Lecture 42

Circuit Averaging Models

A. Circuit Averaging of Three Key PWM dc-dc Converters

1. **Boost dc-dc Converter** Time Invariant Circuit Model
   a. Switching or Computation cycle $<>_{Ts}$ average
   b. Perturb and linearize independent variables
      (1) Primary: dependent V source
      (2) Secondary: dependent I source
      (3) Combine I/O circuits to achieve
         (a) One ac/dc transformer model with dependent sources
         (b) Two independent sources
2. **Buck dc-dc Converter** Time Invariant Circuit Model
   a. Choice of ind. and dep. variables and $<>_{Ts}$
   b. Perturb, Linearize, and Neglect Second Order Effects
      (1) Dependent I source in primary
      (2) Dependent V source in secondary
      (3) Combine I/O to achieve one AC/DC transformer
3. **Buck-Boost** dc-dc Time Invariant Circuit Model
   a. HW 7.12, 7.17 for grad. students <u>only</u>
4. Comparison of Three Different Average Small Signal Switch Models

B. Review of (Switch/Circuit) Averaging

1. Methodology of Switch Averaging
2. Three Basic Converters

C. **Dynamic Switching Losses Revisited** for HW Pbm. 7.15

1. Piecewise Linear Approximation
2. Buck Converter Dynamic Switching Loss
   a. Dependent, indep. variables
   b. $<>_{Ts}$ of dependent variables
   c. I_{sink}(loss) on output circuit
   d. V_{sink}(loss) on output circuit
   e. Ideal vs. lossy V_o/V_g and η

All students for HW#3 go through the solution Pbm. 7.15 outline and fill in the missing portions and hand it in.

Lecture 42
Circuit Averaging Models

A. Circuit Averaging of Three Key PWM dc-dc Converters

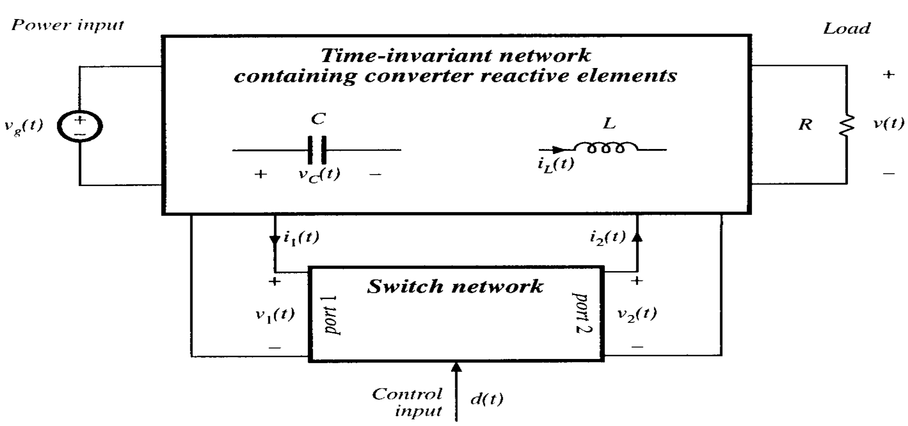
1. **Boost dc-dc Converter** Time Invariant Circuit Model

a. General Comments on Circuit Averaging

Circuit Averaging is a second method to derive the SAME AC converter models. It also must be done for each circuit topology separately. **We will average the boost, buck, and buck-boost waveforms** and topologies, rather than the circuit differential equations. It is amazingly simple, allowing the creation of the AC model by inspection. It also has broad applicability. It works not only for dc-dc converters but also for:

- Resonant Converters
- Three Phase inverters and Rectifiers
- Phase Controlled Rectifiers
- All DCM models as well as CCM Models
- Current Controlled Converters

The main idea is to replace the time varying switches with



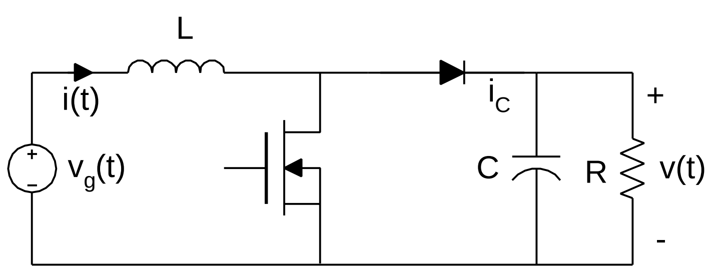
time invariant voltage and current sources. That is be break out the time invariant circuit from the switch network as shown above.

We can make some general comments before starting the process of circuit averaging.
The definition of independent inputs is very arbitrary and depends upon circuit intuition for the initial choice. Some

- The number of ports in the switch network is less than or equal to the number of SPST switches
- Simple dc-dc case, in which converter contains two SPST switches: switch network contains two ports
  - The switch network terminal waveforms are then the port voltages and currents: $v_1(t)$, $i_1(t)$, $v_2(t)$, and $i_2(t)$.
  - Two of these waveforms can be taken as independent inputs to the switch network; the remaining two waveforms are then viewed as dependent outputs of the switch network.
- Definition of the switch network terminal quantities is not unique. Different definitions lead equivalent results having different forms

choices result in quicker progress in model formation than others.

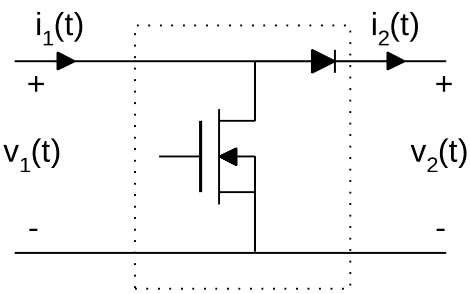
b. Boost Converter: Time Invariant Network



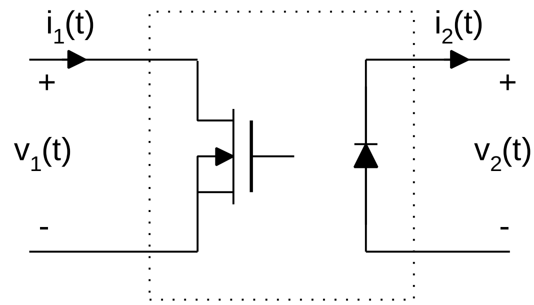
The topology of the Two port switch has two ways to be realized

The switched inductor with the right terminal going from ground to V_{out} will be replaced by a time invariant two port containing: a DC transformer, a dependent voltage source in the input and a dependent current source in the output.

Natural, best, intuitive



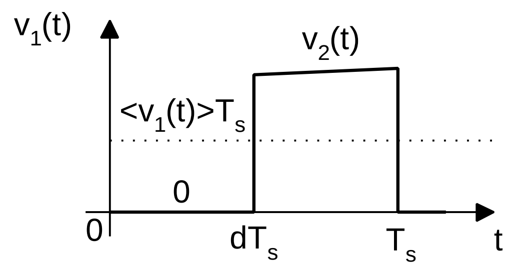
Alternative, but not the best



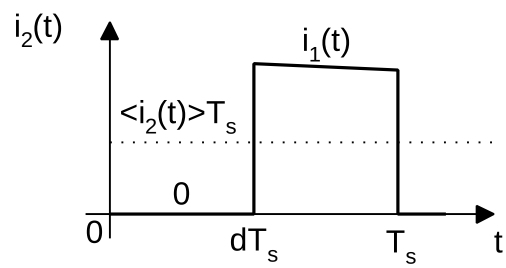
i_1 and v_2 are best choices for independent variables and we can readily plot $v_2(t)$ and $i_1(t)$ from circuit conditions. Moreover, both v_2 and i_1 are not varying when on.

