Lecture 41
SIMPLE AVERAGING OVER T_{SW} to ACHIEVE LOW FREQUENCY MODELS

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. HOMEWORK HINTS

Selected Problems will be gone over partially.

We have for Assignment #3 Problems1,2,3,12 and17 as well as all questions in the lectures up to lecture 44

Lecture 41
SIMPLE AVERAGING OVER T_{SW} to ACHIEVE LOW FREQUENCY MODELS

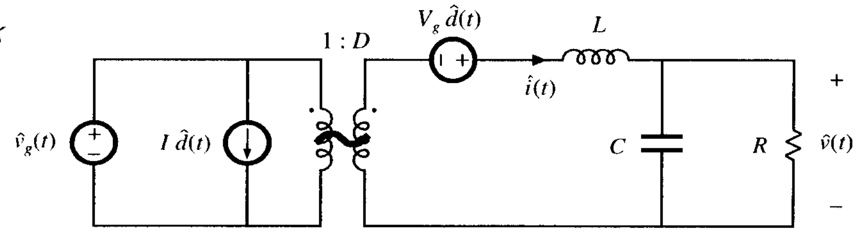
A. Goals and Methodology to Get There

1. Goals

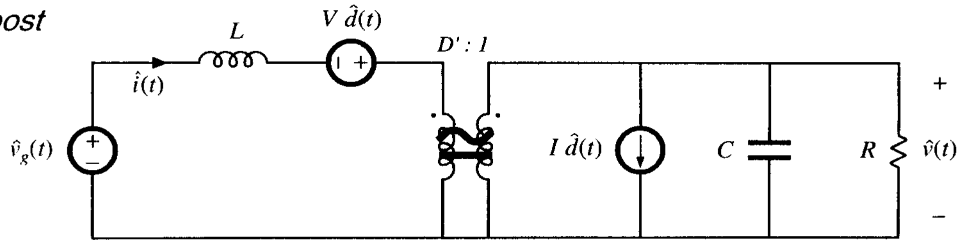
We seek small signal models of the three basic converter circuits that are valid at frequencies< f_{SW}. While the input voltage in a switch mode circuit is a continuous function of time, the switches operate at f_{SW}. In practice this limits any AC model to frequencies less than the Nyquist rate,$f_{SW}/2$, due to the finite sampling that occurs. In all of our work below we aim for AC models that are valid only for f<f/2 by some margin. However, these AC models will be very adequate for the control loop simulations, as control functions take place at much lower frequencies than the switch frequency. We will represent small AC variations about a DC operating point, X, as the variable x. Thus the input voltage, V_g, might have an effective DC value but any variations about that value are represented by v_g. Duty cycle is a second example with D as the effective DC value and d as the AC variation. What is tricky is that the effective DC values may change from one switch cycle to the next.

Our goal is simple AC circuit models for the major converter topologies such as the buck and boost shown below on page 2. The goal of the circuit models is to easily calculate AC chances in the output voltage,v, of the converter in terms of either AC changes in the input voltage, v_g, or AC changes in the duty cycle, d. The equivalent circuit models can easily provide the two transfer functions v/d or v/v_g that we will need for AC control analysis. All models of dc-dc PWM converters will have both DC transformers with the equivalent DC operating duty cycle, D, and small signal models of DEPENDENT current and voltage sources. The later dependent sources depend on the product of DC and AC quantities as shown below.

buck



boost



The model output will contain only signals with frequencies well below f_{SW}. The converter waveforms will be controlled by the AC variation of the duty cycle.

1. Methodology

The small signal model will be created, by averaging all waveforms over the switch period, T_{SW}. This results in a new set of equations to represent the converter circuit. Averaging the L and C relations over T_{SW} is done first and then the input current or output voltage is averaged over the switch cycle. This results in a new set of averaged but NON-LINEAR equations that we must linearize. The top of page 3 displays an illustrative set of non-linear equations, that corresponds to a particular DC operating point. It is about this operating point that the linearization must occur. As a consequence the AC model parameters do depend upon the chosen DC operation point as we saw in Lecture 40. This is evident in the absolute values of the dependent sources, which do indeed depend on the DC levels.

Next on the top of page 3 we show the three averaged equations we would get for a buck-boost.

Converter averaged equations:

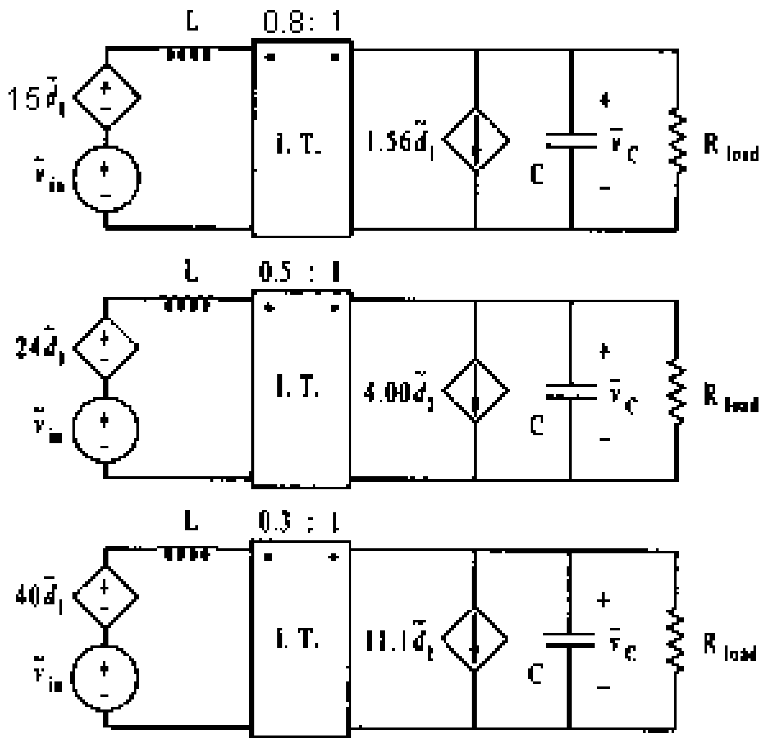
$$L \frac{d\langle i(t)\rangle_{T_s}}{dt} = d(t)\,\langle v_g(t)\rangle_{T_s} + d'(t)\,\langle v(t)\rangle_{T_s}$$

$$C \frac{d\langle v(t)\rangle_{T_s}}{dt} = -\,d'(t)\,\langle i(t)\rangle_{T_s} - \frac{\langle v(t)\rangle_{T_s}}{R}$$

$$\langle i_g(t)\rangle_{T_s} = d(t)\,\langle i(t)\rangle_{T_s}$$

—nonlinear because of multiplication of the time-varying quantity $d(t)$ with other time-varying quantities such as $i(t)$ and $v(t)$.

Recall the three cases for D=0.8,0.5 and 0.3 from lecture 40? Look at the AC model changes in dependent sources.



In practice, one DC operating point is usually employed for a given desired output level and one AC model results.

The mathematical symbols we will employ for averaging as well as the steady state conditions are reviewed below.

