LECTURE 34

HIGH FREQUENCY TRANSFORMER

A. Transformer Basics

1. Geometry of Cu Wire Windings and Core Wire Winding Window

2. Single Wire Skin Effect and Multi-wire Proximity Effects Alter Both Primary and Secondary R(wire coil) via Effective A_{cu} (wire) Changes

3. Transformer Inductance's: **L_m and L_l**
   a. Magnetizing Inductance, **L_m: Core Flux**
   b. Leakage inductance, **L_l: Air Window Flux**
   A crude estimate for **L_l** is $(1*L_m)/\mu$ and **Interleaved Primary/Secondary** Winding effects occur

4. Total Core Loss: R_m
   a. Hysteresis Loss due to frictional movement
   b. Eddy Current Loss due to core currents

5. How is the Transformer Magnetization Current, i_m ,Created?
   a. i_m is **Not effected by load current** at all
   b. $i_m = \int v_L dt/L_m$ volt-second driven only

7. T(max) of the core of a LossyTransformer
   a. Heat Balance Equation
   T(core) = T(ambient) + P(core and wire loss)*R(core and Winding Structure)

8. Comparing Inductors vs Transformers

B. Special Case for Wire Losses in Transformers with Multiple Windings

1. V ~ Number of Wire Turns, I ~ Depends on Load Impedance in the secondary
2. Optimum Winding Area for the k'th coil
   $\alpha_K = P_K/P_T$
3. Example of PWM Converter
   A_K (winding) = f(P_K,D), that is the winding area required depends on the duty cycle employed in the electrical circuit. Generally, this is a range of D values.

LECTURE 34
High Frequency TRANSFORMERS

In part A we will focus on transformers and the relation between magnetic and electrical properties. We then use the information gleaned from relating the copper wire turns loss of a transformer to more accurately treat the case of an inductor which carries only AC currents in part B. Previously we treated only the filter inductor, which had small AC currents. **For HW #1 do problems 1 and 5 of Chapter 14 from Erickson's text and the questions we have raised in the lecture notes to date. THIS IS DUE NEXT WEEK**

A. Transformer Basics

Two or more wire windings placed around a common magnetic core is the physical structure of a transformer. It's electrical purpose is to transfer power from the primary winding to the other windings with no energy storage or loss.

For HW# 1 show the B-H curve for a transformer with transferred and core loss energy indicated.

The choice of circuit topology obviously has great impact on the transformer design. Flyback transformer circuits are used primarily at power levels in the range of 0 to 150 Watts, Forward converters in the range of 50 to 500 Watts, half-bridge from 100 to l000 Watts, and full bridge usually over 500 Watts. The waveform and frequency of currents in transformers employed in these unique circuit topologies are all unique. Intuitively we expect all windings employed on a given transformer to take up a volume consistent with their expected power dissipation. What is not intuitively clear is **that there is an optimum core flux density, B_{OPT} , where the total of copper and core losses will be a minimum**. This B_{OPT} will guide transformer and AC inductor design. For if we hit the B_{OPT} target we will operate with minimum transformer losses and achieve minimum temperature of the core.