The dependent variables then become v_2 and i_1.

$v_1 = f_1(i_1, v_2)$

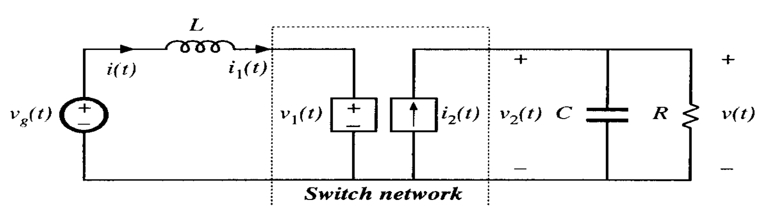


$i_2 = f_2(i_1, v_2)$



By $\langle\rangle_{Ts}$ averaging, this leads to a time invariant switch network topology as described below. That is in anticipation:

Replace the switch network with dependent sources, which correctly represent the dependent output waveforms of the switch network

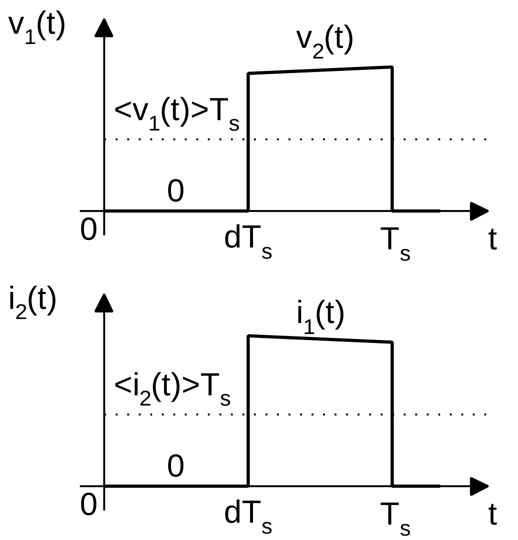


Boost converter example

To get there requires us do the very simple waveform averaging. For the boost converter averaging over T_s (the switching period) with the recognition that we are reemoving the switching harmonics BUT PRESERVING the the low frequency components, $f<f_{SW}$. The work required is to average the switch dependent waveforms.

$$\begin{array}{c} RC > T_s \\ L/R > T_s \end{array} \Rightarrow$$

Has the effect of efficiently removing <u>all</u> switching harmonics from the signals



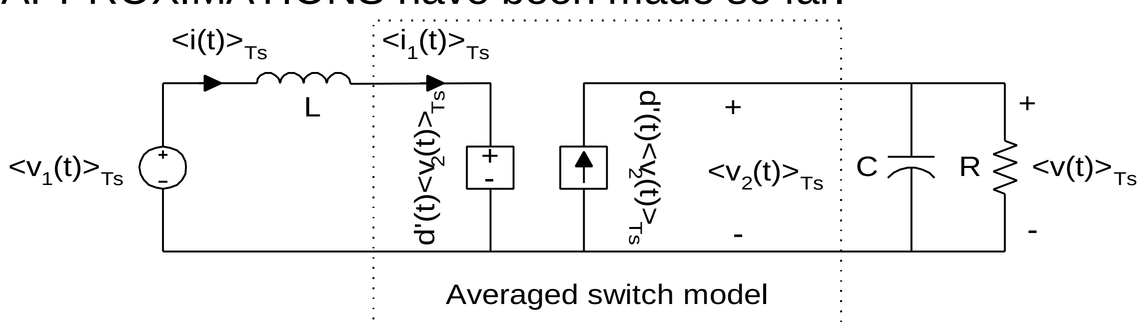
$$< v_1 >_{T_s} = d' < v_2 >_{T_s}$$

$$< i_2 >_{T_s} = d' < i_1 >_{T_s}$$

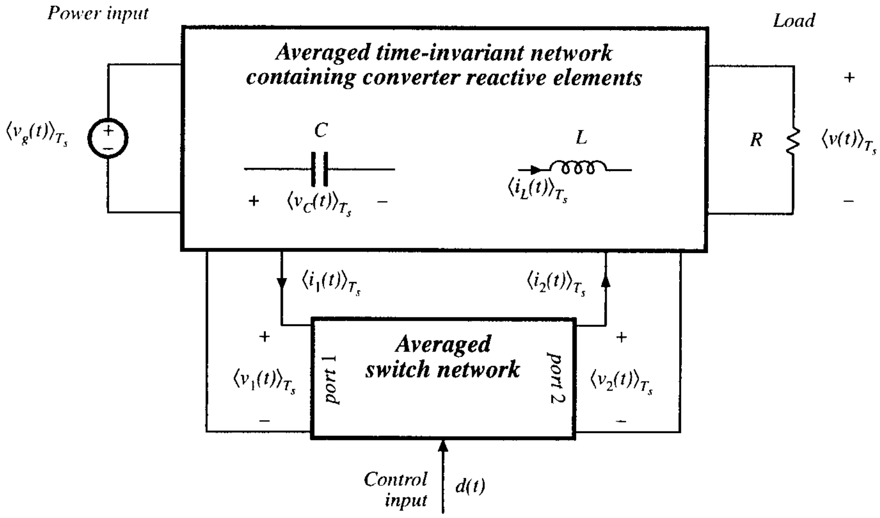
Fig. 7.40 The waveforms of the dependent voltage and current sources are defined to be identical to the corresponding waveforms of the original circuit.

More complex independent variable waveforms use area under curves to calculate dependent variables.

This approach creates a large signal model in which NO APPROXIMATIONS have been made so far.



In essence we have only done the simple step shown below.



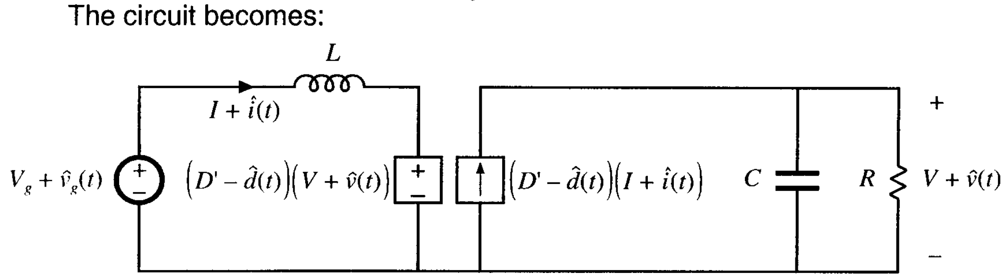
b. Perturbation & Linearization of Large Signal Model

The large signal model does have non-linear terms arising from the product of two time varying quantities. We can linearize the model by expanding about the operating point and removing second order terms:

As usual, let:

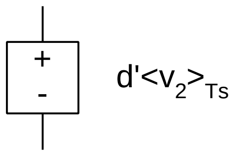
$$\langle v_g(t)\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t) \quad \Rightarrow d'(t) = D' - \hat{d}(t)$$
$$\langle i(t)\rangle_{T_s} = \langle i_1(t)\rangle_{T_s} = I + \hat{i}(t)$$
$$\langle v(t)\rangle_{T_s} = \langle v_2(t)\rangle_{T_s} = V + \hat{v}(t)$$
$$\langle v_1(t)\rangle_{T_s} = V_1 + \hat{v}_1(t)$$
$$\langle i_2(t)\rangle_{T_s} = I_2 + \hat{i}_2(t)$$

The circuit becomes:



We will do the linearization in terms of the input and output portions of the circuit separately.