Average over one switching period to remove switching ripple:

$$L \frac{d\langle i_L(t)\rangle_{T_s}}{dt} = \langle v_L(t)\rangle_{T_s}$$

$$C \frac{d\langle v_C(t)\rangle_{T_s}}{dt} = \langle i_C(t)\rangle_{T_s}$$

where

$$\langle x_L(t)\rangle_{T_s} = \frac{1}{T_s}\int_t^{t+T_s} x(\tau)\, d\tau$$

Note that, in steady-state,

$$\langle v_L(t)\rangle_{T_s} = 0$$

$$\langle i_C(t)\rangle_{T_s} = 0$$

by inductor volt-second balance and capacitor charge balance.

A. BUCK-BOOST ILLUSTRATIVE EXAMPLE

1. Overview of Methodology

a. Average inductor current and capacitor voltage equations over T_{SW}

a. Average the input current, I_g, over T_{SW}.

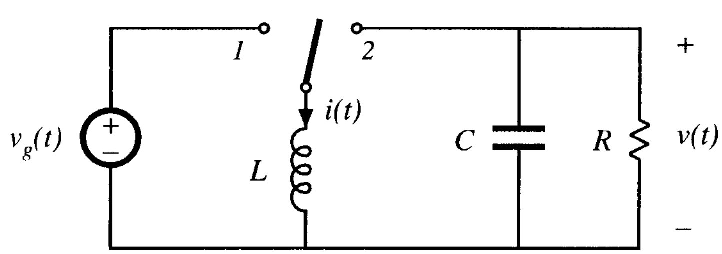
a. Write down the non-linear system equations

a. Linearize the equations about an operating point

a. Construct an AC Circuit Model from First Order Terms Only

1. Buck-Boost Example

Buck-boost converter example



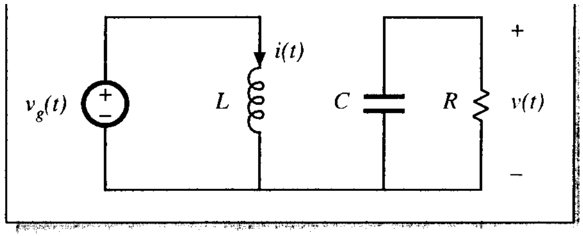
We will sketch the solution pathways below.
There are two switch positions.

First position 1:

Inductor voltage and capacitor current are:

$$v_L(t) = L\frac{di(t)}{dt} = v_g(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = -\frac{v(t)}{R}$$



Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\frac{di(t)}{dt} \approx \left\langle v_g(t) \right\rangle_{T_s}$$

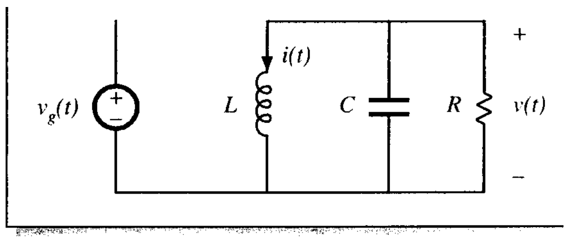
$$i_C(t) = C\frac{dv(t)}{dt} \approx -\frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

Then position 2:

Inductor voltage and capacitor current are:

$$v_L(t) = L\frac{di(t)}{dt} = v(t)$$

$$i_C(t) = C\frac{dv(t)}{dt} = -i(t) - \frac{v(t)}{R}$$



Small ripple approximation: replace waveforms with their low-frequency averaged values:

$$v_L(t) = L\frac{di(t)}{dt} \approx \left\langle v(t) \right\rangle_{T_s}$$

$$i_C(t) = C\frac{dv(t)}{dt} \approx -\left\langle i(t) \right\rangle_{T_s} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$

If we average over T_{SW} , we can eliminate the switch frequency ripple and determine the non-linear low frequency inductor equation as well as the capacitor equation. These equations together with the input current equation will be sufficient to fully specify the problem.

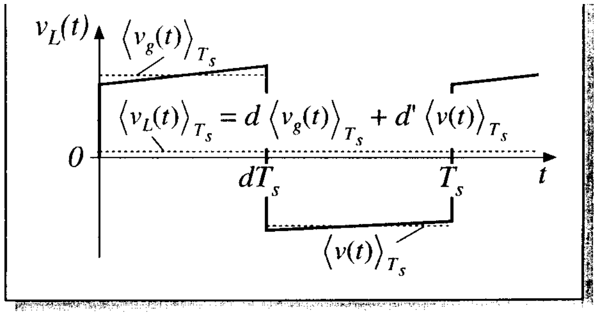
a. Inductor Equation

Inductor voltage waveform

Low-frequency average is found by evaluation of

$$\left\langle x_L(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} x(\tau) d\tau$$

Average the inductor voltage in this manner:



$$\left\langle v_L(t) \right\rangle_{T_s} = \frac{1}{T_s} \int_t^{t+T_s} v_L(\tau) d\tau \approx d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s}$$

Insert into Eq. (7.2):

$$L \frac{d\left\langle i(t) \right\rangle_{T_s}}{dt} = d(t) \left\langle v_g(t) \right\rangle_{T_s} + d'(t) \left\langle v(t) \right\rangle_{T_s}$$

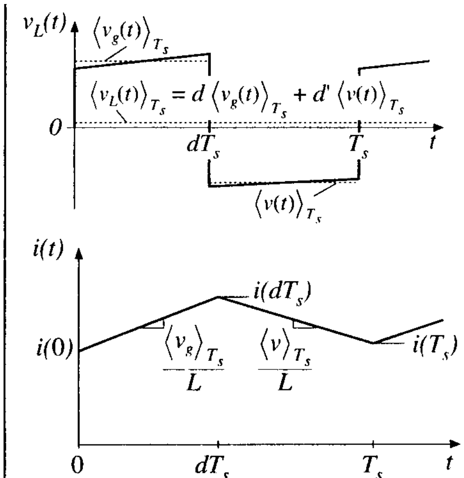
This equation describes how the low-frequency components of the inductor waveforms evolve in time.

The approximation is that:

Use of the average inductor voltage allows us to determine the net change in inductor current over one switching period, while neglecting the switching ripple.

In steady-state, the average inductor voltage is zero (volt-second balance), and hence the inductor current waveform is periodic: $i(t + T_s) = i(t)$. There is no net change in inductor current over one switching period.

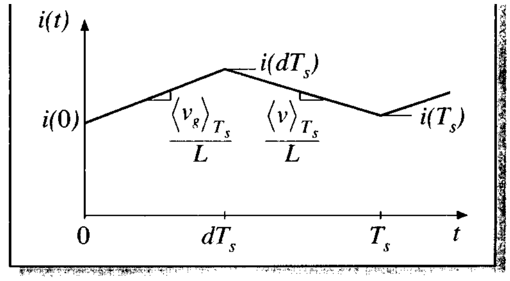
During transients or ac variations, the average inductor voltage is not zero in general, and this leads to net variations in inductor current.



Inductor voltage and current waveforms

We learned before the value of the linear ripple approximation, which greatly simplifies the mathematics of averaging as we show next. Our goal is to express $I(t_{sw})$ in terms of the initial current I(0).