1. Geometry of Copper Windings and Core Wire Winding

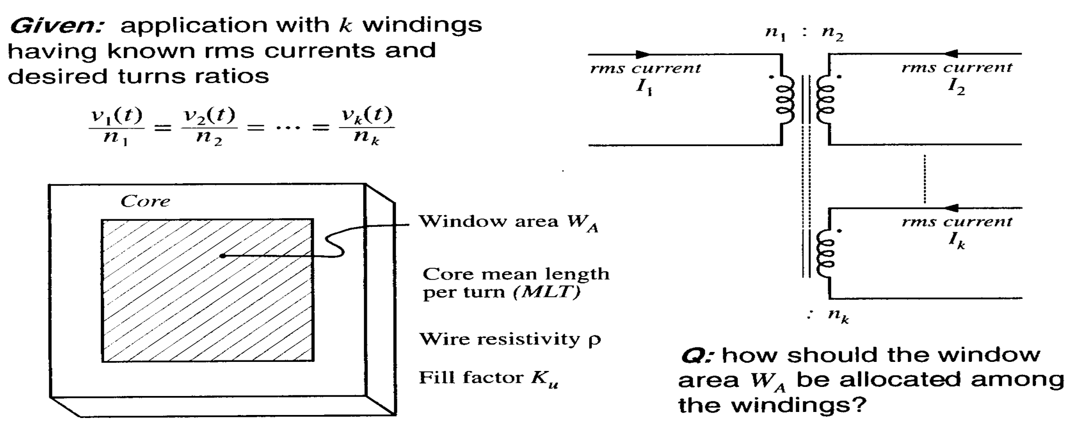
Window

a. Overview

Lets for simplicity consider only two wire windings wound upon one magnetic core, which acts to couple the magnetic flux between the two coils with near unity transfer. The main purpose of a power transformer in Switch Mode Power Supplies is to transfer power efficiently and instantaneously from an external electrical source to external loads placed on the output windings. In doing so, the transformer also provides important additional capabilities:

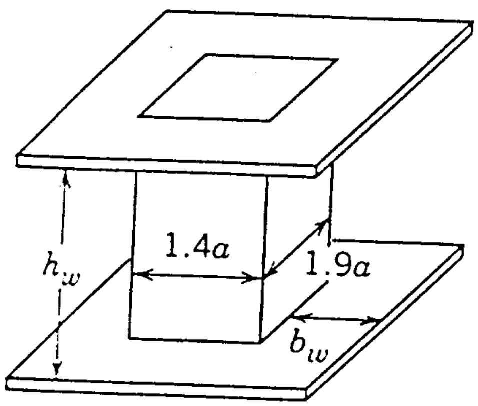
- The primary to secondary turns ratio can be established to efficiently accommodate widely different input/output voltage levels.
- Multiple secondaries with different numbers of turns can be used to achieve multiple outputs at different voltage levels and different polarities.
- Separate primary and secondary windings facilitate high voltage input/output isolation, especially important for safety.
- A set of k wire windings introduces some complexity to the issues for an **optimum transformer winding partition**

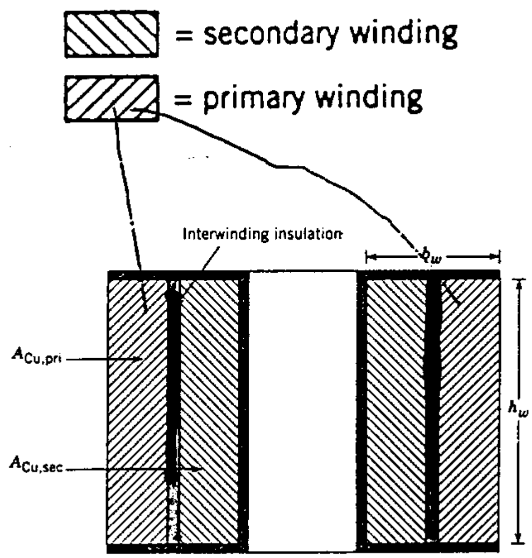
For the case of multiple windings the competing issues are more



complex as we will see at the end of this lecture. It will require Lagrangian optimum analysis using one variable for each wire coil. In the end the winding area allotted to each coil winding will vary as its power handling requirement compared to the total power level.

All the wire windings, wound on a given core must fit into its one wire winding *window* which we term either A_W or W_A in the text below. Both symbols are found in the transformer literature. Much of the actual winding area is taken up by voids between round wires, by wire insulation and any bobbin structure on which the wire turns are mounted as well as insulation between high voltage and low voltage windings. In practice, only about 50% of the window area can actually carry active conductor, This fraction is called the *fill factor*. In a two-winding transformer, this means that each winding can fill not more than 25% of the total wire winding window area





Total area for windings ≡
$A_w = A_{pri} + A_{sec}$

A_w is split into two parts, for a two winding transformer, according to the required wire sizes (AWG#) in each coil which in turn is chosen for the expected current flow to avoid overheating of the wires. In short the primary wire winding area employed in the wire winding window is:

$$A_W(\text{prim}) = \frac{N_{pri}\, A_{cu\ wire}\ (\text{AWG \# of primary})}{K_{cu}(\text{primary})}$$

Note that we use AWG tables for USA wires specified in cir mils. EUC tables for wires are specified in mm^2.