(1) Dependent voltage source in primary

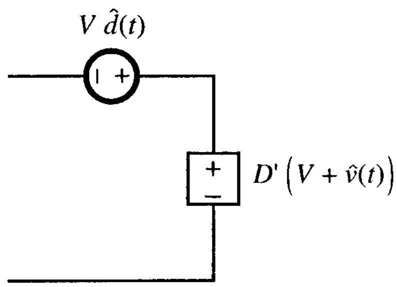


$$d' = D' - d$$

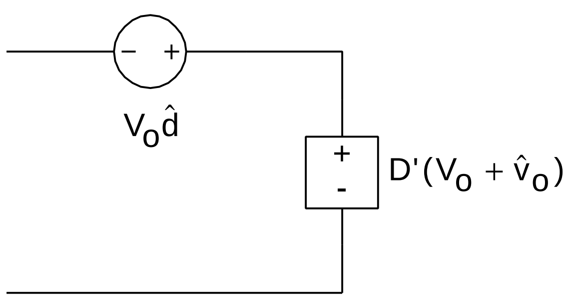
$$< v_2 >_{T_s} = V_o + \hat{v}_o$$

$$\left(D' - \hat{d}(t)\right)\left(V + \hat{v}(t)\right) = D'\left(V + \hat{v}(t)\right) - V\hat{d}(t) - \hat{v}(t)\hat{d}(t)$$

nonlinear, 2nd order

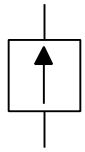


Multiply out the product terms and neglect second order terms in the input circuit. The result is the new input circuit model:



Small signal model is good for both ac and DC conditions

(2) Dependent I source in secondary

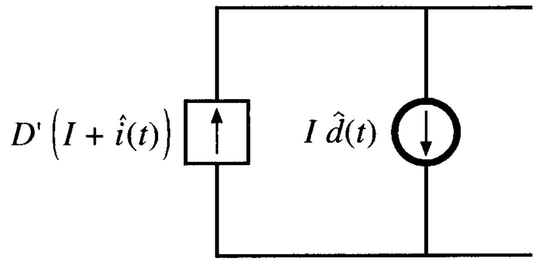


$$(D' - d)(I + \hat{i})$$

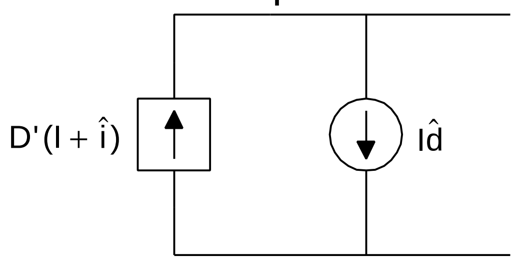
To simplify, multiply out and neglect second order terms in the output circuit.

$$\left(D' - \hat{d}(t)\right)\left(I + \hat{i}(t)\right) = D'\left(I + \hat{i}(t)\right) - I\hat{d}(t) - \cancel{\hat{i}(t)\hat{d}(t)}$$

nonlinear,
2nd order

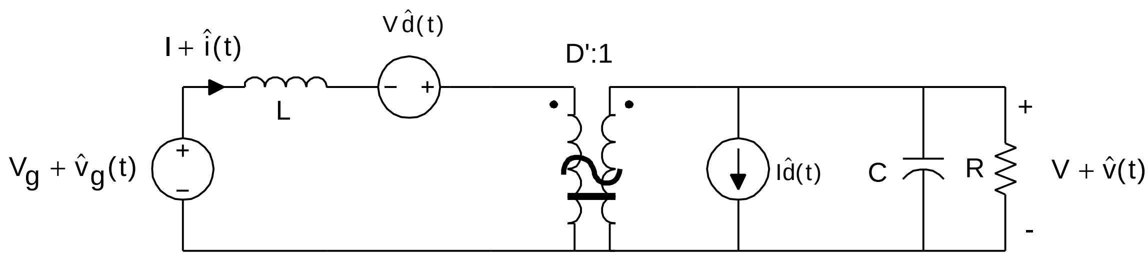


Result is the new output circuit model:



Small signal model good for both ac and DC conditions

(3) Combine the small signal input/output circuits with DC/ac transformer

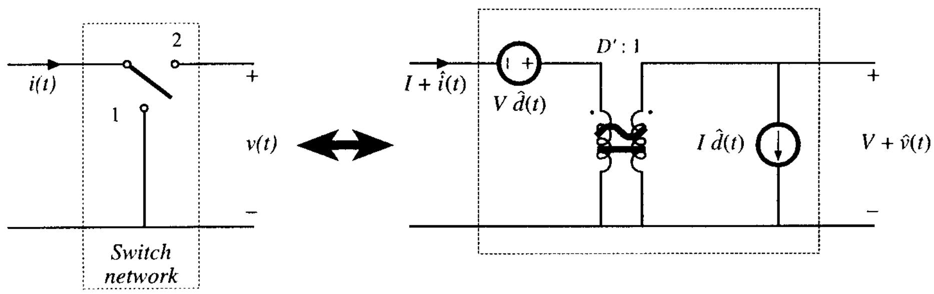


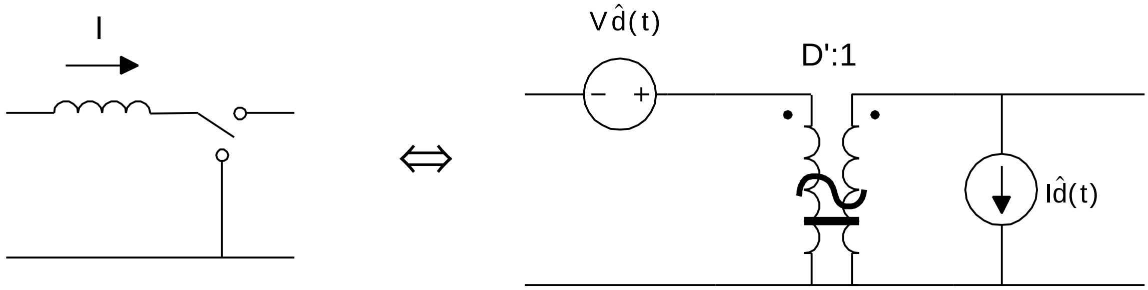
The dependent linear DC sources are replaced by an equivalent ideal DC transformer. This yields the final DC and small-signal ac circuit-averaged model.

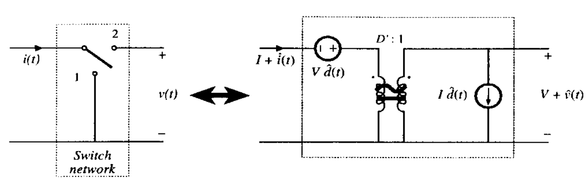
<u>As a Check</u>:
For a DC model, let $\hat{d} \to 0$ and we get the old DC only model.