Let's compute the actual inductor current waveform, using the linear ripple approximation.



With switch in position 1:

$$\underbrace{i(dT_s)}_{} = \underbrace{i(0)}_{} + \underbrace{(dT_s)}_{} \underbrace{\left(\frac{\langle v_g(t) \rangle_{T_s}}{L}\right)}_{}$$

(final value) = (initial value) + (length of interval) (average slope)

With switch in position 2:

$$\underbrace{i(T_s)}_{} = \underbrace{i(dT_s)}_{} + \underbrace{(d'T_s)}_{} \underbrace{\left(\frac{\langle v(t) \rangle_{T_s}}{L}\right)}_{}$$

(final value) = (initial value) + (length of interval) (average slope)

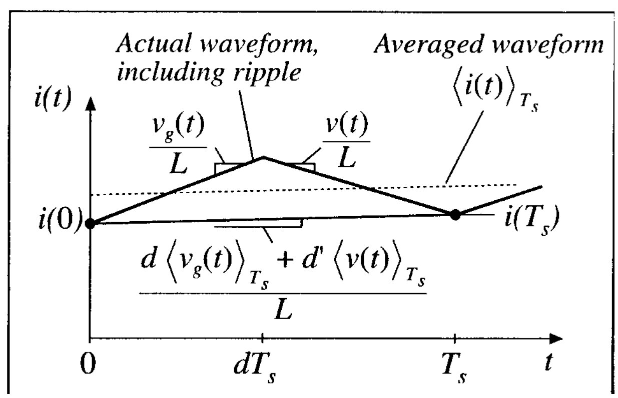
Solving for I(TS):

Eliminate $i(dT_s)$, to express $i(T_s)$ directly as a function of $i(0)$:

$$i(T_s) = i(0) + \frac{T_s}{L} \underbrace{\left(d(t) \langle v_g(t) \rangle_{T_s} + d'(t) \langle v(t) \rangle_{T_s}\right)}_{\langle v_L(t) \rangle_{T_s}}$$

The intermediate step of computing $i(dT_s)$ is eliminated.

The final value $i(T_s)$ is equal to the initial value $i(0)$, plus the switching period Ts multiplied by the average slope $\langle v_L \rangle_{T_s}/L$.



. Capacitor Equation

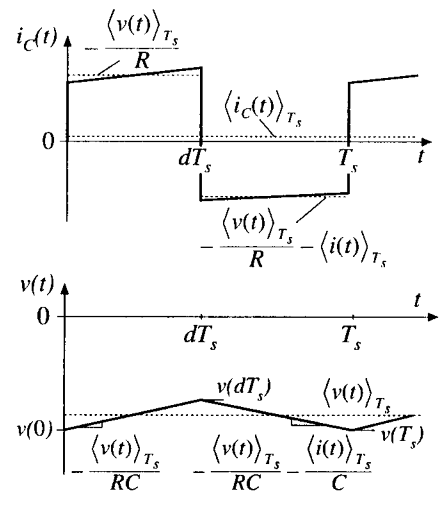
We now consider in the buck-boost circuit the average over the switch cycle of both the capacitor current and the output voltage in order to determine the averaged capacitor equation.

Average capacitor current:

$$\left\langle i_C(t) \right\rangle_{T_s} = d(t)\left(-\frac{\left\langle v(t) \right\rangle_{T_s}}{R}\right) + d'(t)\left(-\left\langle i(t) \right\rangle_{T_s} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}\right)$$

Collect terms, and equate to $C\, d\langle v \rangle_{T_s}/dt$:

$$C\frac{d\left\langle v(t) \right\rangle_{T_s}}{dt} = -\,d'(t)\left\langle i(t) \right\rangle_{T_s} - \frac{\left\langle v(t) \right\rangle_{T_s}}{R}$$



Capacitor voltage and current waveforms

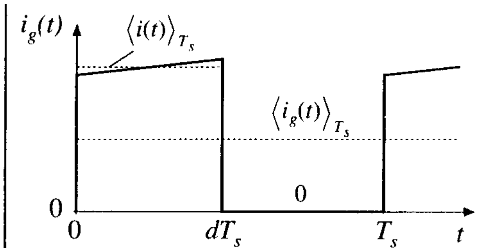
. Input Current Average Equation

In a similar fashion the input current, I_g ,can be averaged:

We found in Chapter 3 that it was sometimes necessary to write an equation for the average converter input current, to derive a complete dc equivalent circuit model. It is likewise necessary to do this for the ac model.

Buck-boost input current waveform is

$$i_g(t) = \begin{cases} \left\langle i(t) \right\rangle_{T_s} & \text{during subinterval 1} \\ 0 & \text{during subinterval 2} \end{cases}$$



Converter input current waveform

Average value:

$$\left\langle i_g(t) \right\rangle_{T_s} = d(t)\left\langle i(t) \right\rangle_{T_s}$$

We now have the three equations we sought at the onset averaged over the switch time, T_{SW}. They are summarized on the top of page 3. In practice we operate the buck-boost at one selected value of D in order to achieve the desired output level on a DC or steady-state basis.

For the buck-boost the top of page 9 summarizes:

d. Perturbation About the Operating Point

If the converter is driven with some steady-state, or quiescent, inputs

$$d(t) = D$$
$$\left\langle v_g(t) \right\rangle_{T_s} = V_g$$

then, from the analysis of Chapter 2, after transients have subsided the inductor current, capacitor voltage, and input current

$$\left\langle i(t) \right\rangle_{T_s}, \left\langle v(t) \right\rangle_{T_s}, \left\langle i_g(t) \right\rangle_{T_s}$$

reach the quiescent values I, V, and I_g, given by the steady-state analysis as

$$V = -\frac{D}{D'} V_g$$
$$I = -\frac{V}{D' R}$$
$$I_g = D\, I$$

The DC and AC components of the signals are:

So let us assume that the input voltage and duty cycle are equal to some given (dc) quiescent values, plus superimposed small ac variations:

$$\left\langle v_g(t) \right\rangle_{T_s} = V_g + \hat{v}_g(t)$$
$$d(t) = D + \hat{d}(t)$$

In response, and after any transients have subsided, the converter dependent voltages and currents will be equal to the corresponding quiescent values, plus small ac variations:

$$\left\langle i(t) \right\rangle_{T_s} = I + \hat{i}(t)$$
$$\left\langle v(t) \right\rangle_{T_s} = V + \hat{v}(t)$$
$$\left\langle i_g(t) \right\rangle_{T_s} = I_g + \hat{i}_g(t)$$

We can extract three linearized equations: one for the inductor, one for the capacitor and one for the input current. Substituting the Dc and AC components and expanding the equations, we are able to justify neglecting all SECOND order terms to get three linear equations for SMALL SIGNAL analysis at $f<f_{SW}$.