Again K_{Cu} is an estimate of what % of the wire volume is actually Cu. K_{cu} is specified for each type of wire and spans a range: K_{cu}(Litz wire) = 0.3 and K_{cu}(Cu foil) = 0.8-.9. This is nearly a factor of 3 possible variation. The same is true for the secondary wire area.

$$A_w(\text{sec}) = \frac{N_{sec} A_{cu\ wire}\ (\text{secondary } AWG\ \#)}{K_{cu}(\text{secondary})}$$

With different wire type choices for the primary and secondary $K_{cu}(\text{primary}) \neq K_{cu}(\text{secondary})$. If we want heat from I^2R or ρJ^2 losses to be uniformly distributed over core window volume, then we usually choose the same type wire for primary & secondary windings to get similar K_{cu} ,but vary AWG # to achieve the same J in the two sets of windings. That way both windings heat uniformly if

$$\rho_{cu} J^2(\text{primary}) = \rho_{cu} J^2(\text{secondary})$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$\left(\frac{I_{prim}}{A_{cu}(\text{primary})}\right)^2 \qquad \left(\frac{I_{sec}}{A_{cu}(\text{secondary})}\right)^2$$

Area of Cu wire or AWG# must be different for primary vs secondary wires since $I_{sec} \neq I_{pri}$. So we find in a usual transformer AWG #(primary) ≠ AWG #(secondary) and for no skin effects follow the rule:

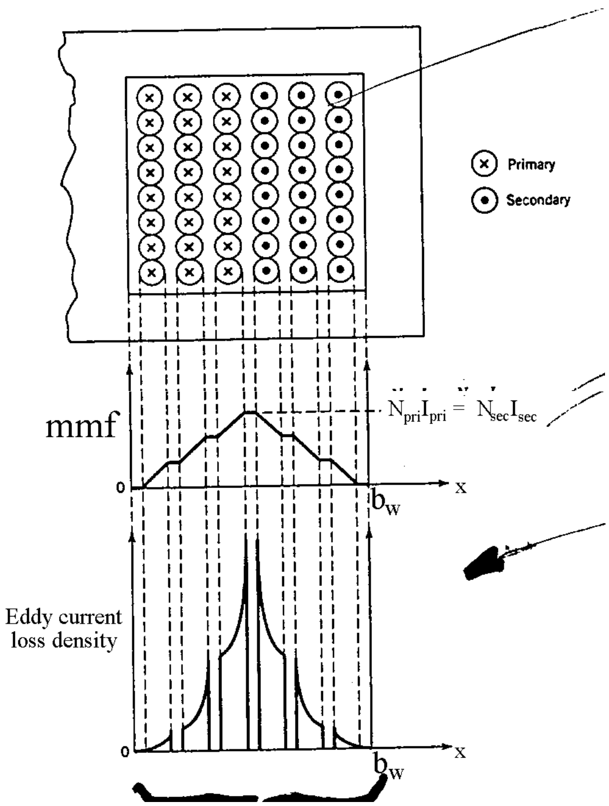
$$\frac{I_{prim}}{I_{sec}} = \frac{A_{cu}(\text{primary wire})}{A_{cu}(\text{secondary wire})} = \frac{N_{sec}}{N_{prim}}$$

Since both wire windings must fit in the core wire window area A_w or W_A we have a constraint: $A_w(\text{window for wire}) = A(\text{prim}) + A(\text{sec})$

In short, wire size is selected to support both the desired level of current and to fit into the core's *window area.* Low frequency windings often use the largest wire that will fit into the window. This minimizes losses, and maximizes the power rating for a transformer. High frequency windings are more complex to deal with for reasons as described below.

b. Single Wire Skin Effect multi-wire and proximity effects. The tendency for hf currents in wire coils to go to the wire surface, alters the effective A(wire) in turn causing increases in Cu loss expected from windings at high frequencies as compared to low frequencies!

Proximity effects, due to the collective magnetic field from many wires, also increases wire losses via cumulative MMF effects.



J flow only on the surface causes increase in Cu wire loss!