Now when you see a switched inductor of the boost type you can replace it, by inspection!, by a two port small signal model that is time invariant:

Circuit averaging of the boost converter: in essence, the switch network was replaced with an effective ideal transformer and generators:







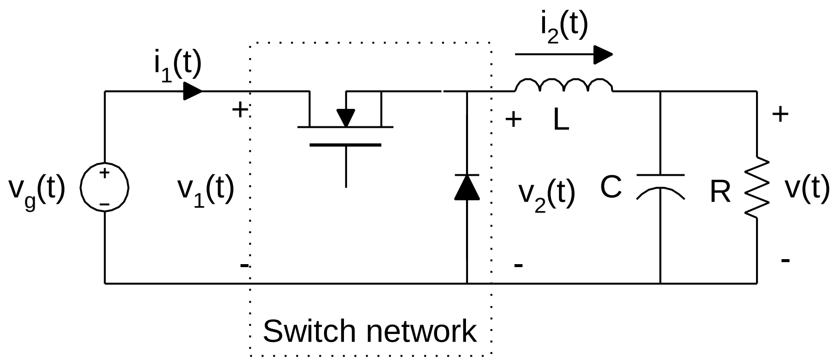
For the boost example, we can conclude that the switch network performs two basic functions:

- Transformation of dc and small-signal ac voltage and current levels, according to the D':1 conversion ratio
- Introduction of ac voltage and current variations, drive by the control input duty cycle variations

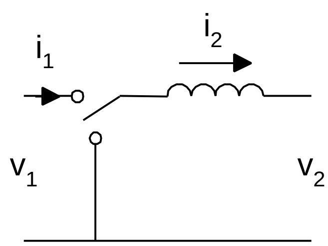
Circuit averaging modifies only the switch network. Hence, to obtain a small-signal converter model, we need only replace the switch network with its averaged model. Such a procedure is called *averaged switch modeling.*

2. **Buck Converter Time Invariant Network**

The switched inductor with the left side commutating from Vg to ground will be replaced by a time invariant two port containing: a DC transformer, a dependent current source in the primary and a dependent voltage source in the secondary.



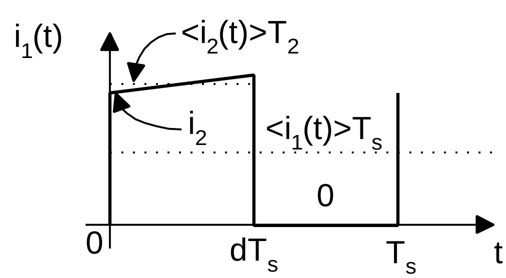
Two port switch with four variables



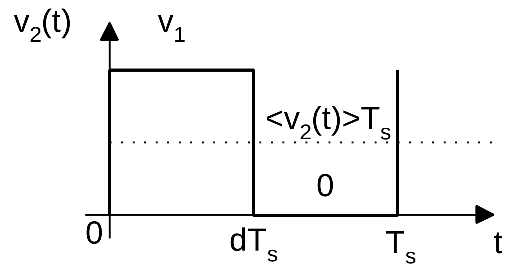
Intuitive best choice of independent variables is i_2 and v_1

Dependent variables are i_1 and v_2. Again we can intuitively plot both $i_2(t)$ and $v_1(t)$ versus time easily.

$$i_1 = f_x(i_2, v_1)$$



$v_2 = f_y(i_2, v_1)$



a. By time averaging over T_s we get a path to a time invariant switch model.

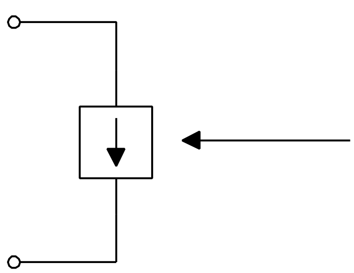
Dependent	Independent	where:
$\langle i_1 \rangle_{Ts} =$	$d\langle i_2 \rangle_{Ts}$	$d = D + \hat{d}$
$\langle v_2 \rangle_{Ts} =$	$d\langle v_1 \rangle_{Ts}$	

The above large signal models are next linearized.

b. Perturb & linearize, removing 2nd order terms

(1) Dependent I source in primary

Starting point of large signal model:

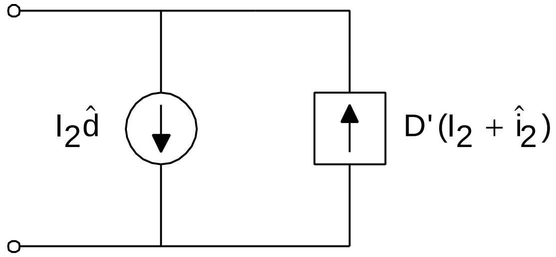


$(D'+\hat{d})(I_2 + \hat{i}_2)$

To simplify, multiply out and neglect second order terms, $\hat{d}\,\hat{i}_2$, to get small signal model input circuit.

$$\Rightarrow D(I_2 + \hat{i}_2) + I_2\hat{d}$$

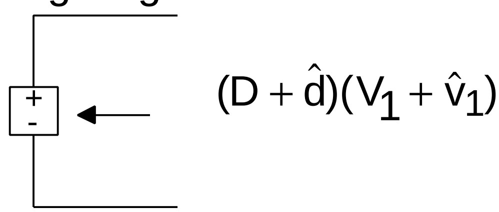
Result:



Small signal model good for both ac and DC conditions

(2) Dependent V source in secondary

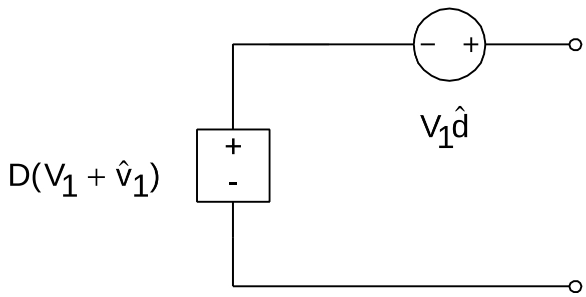
Starting point is the large signal model:



To simplify, multiply out and neglect second order terms to get small signal output model.

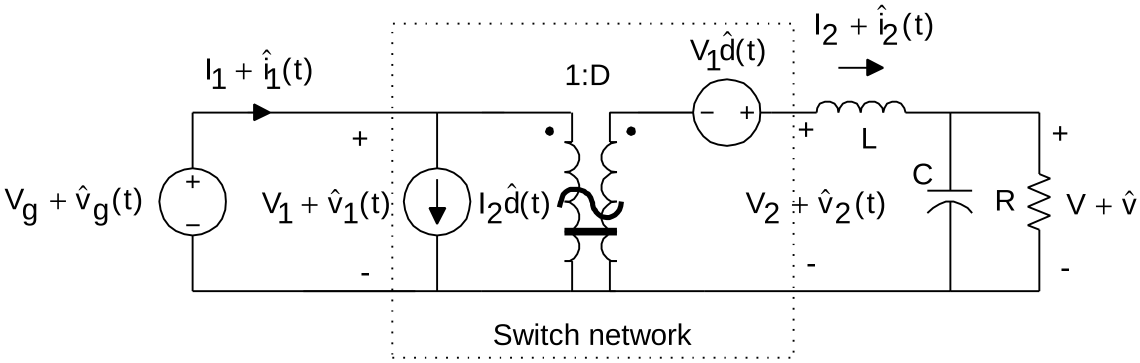
$$\Rightarrow D(V_1 + \hat{v}_1) + V_1\hat{d}$$

Result is the new output circuit model:



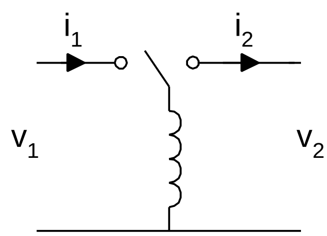
Small signal model is good for both ac and DC conditions

(3) Combine small signal input/output circuits via ac/DC transformer



<u>Check</u>:
For a DC only model, let d → 0 and we get the old DC Buck model of Lecture 5.