0. Linear I_g Equation

Collect terms:

$$\underbrace{I_g}_{\text{Dc term}} + \underbrace{\hat{i}_g(t)}_{1^{st}\text{ order ac term}} = \underbrace{(DI)}_{\text{Dc term}} + \underbrace{\left(D\hat{i}(t) + I\hat{d}(t)\right)}_{\substack{1^{st}\text{ order ac terms} \\ \text{(linear)}}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\text{ order ac term} \\ \text{(nonlinear)}}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following first-order terms remain:

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

This is the linearized small-signal equation which described the converter input port.

2. Linear "L" Equation

Upon discarding second-order terms, and removing dc terms (which add to zero), we are left with

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

This is the desired result: a linearized equation which describes small-signal ac variations.

Note that the quiescent values D, D', V, V_g, are treated as given constants in the equation.

0. 3. C Linear Equation

Perturbation leads to

$$C\frac{d\left(V + \hat{v}(t)\right)}{dt} = -\left(D' - \hat{d}(t)\right)\left(I + \hat{i}(t)\right) - \frac{\left(V + \hat{v}(t)\right)}{R}$$

Collect terms:

$$C\left(\frac{dV}{dt}^{0} + \frac{d\hat{v}(t)}{dt}\right) = \underbrace{\left(-D'I - \frac{V}{R}\right)}_{\text{Dc terms}} + \underbrace{\left(-D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)\right)}_{\substack{1^{st}\text{ order ac terms} \\ \text{(linear)}}} + \underbrace{\hat{d}(t)\hat{i}(t)}_{\substack{2^{nd}\text{ order ac term} \\ \text{(nonlinear)}}}$$

Neglect second-order terms. Dc terms on both sides of equation are equal. The following terms remain:

$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

This is the desired small-signal linearized capacitor equation.

Putting all three linearized equations together we see a set of three equations from which we can build a circuit model.

The linearized small-signal converter equations:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) + D'\hat{v}(t) + \left(V_g - V\right)\hat{d}(t)$$

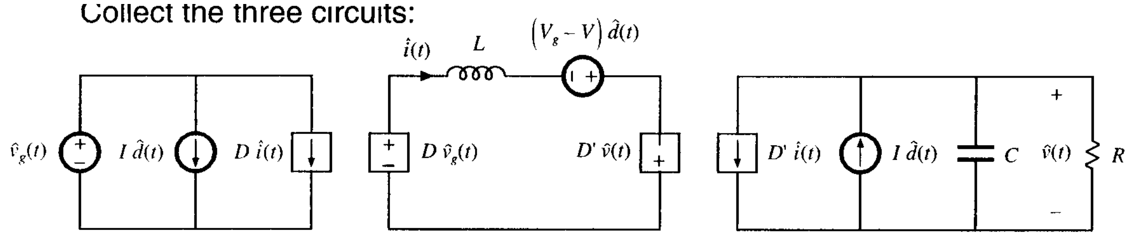
$$C\frac{d\hat{v}(t)}{dt} = -D'\hat{i}(t) - \frac{\hat{v}(t)}{R} + I\hat{d}(t)$$

$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

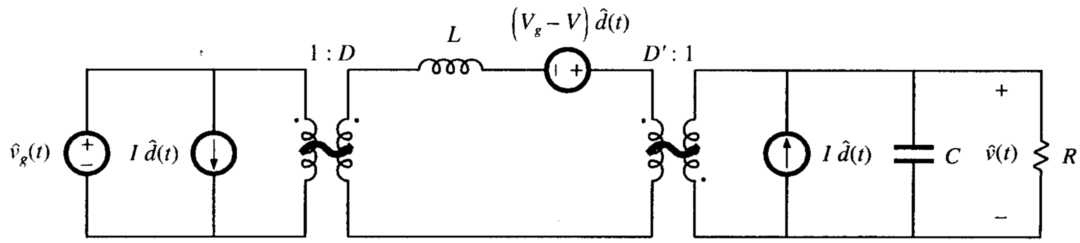
Reconstruct equivalent circuit corresponding to these equations, in manner similar to the process used in Chapter 3.

Each equation gives rise to a loop as shown below which when coupled together by DC transformers gives us the buck-boost converter linear circuit model below.

Collect the three circuits:



Replace dependent sources with ideal dc transformers:



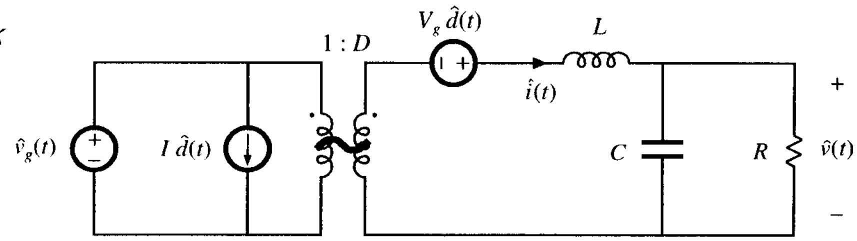
Small-signal ac equivalent circuit model of the buck-boost converter

We can repeat this tedious process for the buck, boost and flyback circuits to achieve the AC circuit models of page 12. All such models are only accurate for $f<f_{SW}$ and will allow transfer functions for V/d or V/v_g to be easily made via superposition arguments.

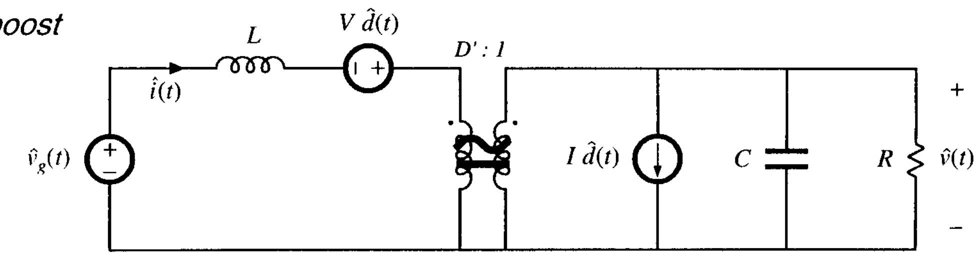
0. Models for Buck and Boost

We leave for HW the derivations.

buck



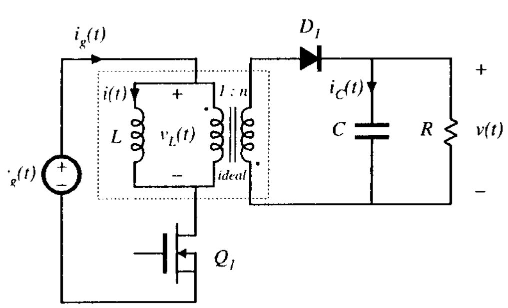
boost



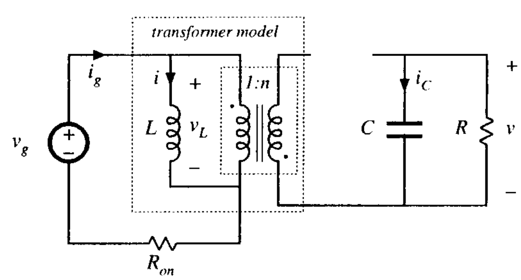
4. Flyback with DC Switch Losses: Simplest Case

We consider as the only switch loss R_{ON} of the MOSFET and derive the AC model below starting with the two switch states and associated circuit topologies.

