mass transfer Peclet number is 25. Hint: Graph \( \Psi_{A,\text{bulk}} \) vs. \( \zeta \) to obtain the answer.
Predict the thickness of the mass transfer boundary layer \( \delta_{\text{MTBLT}}(\zeta) \), measured inward from the catalytically active surface toward the centerline of the tube, as a fraction of the tube radius \( R \) when \( \zeta = 1 \), \( \beta = 150 \), and \( \text{Pe}_{\text{MT}} = 25 \). Hint: Graph \( \Psi_A(\eta, \zeta=1) \) vs. \( \eta \). Within the mass transfer boundary layer; \( \Psi_A(R-\delta_{\text{MTBLT}}, \zeta=1) \leq 0.98 \)

Problem#5
Develop a plug-flow version of the differential thermal energy balance that must be solved to calculate the temperature profile within a tubular reactor \( T(z) \) as a function of independent variable \( z \) which increases in the direction of flow of the reactive fluid through a straight cylindrical tube of length \( L \) and radius \( R \). There is one pseudo-first-order homogeneous irreversible chemical reaction that converts reactants to products. The control volume is \( dV = \pi R^2 dz \), the wall of the tube at radius \( r=R \) is not insulated from the surroundings, pressure effects are negligible, and the heat transfer Peclet number is only 10 because the Reynolds number is very small in the creeping flow regime. The units of each term in the thermal energy balance should be "energy per volume per time".

Coupled Heat & Mass Transfer in Batch Reactors
see Chapter#6 in TPfCRD

Isothermal analysis of calorimetric rate data
Formalism for multiple chemical reactions
Batch Reactor Problem

Conversion vs. time for a variable-volume batch reactor that produces methanol in the gas phase at constant $T$ and $p$

A stoichiometric feed of carbon monoxide and hydrogen is injected into a batch reactor that operates isothermally at 298K and isobarically at low pressure (i.e., 50 torr). The overall objective is to produce methanol, and simulate the time dependence of the conversion of the key limiting reactant, carbon monoxide. This gas phase reaction is elementary, reversible, and it requires low operating pressures to quench the initiation of undesirable side reactions. The normal boiling point of methanol is 338K at 1 atmosphere pressure. Of course, methanol boils at a much lower temperature when the pressure is 50 torr, so it is acceptable to operate the reactor at 298K and be assured that methanol remains in the gas phase during the course of the reaction. At 298K and 50 torr, the compressibility factor for the gas mixture is $Z = 0.1$ at 298K and 50 torr. The kinetic rate constant for the forward reaction, based on partial pressures in the gas phase, is described by the following Arrhenius parameters; the pre-exponential factor is $2 \times 10^5$ g-mol/cm$^3$-min-(atm)$^3$ and the activation energy divided by the gas constant $R$ is 5000K. Thermodynamic data at 298K reveal that the standard state free energy of formation is $-32,800$ cal/g-mol for carbon monoxide, $-38,700$ cal/g-mol for methanol, and 0 for hydrogen because gaseous H$_2$ is the standard state for hydrogen at 298K. The feed to this batch reactor contains 5 grams of gaseous carbon monoxide. Useful values of the universal gas constant $R$ are 1.987 cal/mol-K and 0.082 litre-atm/mol-K.

(a) Generate graphs that illustrate the time dependence of the following quantities for the operating conditions described above: (i) conversion of carbon monoxide, (ii) mole fraction of each component in the gas mixture, and (iii) reactor volume.

(b) How long must the batch reactor operate at 298K and 50 torr to achieve 50% conversion of carbon monoxide?

(c) What is the maximum possible conversion of CO that can be achieved for the operating conditions described above?

(d) Design the size of the reactor. Provide only one numerical answer in litres.
(e) Demonstrate that it is possible to achieve 50% conversion of CO in half of the time based on your answer from part (b), and quantitatively describe two different sets of operating conditions that will achieve this goal.

(f) Qualitatively, describe at least 2 strategies that could be implemented to increase the maximum possible conversion of CO in this batch reactor. Then, provide quantitative evidence that both strategies are feasible.

Polymath solution to this batch reactor problem for the production of methanol from a stoichiometric feed of carbon monoxide and hydrogen

\[
d(x)/d(t)=k_{\infty}*\exp(-E_{\text{activation}}/T)*R_{\text{Rate}}*\text{ReactorVolume}*(10**3)/\text{InitialMolesA}
\]
\[
y_{CO}=(1-x)/(1+\Theta_{H2}-2*x)
\]
\[
y_{H2}=(\Theta_{H2}-2*x)/(1+\Theta_{H2}-2*x)
\]
\[
y_{CH3OH}=(x)/(1+\Theta_{H2}-2*x)
\]
\[
R_{\text{Rate}}=(p**3)*(y_{CO}*y_{H2}**2-(p**(-2))*y_{CH3OH}/K_{\text{equilibrium}})
\]
\[
K_{\text{equilibrium}}=\exp(5900/(1.987*T))
\]
\[
\text{ReactorVolume}=(0.1*0.082*T/p)*\text{InitialMolesA}*(1+\Theta_{H2}-2*x)
\]
\[
k_{\infty}=200000
\]
\[
E_{\text{activation}}=5000
\]
\[
\Theta_{H2}=2
\]
\[
\text{InitialMolesA}=5/28
\]
\[
T=298
\]
\[
p=50/760
\]

Problem focusing on the microscopic approach to mass transfer with chemical reaction in tubular reactors
The appropriate description of a realistic mass transfer problem is posed in terms of the following second-order, non-linear, partial differential equation with variable coefficients;

\[
V_{z,\text{max}}\left\{ 1 - \left( \frac{r}{R} \right)^2 \right\} \frac{\partial C_{A}}{\partial z} = D_{AB} \left\{ \frac{\partial^2 C_{A}}{\partial r^2} + \frac{1}{r} \left( \frac{\partial C_{A}}{\partial r} \right) \right\} + k_{r}(C_{A})^{3/2}
\]

a) Identify a mass transfer mechanisms for each of the four terms in the equation

b) Is the mass transfer model written explicitly for a reactant or a product?

c) What is the apparent order of the chemical reaction?
d) Is the reaction reversible or irreversible?

e) Does the mass transfer equation represent a plug-flow model?

f) Write an expression for the control volume, \( dV = ? \), that was used to generate the equation above.

g) In what coordinate direction or directions must the control volume be differentially thick to obtain the governing mass balance?

h) Will residence time distributions affect the performance of this model in a tubular reactor? Provide a brief explanation.

**Problem on Transport Analogies**

Complete the table below which focuses on analogies between heat, mass, and momentum transport for non-reactive systems:

<table>
<thead>
<tr>
<th>Momentum Transport</th>
<th>Heat Transfer</th>
<th>Mass Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re is the dimensionless scaling factor in the Equation of Motion</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

The friction factor vs. Re is a dimensionless correlation obtained by focusing on the interface and calculating forces that are exerted by the fluid on a stationary solid surface |

? | ? |

? | ? |

? | Fourier's Law of Heat Conduction |

Schmidt Number |

Peclet Number |

167 Introduction to Transport Phenomena
Ch406/Detailed Lecture Topics
"Education teaches us how little we know. It teaches us the ability to recognize our inadequacies, and this enables us to mature and grow. Commencement means beginning. It should be the beginning of more study and more learning, because that is the way to recognize our full potential. We must never stop being students. Always strive to know more, to do better, and be better. Learning is a job that should remain forever unfinished, because when we are through improving ourselves, we are through".