3. Buck-Boost Converter Time Invariant Network

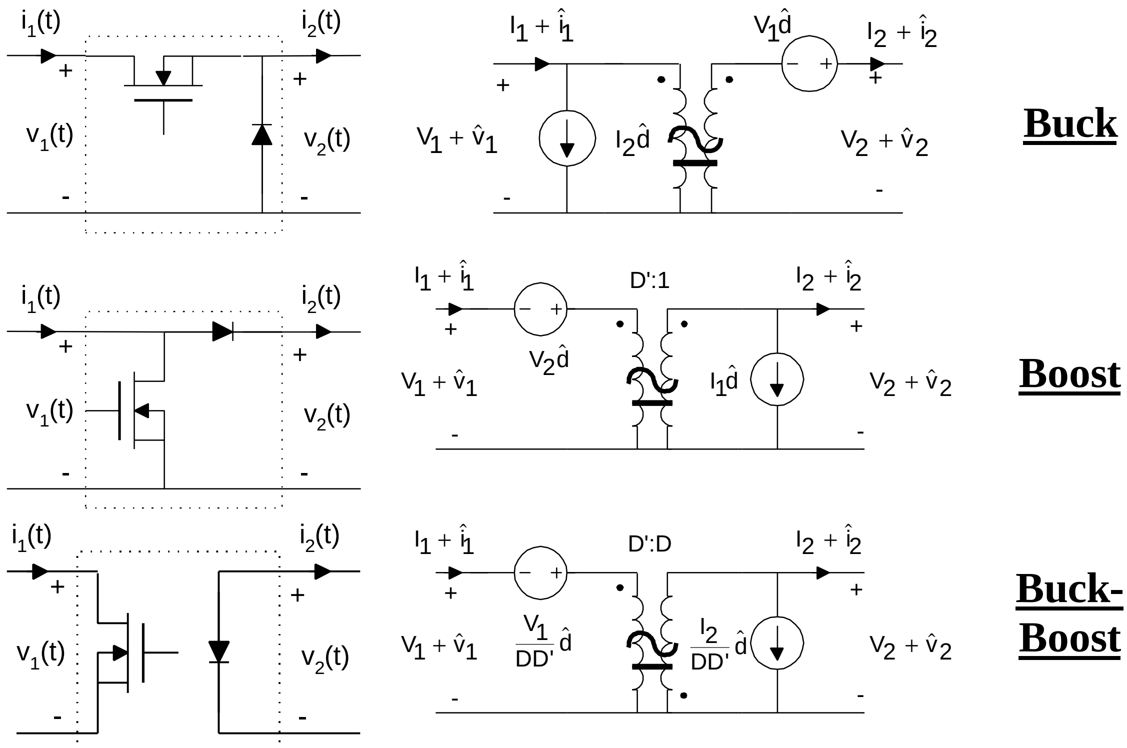


Independent variables: V_1 for sure. Then I_1 or I_2 as both I_1 and I_2 flow in L. Choose V_1 and I_1 as independent or ???

Graduate students DC/ac model of buck-boost for **Homework #3**—Chapter 7 of Erickson

#12-answers also can be done for DC Methods
#17 Buck, Boost, Buck-Boost-include DC loss R_{on} of TR and V_{on} of diode—Hint Fig. 7.35

4. Summary of 3 Switch Averaged Networks/Models for both ac & DC.

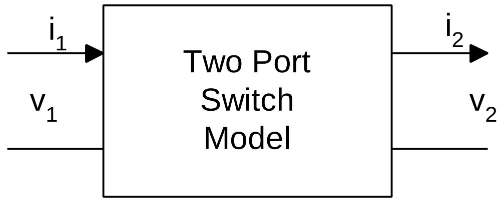


Now, you should be able to recognize these basic switch networks and equivalent circuit models when you encounter them again. Or at least look them up.

⇒For example, Graduate students HW Prob. #17 Chapter 7 of Erickson include R_{on}. and V_{on} models. Below in section C we will outline solution to Erickson Pbm. 7.15.

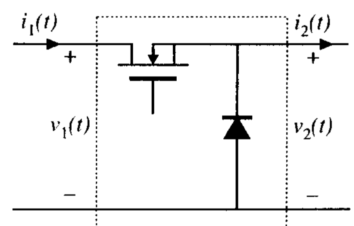
B. Methodology of Switch Averaging

We made time independent two port models for the case of two time varying single-pole, single-switches in three circuit topologies. This was done however, only for the CCM of operation. Averaging of waveforms changed only the switch network. This means to obtain AC models we need only REPLACE THE SWITCH with its averaged model. This procedure is called AVERAGED SWITCH MODELING.



Average models only provide information about the low frequency action of a PWM converter. We cannot model ripple, switch commutation and other FAST transients. Still these crude models give insight into DC operation, voltage regulation and dynamic or transient response. Averaged system equations do not show ripple at f_{sw}.
In its simplest form this involves two broad-brush stroke procedures:

1. Define a switch network and its terminal waveforms. For a simple transistor-diode switch network as in the buck, boost, etc., there are two ports and four terminal quantities: v_1, i_1, v_2, i_2.The switch network also contains a control input d. Buck example:

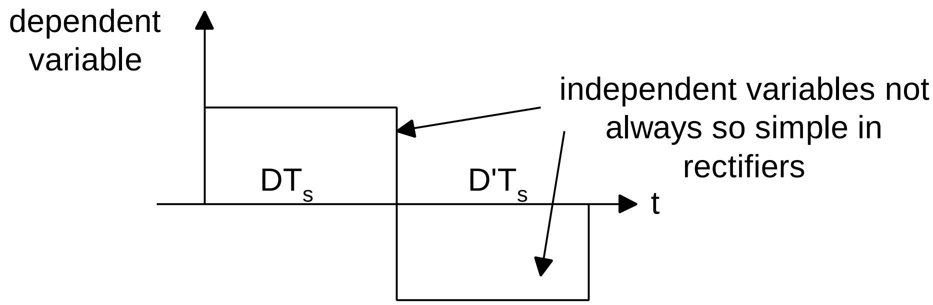


2. To derive an averaged switch model, express the average values of two of the terminal quantities, for example $\langle v_2 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$, as functions of the other average terminal quantities $\langle v_1 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$. $\langle v_2 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$ may also be functions of the control input d, but they should not be expressed in terms of other converter signals.

In Chapter 6 of Erickson we also found a five step method to do so:

1. Choose (two independent/two dependent) variables from the PWM converter circuit depending on where switch lies in the circuit topology - this is an art!

2. Sketch waveforms over T_s for the two dependent variables in terms of the independent variables during DT_s and $D'T_s$.



3. Average the dependent variable over the switch period T_s and express the average in terms of a function of the duty cycle and circuit parameters f(independent variables, D, R_D, V_D, R_{on}). This yields a set of equations for the ac model.

4. Perturb and linearize the large signal equations of step (3). Place emphasis on the duty cycle changes d(t) = D(DC) + $\hat{d}$ (ac). Neglect higher order terms in both the input circuit and output circuit equations. We are left with only DC and first order ac models for the switch action. Avoid for the moment the fact that switching is **NOT** a small perturbation about a DC value.

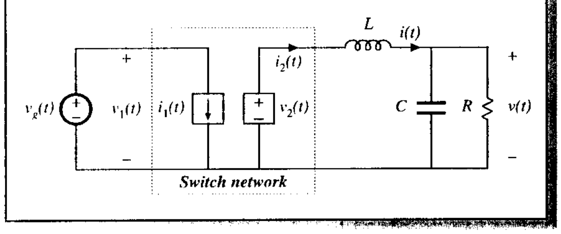
5. Draw input equivalent circuit containing current and voltage sources and the output equivalent circuit. Combine the two for a full switch model. The sources are functions of the dc operating point chosen.

6. Review of CCM Models for Switches in PWM dc-dc Converters

The time varying switch circuits on the left were replaced by time independent averaged switched models on the right. Models were accurate in the range f << f_{sw}. That is, in the infinite switch frequency approximation where the ripple is always triangular and the time derivatives are therefore only constant.