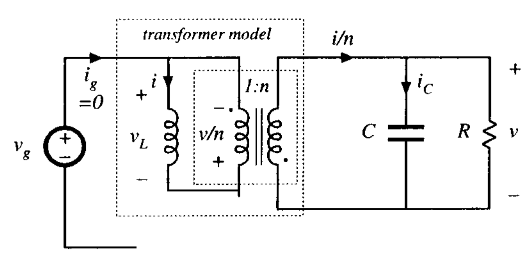
Flyback converter, with transformer equivalent circuit



Subinterval 1



Subinterval 2



On the top of the next page we summarize both DC and AC equations for the flyback with R_{ON} included in the mix. We will find three linearized small signal equations, each of which gives rise to a circuit loop.

Dc equations:

$$0 = DV_g - D'\frac{V}{n} - DR_{on}I$$
$$0 = \left(\frac{D'I}{n} - \frac{V}{R}\right)$$
$$I_g = DI$$

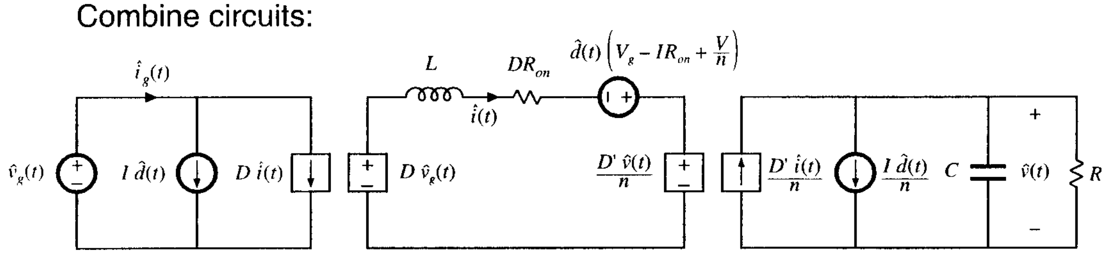
Small-signal ac equations:

$$L\frac{d\hat{i}(t)}{dt} = D\hat{v}_g(t) - D'\frac{\hat{v}(t)}{n} + \left(V_g + \frac{V}{n} - IR_{on}\right)\hat{d}(t) - DR_{on}\hat{i}(t)$$
$$C\frac{d\hat{v}(t)}{dt} = \frac{D'\hat{i}(t)}{n} - \frac{\hat{v}(t)}{R} - \frac{I\hat{d}(t)}{n}$$
$$\hat{i}_g(t) = D\hat{i}(t) + I\hat{d}(t)$$

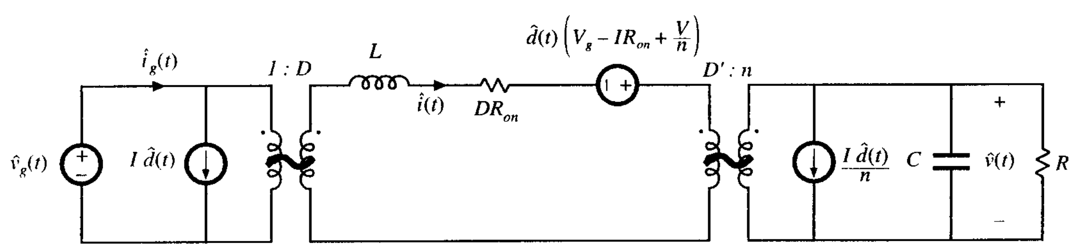
Next step: construct equivalent circuit models.

Each equation contributes a loop as shown below:

Combine circuits:



Replace dependent sources with ideal transformers:



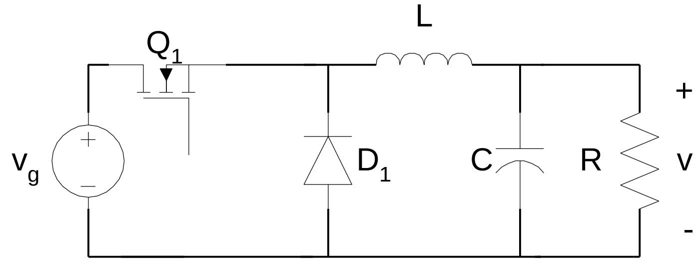
0. 5. Other Circuit Cases with Losses

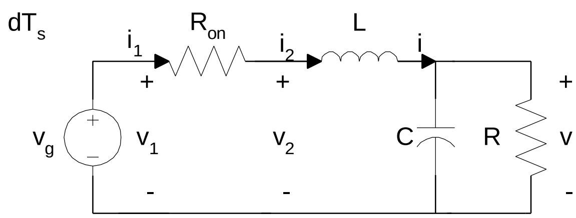
. Erickson Problem 7.17

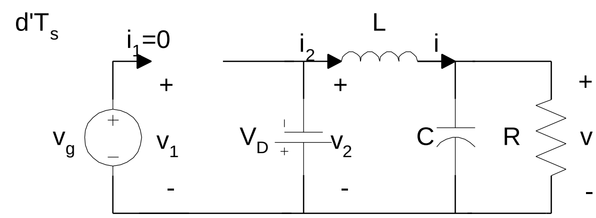
Transistor on resistance, R_{on}, and diode forward voltage drop, V_D, included for Both the Buck and Boost Circuits.

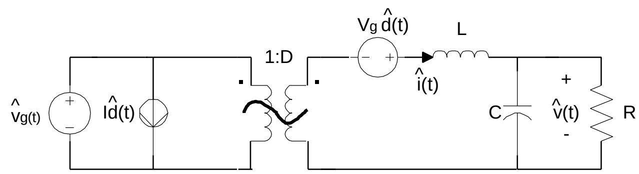
1) Buck:

Model For the Buck Converter

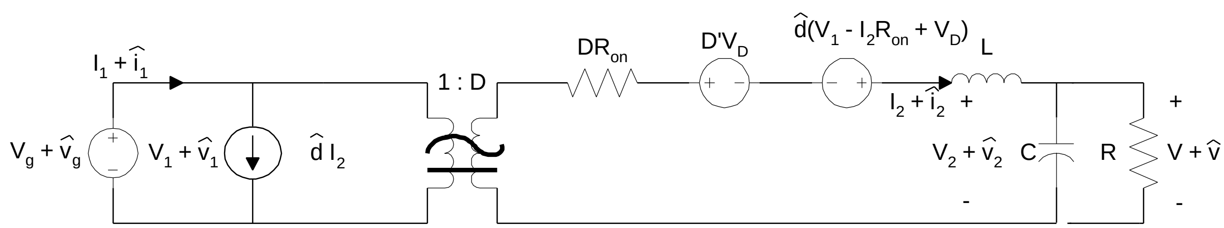




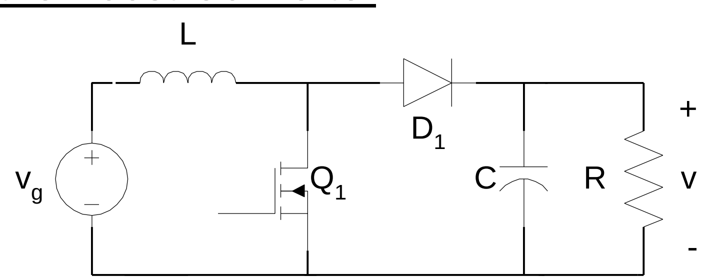


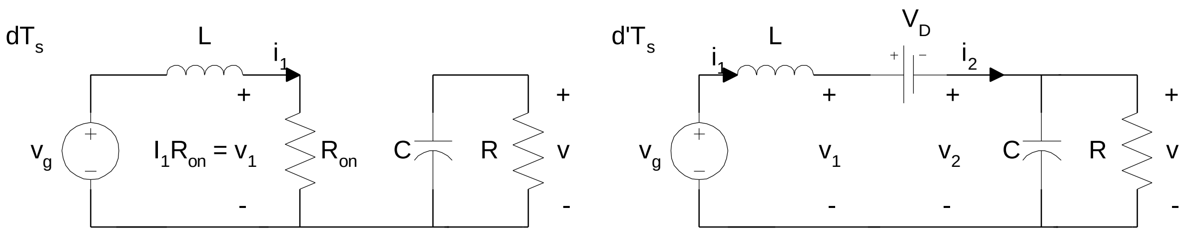


An ideal buck would look:
Adding both R_{ON} (MOSFET)and V_D (DIODE) we find:

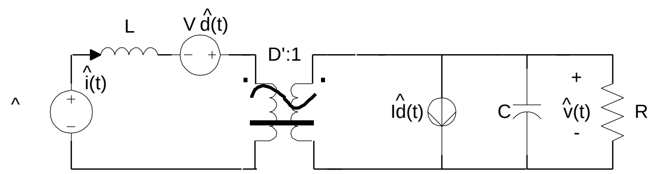


2) Model For the Boost Converter

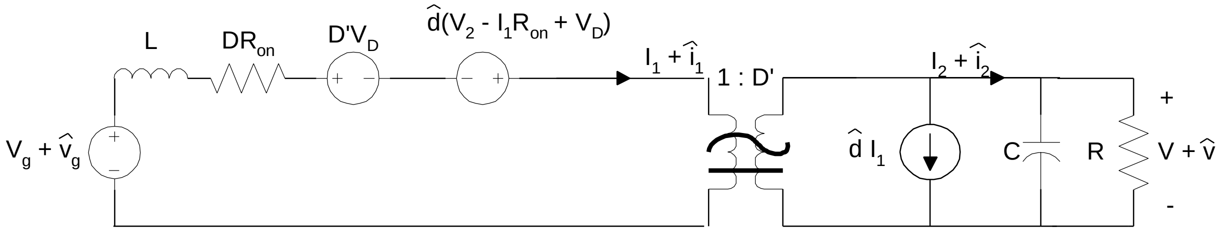




The lossless boost is:



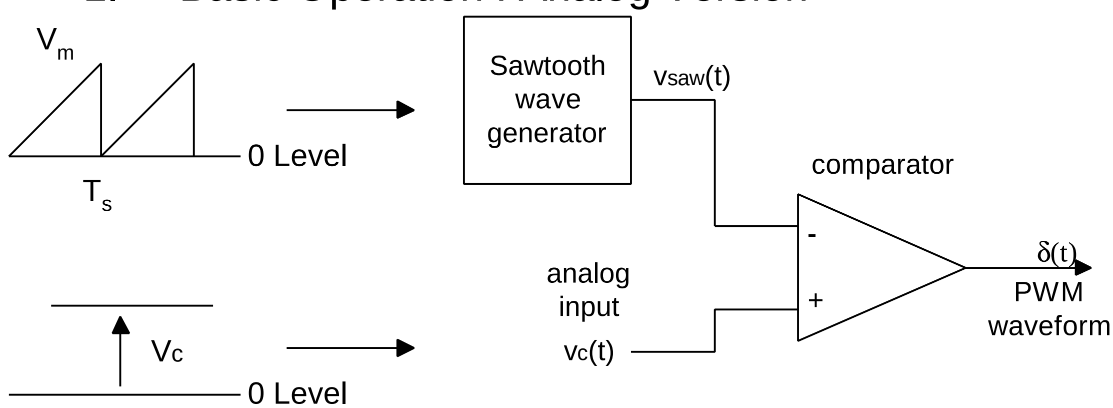
When we include R_{ON} and V_D we find:



Problem 17c asks the results for the buck-boost with these same losses included.

C. Pulse Width Modulators

1. Basic Operation : Analog Version



The output from the comparator will vary as follows:

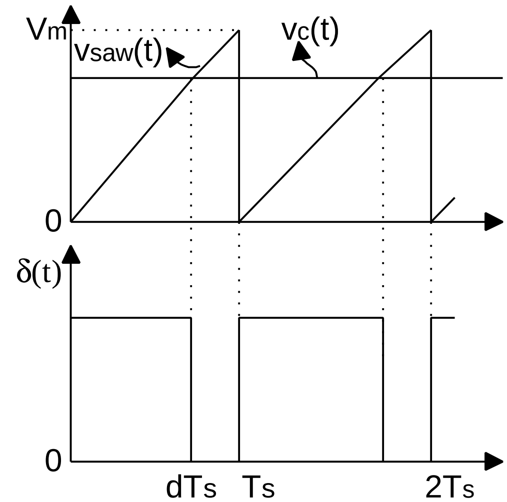
If V_c is negative duty cycle $\equiv 1$

V_c is $> V_m$ duty cycle $\equiv 0$

In between values of V_c generate $0 \leq d \leq 1$.

$$d(t) = \frac{V_c(t)}{V_M} \quad 0 \le V_c \le V_M$$

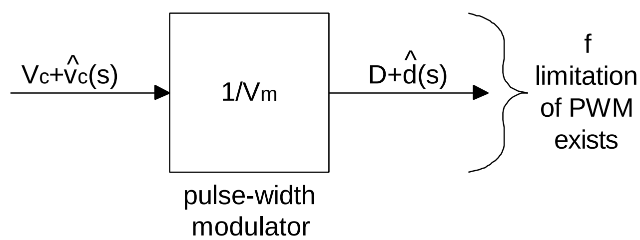
Clearly the duty cycle, d, out from the comparator has $d \propto V_c(t)$