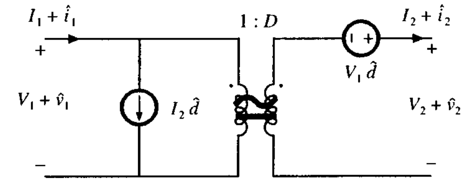
The key is averaging all parameters over the switch cycle, <>$_{Ts}$.

Circuit-averaged model

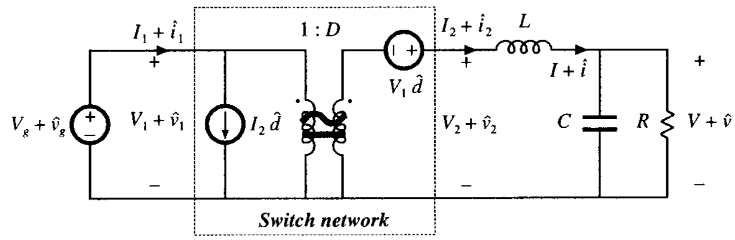


Perturbation and linearization of switch network:

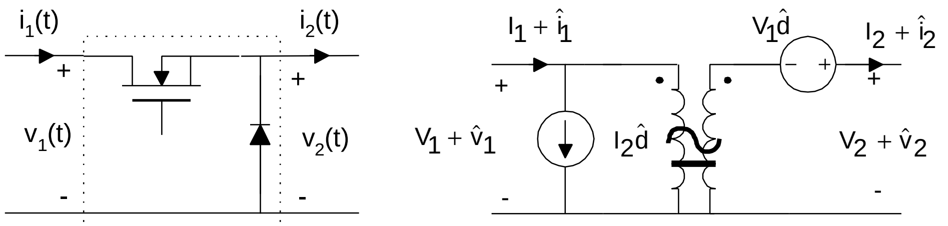
$$I_1 + \hat{i}_1(t) = D\left(I_2 + \hat{i}_2(t)\right) + I_2\,\hat{d}(t)$$
$$V_2 + \hat{v}_2(t) = D\left(V_1 + \hat{v}_1(t)\right) + V_1\,\hat{d}(t)$$



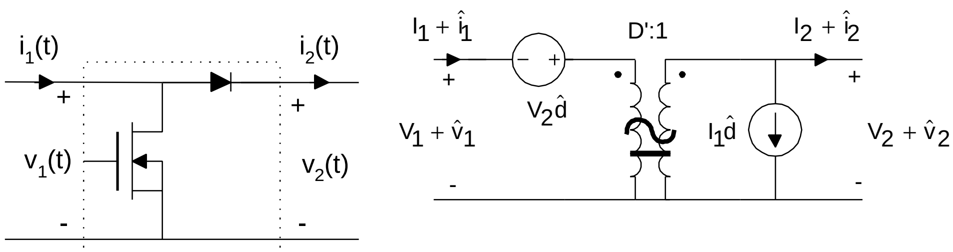
Resulting averaged switch model: CCM buck converter



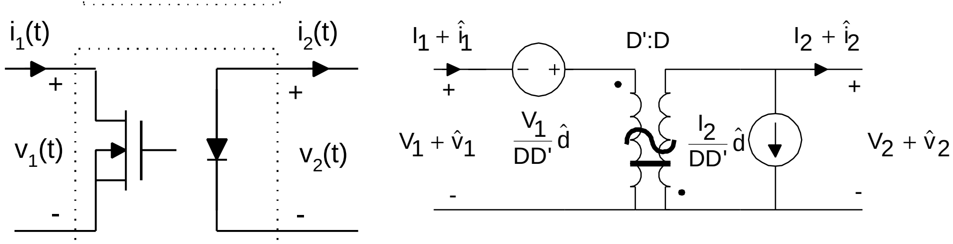
Summary of 3 Switch Networks/Models for both ac & DC:



Buck



Boost



Buck-Boost

Again notice that for each DC operating point the model values will change compared to another. Why bother with ac models? Because they support control design analysis based on established linear systems techniques such as Laplace Transforms, Bode plots and Nyquist criteria.

C. Dynamic Switching Losses Revisited

1. Piecewise Linear Approximation

Previously we included effects only from static state DC losses in the converter AC models:

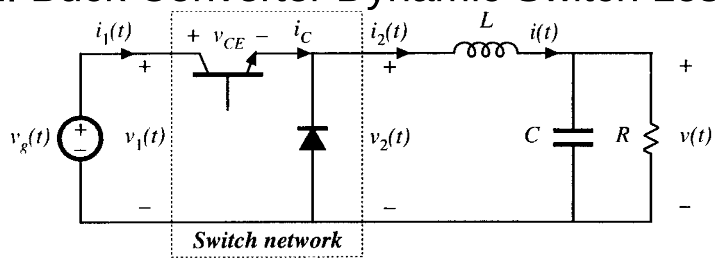
R_L(inductor) $\rightarrow$ new $M(D, R_L)$ $\leftarrow$includes core &
$\rightarrow$ $\eta(D', R, R_L)$ $\leftarrow$winding loss

<u>DC device effects</u>:

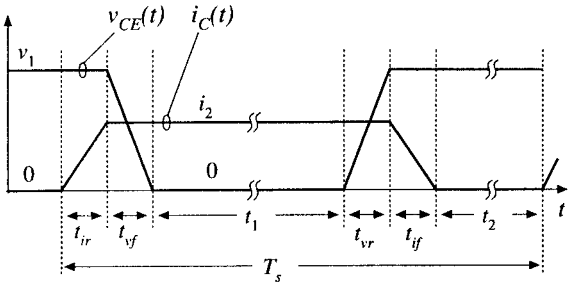
V_D, R_{on}, $R_D \rightarrow$ new $M(D, R_{on}, R_D, V_{on})$
$\rightarrow$ $\eta(D', R_{on}, R_D, V_D)$

These DC losses are <u>not</u> transient switch losses. We dealt with transient switch loss by first calculating the energy lost per switch event. We did this via the piecewise linear approximations on the switch wave forms for both TR(on/off) and TR(off-on) transitions between static switch states. We also introduced diode stored charge to better calculate loss due to large transient currents when we go from the transistor-off and diode-on to the transistor-on and diode-off. Lets look now in more detail. If this doesn't ring a bell go back to the notes on solid state switches and refresh. We will now proceed to show how switch loss can be modeled via AVERAGED SWITCH MODELING

2. Buck Converter Dynamic Switch Loss

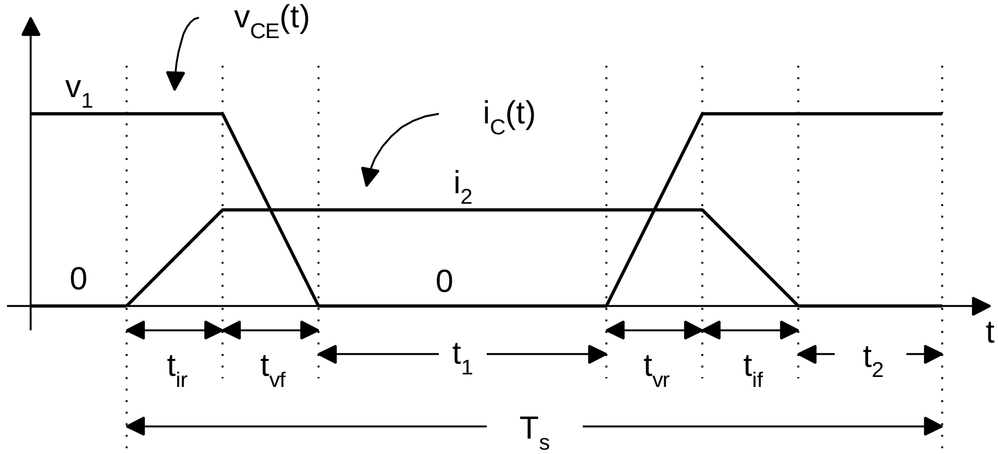


$$i_1(t) = i_C(t)$$
$$v_2(t) = v_1(t) - v_{CE}(t)$$



Switch network terminal waveforms: v_1, i_1, v_2, i_2. To derive averaged switch model, express $\langle v_2 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$ as functions of $\langle v_1 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$. $\langle v_2 \rangle_{T_s}$ and $\langle i_1 \rangle_{T_s}$ may also be functions of the control input d, but they should not be expressed in terms of other converter signals.