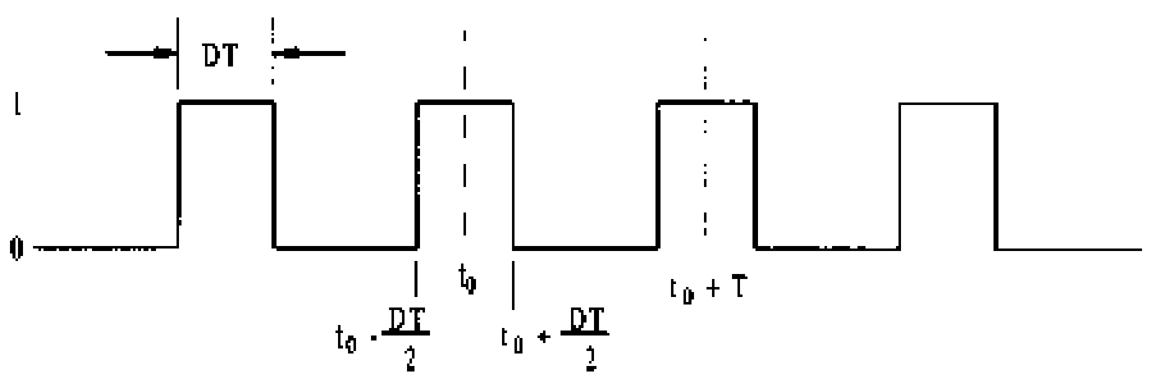
One can visualize PWM operation as sampling V_c@ the switch frequency f_{sw}

V_c@ f_{sw}

⇒ Only see changes in V_c at $f \le f_{sw}$



The periodic pulse train shown below:



has a Fourier series expansion

$$d(t) = D + \frac{2}{p}\sum_{n=1}^{\infty}\frac{Sin(npD)}{n}Cos(nwt - n f_o)$$

Three parameters specify d(t)

0. The duty ratio D
0. The radian frequency w
0. The reference time t_o or phase ϕ_o

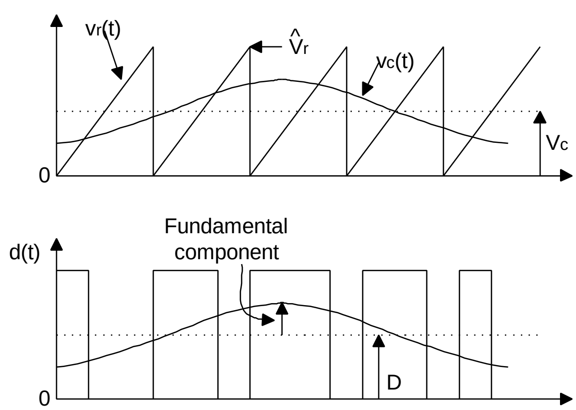
All three pulse parameters are employed to vary switch commutation with parameter(1) most popular for PWM converters and parameter(3) most popular for ac commutation of SCR's etc. Parameter (2) finds little use because of the need for tight constraints on f_{sw} for frequency modulation, FM, control.

2. Transfer Function : $T(s) = \dfrac{1}{V_M}$

V_c is output of control voltage or error amplifier

$$V_c(t) = V_c(dc) + \hat{v}_c$$

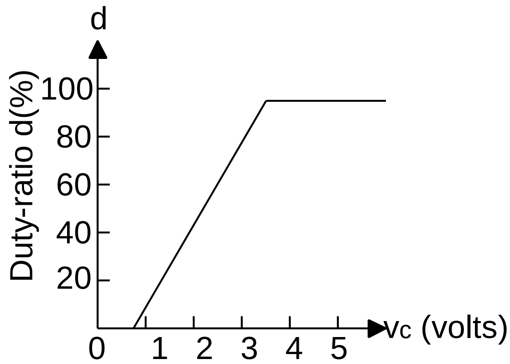
$$\hat{v}_c \ll V_c$$



$$\hat{v}_c \sim A \sin(wt - f)$$

$$d(t) + \frac{V_c}{V_M} + \frac{A\sin(wt - f\ )}{V_m}$$

$$T(s) = \frac{d(s)}{V_c(s)} = \frac{1}{V_M} = D + d(t)$$



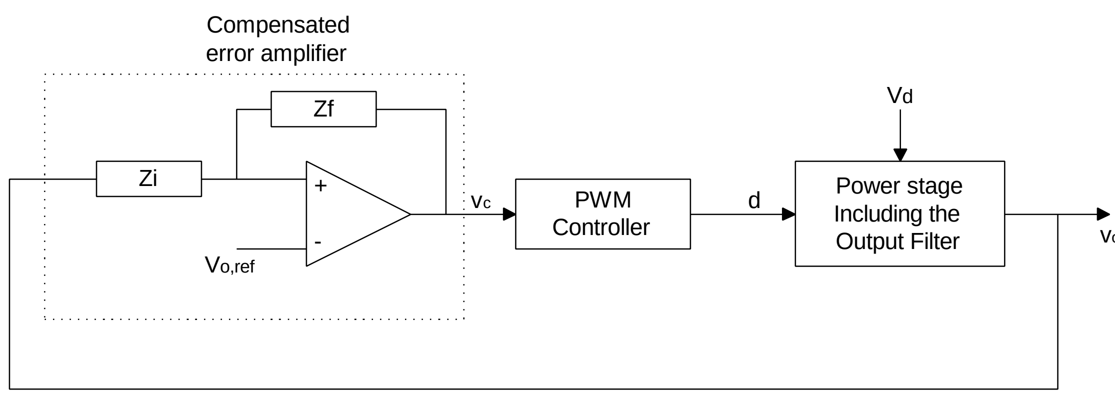
From an actual IC chip controller output voltage plot versus d we find

$d = 0$ @ $V_c = .8$ V

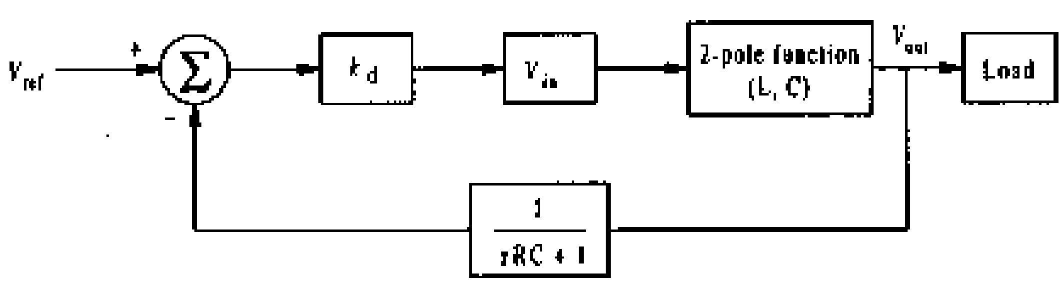
$d = .95$ @ $V_c = 3.6$V

$$\frac{\hat{d}}{\hat{V}_c} = \frac{\Delta d}{\Delta V_c} = \frac{.95-0}{3.6-0.8} = \frac{1}{2.94}$$

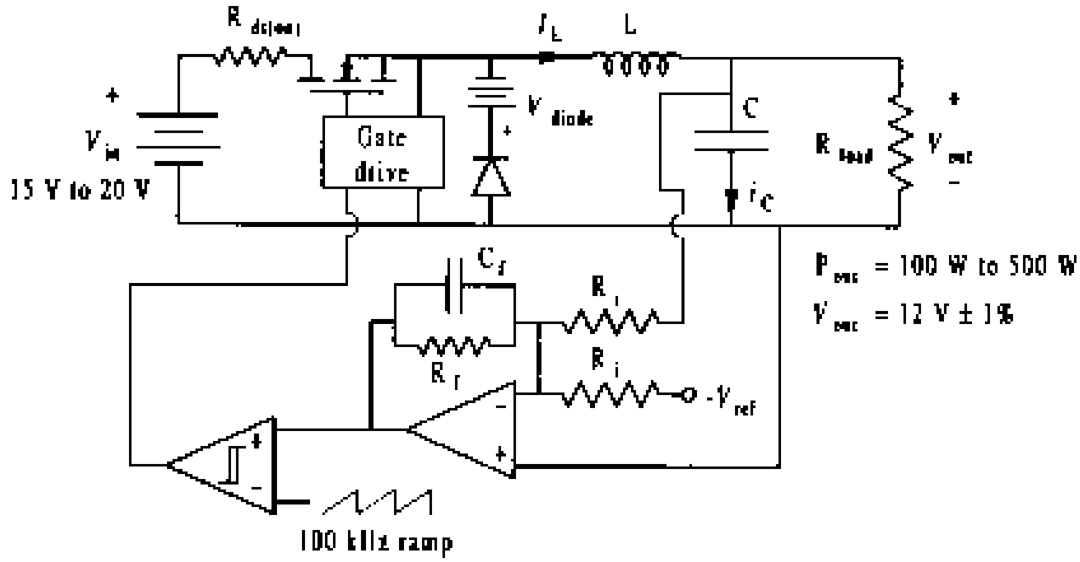
This analysis **neglects** any comparator time delays! So a full system with feedback could look schematically like:



A block diagram of a buck converter with voltage feedback appears as shown below.



A low pass filter is added to the feedback to eliminate switch ripple feedback. The closed loop transfer function would contain three poles and could oscillate or show instability. A more detailed layout of the voltage-mode feedback is shown below.



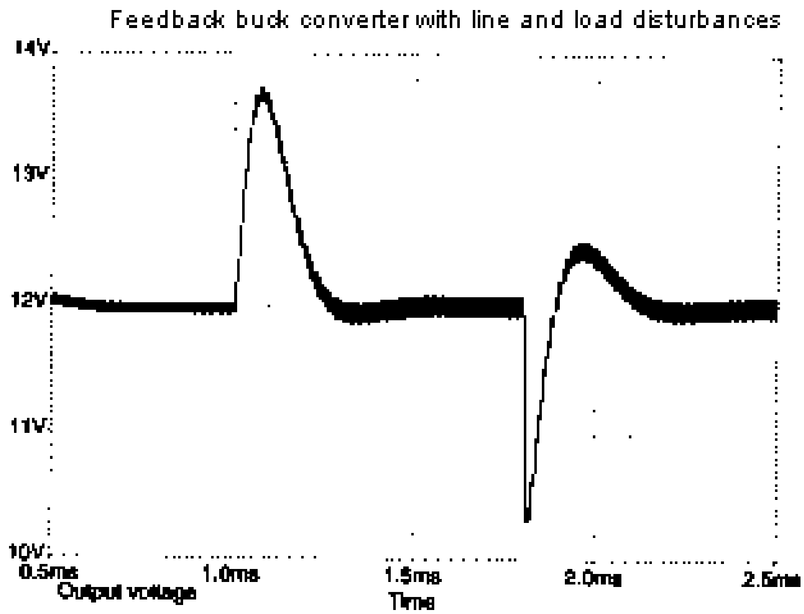
With the above control loop we can change loads and input voltages as follows yet maintain V_{out} at 12V.

<u>Changes</u>

0. $V_g = V_{in}$ changes at t = 1ms from 15 to 18V yet V_o returns to 12V in ½ ms. See below.

0. I(load) changes from 20 to 24A yet V_o returns to 12V in

   ½ ms. See on page 20
below.

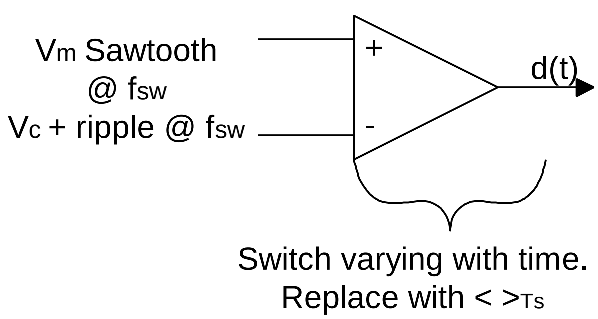


In recent years, PWM Controller is made either:

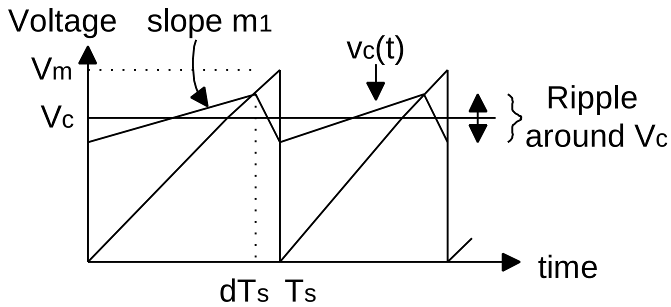
a. Digitally for increased environmental stability to T, power supplies, aging, etc. Usually this **<u>lowers</u>** parts count.

b. Using software and hardware for minimizing error and for faster transient response.

4. Influence of switching Ripple on $V_c(t)$ and d(t).



Problem 7.15 Consider a linear ripple voltage on V_c(control)



a) Determine d(t) /$<V_c(t)>_{Ts}$

1) $V_c(nT_s) = V_c[(n + 1)Ts]$ No net drift of V_c occurs!

2) $V_c[(n + d(t))\, Ts] = <V_c(t)>_{Ts} + M_1\, d(t)/2\, Ts$

↓ DC

↓ amount above/below average

$$= \frac{V_m}{Ts} \bullet d(t)\, Ts$$

$$< V_c(t) >_{Ts} = d(t) T_s \left(\frac{V_m}{Ts} - \frac{m_1}{2}\right) \rightarrow \frac{d(t)}{< V_c >_{Ts}} = \frac{1}{Ts\left(\frac{V_m}{Ts} - \frac{m_1}{2}\right)}$$

$$\therefore \frac{d(t)}{< V_c(t) >_{Ts}} = \frac{1}{V_m - \frac{m_1}{2} Ts}$$

b) Does ripple increase or decrease the modulator gain?

$m_1 > 0 \Rightarrow$ with ripple w/o ripple

$$\frac{1}{V_m - \frac{m_1}{2} Ts} > \frac{1}{V_m}$$

$\therefore$ Modular gain is **increased** with an ac ripple of linear slope m_1.

This will also be important in Chapter 11 Current programmed mode. In problem 7.15 c) Is the modulator still linear with linear ripple on V_c? Over what range of $<V_c(t)> T_s$ is it linear?

Modulator gain is constant until the largest value of $<V_c(t)>_{Ts}$. Then, $V_c[(n + d)Ts] = V_m$

$<V_c>_{ts}$ + mid Ts/2 = V_m

$<V_c>_{Ts} = V_m$ - midTs/2

Therefore it is still linear over the reduced range

$0 < v_c(t)_{Ts} < V_m - m_1 dTs/2$.