For a bipolar transistor switch we approximate the switch trajectories. v_{CE} and i_C as piecewise linear in time as they ramp on and off as shown below. This neglects the diode stored charge which we will add later as its switch trajectory is more complex.



Linear Approximation Switch waveforms, for a buck converter switching loss example.

There are six distinct time intervals within T_s.
The static switch intervals t_1 and t_2 are ideal with no power loss due to V_{on}, R_{on} and I(leakage). The other four time intervals t_{ir}, t_{vf}, t_{vr} and t_{if} have non-zero switching loss.

$t_1 \equiv$ time for which the TR conducts AND $v_{CE} \to 0$

$t_2 \equiv$ TR is off AND $i_2 = 0$

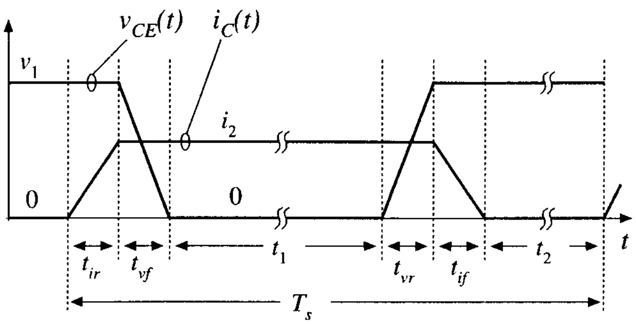
$t_{ir} \equiv$ rise time of current in inductor

$t_{vf} \equiv$ fall time of voltage in transistor

$t_{vr} \equiv$ rise time of voltage across the transistor

$t_{if} \equiv$ fall time of current in inductor

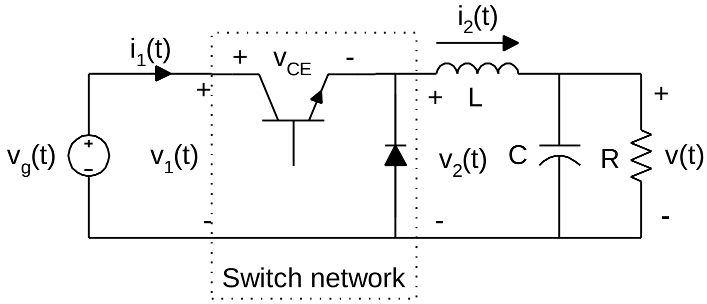
We can approach the problem by averaging i_1 over the switch cycle which depends on the enclosed area under the waveform.

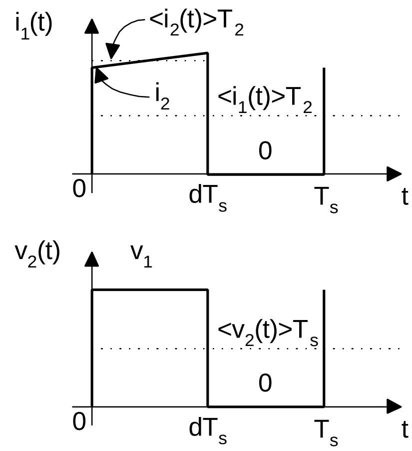


$$\langle i_1(t) \rangle_{T_s} = \frac{1}{T_s} \int_0^{T_s} i_1(t)\, dt$$

$$= \langle i_2(t) \rangle_{T_s} \left(\frac{t_1 + t_{vf} + t_{vr} + \frac{1}{2}\, t_{ir} + \frac{1}{2}\, t_{if}}{T_s} \right)$$

Each time interval has a unique power-time product or contribution to the switch energy.





Note: $v_{CE} \equiv v_1 - v_2$

(a) Independent variables: v_1, i_2

Dependent variables: v_2, i_1

$i_1 = f(v_1, i_2) = i_2$ when transistor is on

(b) Average waveforms over T_s in terms of d, dv and di

$<i1>_{Ts} \equiv$ Area under i_2 curve waveform

Given

$$\langle i_1(t) \rangle_{T_s} = \frac{1}{T_s} \int_0^{T_s} i_1(t)\, dt$$

$$= \langle i_2(t) \rangle_{T_s} \left(\frac{t_1 + t_{vf} + t_{vr} + \frac{1}{2} t_{ir} + \frac{1}{2} t_{if}}{T_s} \right)$$

Let

$$d = \left(\frac{t_1 + \frac{1}{2} t_{vf} + \frac{1}{2} t_{vr} + \frac{1}{2} t_{ir} + \frac{1}{2} t_{if}}{T_s} \right)$$

$$d_v = \left(\frac{t_{vf} + t_{vr}}{T_s} \right)$$

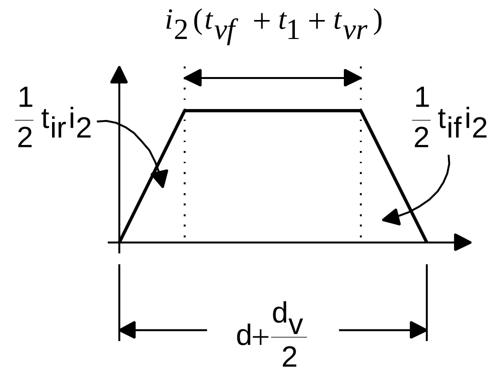
$$d_i = \left(\frac{t_{ir} + t_{if}}{T_s} \right)$$

Then we can write

$$\langle i_1(t) \rangle_{T_s} = \langle i_2(t) \rangle_{T_s} \left(d + \frac{1}{2} d_v \right)$$

We break the interval of conduction change into three parts. dv for switch voltage transition rimes fraction, di for switch current transition time fraction and a time fraction d. Note that the duration i_2 max is $t_{vf} + t_1 + t_{vr}$. Also the area under i_2 has three parts as shown:

$$d = \left(\frac{t_1 + \frac{1}{2} t_{vf} + \frac{1}{2} t_{vr} + \frac{1}{2} t_{ir} + \frac{1}{2} t_{if}}{T_s} \right)$$



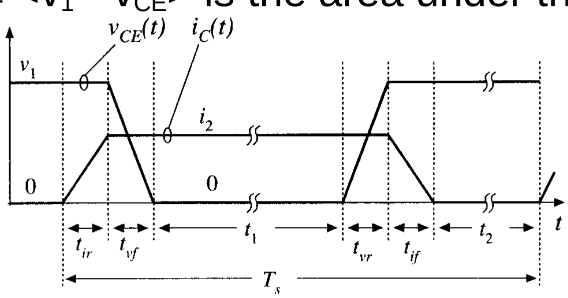
$$d_v = \left(\frac{t_{vf} + t_{vr}}{T_s} \right)$$

$$d_i = \left(\frac{t_{ir} + t_{if}}{T_s} \right)$$

dv is the fractional time v_{ce} rises and falls in a linear fashion.

di is the time i_2 rises and falls in a linear fashion.

We next express average circuit voltages in terms of d, dv and di. $\langle v_2 \rangle_{Ts} \equiv \langle v_1 - v_{CE} \rangle$ is the area under the curve.



$$\left\langle v_2(t) \right\rangle_{T_s} = \left\langle v_1(t) - v_{CE}(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_0^{T_s} \left(- v_{CE}(t) \right) dt + \left\langle v_1(t) \right\rangle_{T_s}$$

$$\left\langle v_2(t) \right\rangle_{T_s} = \left\langle v_1(t) \right\rangle_{T_s} \left(\frac{t_1 + \frac{1}{2} t_{vf} + \frac{1}{2} t_{vr}}{T_s} \right)$$

$$\left\langle v_2(t) \right\rangle_{T_s} = \left\langle v_1(t) \right\rangle_{T_s} \left(d - \frac{1}{2} d_i \right)$$

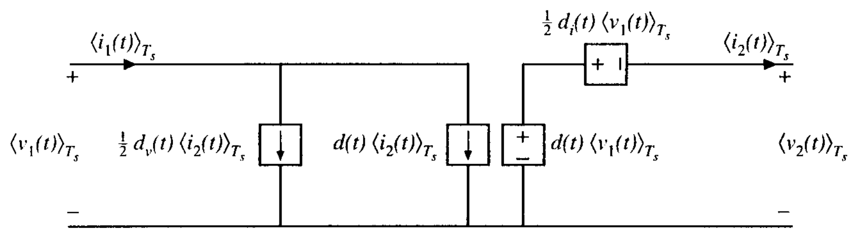
and we find area is = $\langle v_1 \rangle_{Ts}(d - d_i/2)$

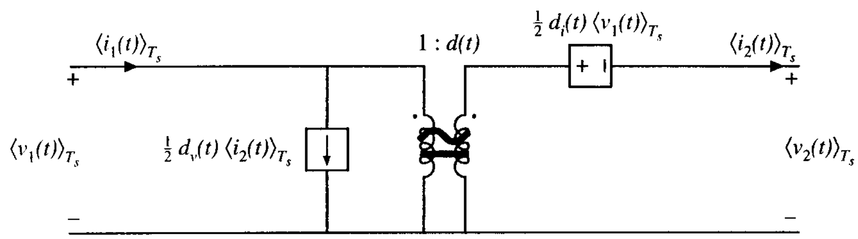
$$\langle v_2 \rangle_{Ts} = \langle v_1 \rangle_{Ts}(d - d_i/2)$$

$$\langle i_1 \rangle_{Ts} = \langle i_2 \rangle_{Ts}(d - d_v/2)$$

Multiply through to get two dependent sources in terms of various d terms and two independent sources. This allows construction of the large-signal averaged-switch model

$$\left\langle i_1(t) \right\rangle_{T_s} = \left\langle i_2(t) \right\rangle_{T_s} \left(d + \frac{1}{2}\, d_v \right) \qquad \left\langle v_2(t) \right\rangle_{T_s} = \left\langle v_1(t) \right\rangle_{T_s} \left(d - \frac{1}{2}\, d_i \right)$$





shown above:

(c) Dependent I-sink on input:

$0.5d_v\langle i_2 \rangle_{Ts}$ @ v_1

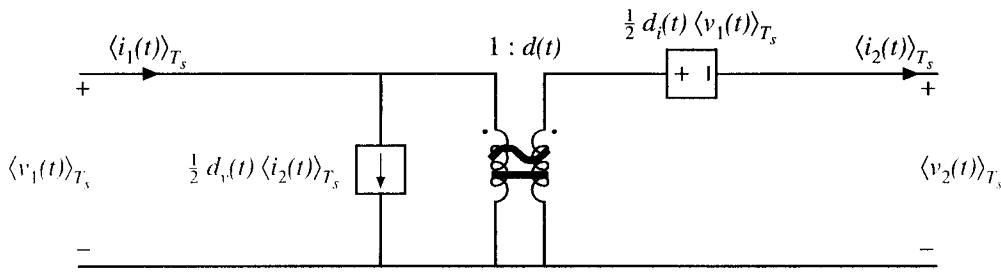
In parallel with the transformer in input node sinks current from source: $i_1 - 0.5d_v\langle i_2 \rangle_{Ts} \equiv$ into transformer

(d) Dependent V-sink on output:

$0.5d_i\langle v_1 \rangle_{Ts}$ @ i_2

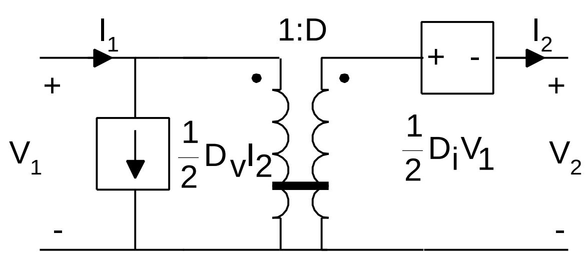
Consumes power from transformer before it reaches load: v_o = v(transf.) - v(loss)

Total switch loss predicted by the averaged switch model is then as shown on the top of page 25:



$$P_{sw} = \frac{1}{2}\left(d_v + d_i\right)\left\langle i_2(t)\right\rangle_{T_s}\left\langle v_1(t)\right\rangle_{T_s}$$

$P_{SW}(\text{total}) = 0.5(d_v + d_i)<i_2>_{Ts}< i_2>_{Ts}$



Use this model in circuit for buck: $d \rightarrow D$

DC equivalent circuit model, buck converter switching loss.

$$V_o = V_2 = V_1\left[D - \frac{D_i}{2}\right]$$

$$I_1 = I_2\left[D + \frac{D_v}{2}\right]$$

DC in steady state

$$V_o \equiv DV_g - \frac{1}{2}D_i V_g$$

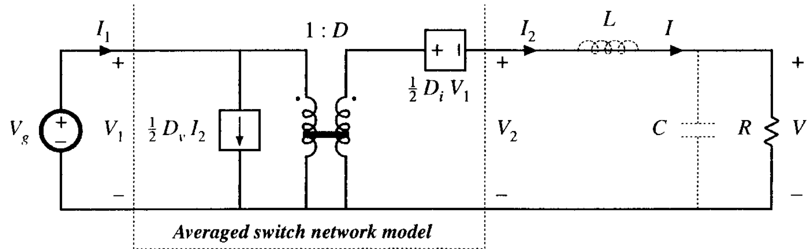
$\uparrow$ $\uparrow$

@secondary switch voltage loss in secondary

$$V_o \equiv DV_g\left[1-\frac{D_i}{2D}\right]$$

$\uparrow$ ideal $\qquad \uparrow$ SW loss in voltage output

The solution of the averaged switch model in steady state



Output voltage:

$$V = \left(D - \tfrac{1}{2}D_i\right)V_g = DV_g\left(1 - \frac{D_i}{2D}\right)$$

Efficiency calcuation:

$$P_{in} = V_g I_1 = V_1 I_2\left(D + \tfrac{1}{2}D_v\right)$$

$$P_{out} = VI_2 = V_1 I_2\left(D - \tfrac{1}{2}D_i\right)$$

$$\eta = \frac{P_{out}}{P_{in}} = \frac{\left(D - \frac{1}{2}D_i\right)}{\left(D + \frac{1}{2}D_v\right)} = \frac{\left(1 - \frac{D_i}{2D}\right)}{\left(1 + \frac{D_v}{2D}\right)}$$

gives:

$$\text{efficiency}\,\frac{P_o}{P_{in}} = \frac{V_o I_2}{V_g I_1} = \frac{V_g I_2\left(D - \frac{D_i}{2}\right)}{V_g I_2\left(D + \frac{D_v}{2}\right)}$$

Note we have separated out D_i and D_v contributions to the efficiency.

$$h = \frac{1 - \frac{D_i}{2D}}{1 - \frac{D_v}{2D}}$$