Skin effects cause non-uniform current density profiles in all wires, BUT Non-interleaved windings allow build-up of mmf in the wire winding window. This mmf is different for each wire position and ruins the assumption that J is uniformly distributed across the wire area the same way for all wires. That is different wires have different J distributions in the real world of hf transformers

Again, the increasing MMF spatial variation makes the R(each wire turn) unique in it's losses according to it's position in the wire winding window. **The Higher the mmf seen by the wire the more non-uniform the J.**

Note the interior wires, where primary and secondary meet, as we have discussed previously, have the highest mmf and the most non-uniform J in the wires causing much higher I^2R loss.

2. Energy Storage in a Transformer

Ideally a transformer stores no energy, rather all energy is transferred instantaneously from input to output coils. In practice, all transformers do store some energy in the two types of inductance's that associated with the real transformer as compared to ideal transformers which have no inductances associated with them. There are two inductances.

- Leakage inductance, L_l , represents energy stored in the wire winding windows and area between wire windings, caused by imperfect flux coupling for a core with finite μ. In the equivalent electrical circuit leakage inductance is in series with the wire windings. and the stored energy is proportional to wire winding current squared.

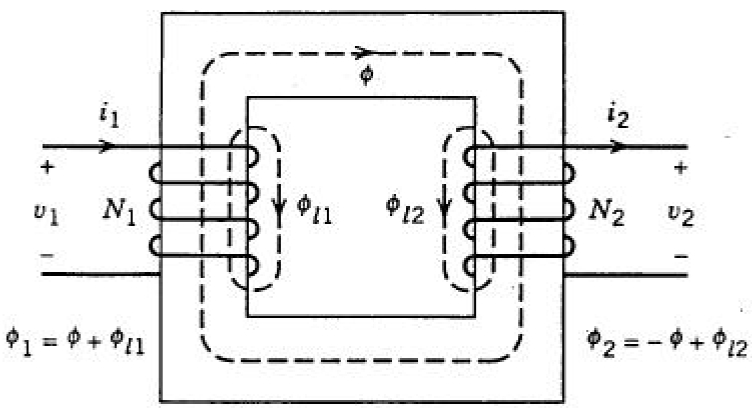
- Magnetizing inductance, L_M , represents energy stored in the magnetic core and in small air gaps which arise when the separate core halves forming a closed magnetic loop core come together. In the equivalent circuit of a real transformer, mutual inductance appears in parallel with the primary windings only. The energy stored in the magnetization inductance is a function of the volt-seconds per turn applied to the primary windings and is independent of load current.

3. Various Frequencies of Interest to Magnetic Devices and Copper Loss

There are several meanings to the term "frequency" in switching power supply applications, and it is easy for confusion to arise. "**Clock frequency"** is the frequency of clock pulses generated in the control IC. Usually, the switching frequency is the same as the clock frequency, but not always. Occasionally, the control IC may divide the clock frequency to obtain a lower switching frequency. It is not unusual for a push-pull control IC to be used in a single-ended forward converter application, where only one of the two switch drivers used, to guarantee 50% max. duty cycle. In this case the switching frequency is half the clock frequency. **"Switching Frequency"**, f_{sw}, is usually defined as the frequency at which switch drive signals are generated. It is sometimes also the frequency seen by the output filter, the frequency of the output ripple and input ripple current, and is an important concept in control loop design. In a single-ended power circuit such as the forward converter, the power switch, the transformer, and the output rectifier all operate at the switching frequency and there is no confusion. The transformer frequency and the switching frequency are the same. This is not true for the bridge converters or center tappped transformers where the current waveforms may have different frequencies, as we saw in prior lectures on transformers in switch mode topologies

4. Estimating $L_{leakage}$(primary) »L_m/m_r via Associated Flux Paths

The magnetizing inductance, L_m , of a transformer is easily understood as the inductance seen at the primary coil winding, $L_m = N_1^2/\Re_c$(core). The assumption for calculating L_m is that the magnetic flux flows primarily in the magnetic core, which is true only for infinite permeability cores. For finite permeability cores a second inductance arises that must also be included in a transformer model, because magnetic flux also flows outside the core in the wire winding window where the wires are located. For μ(core) = 5000 nearly 0.02% of the flux flows in the air outside the core where the wires are wound and about 99.98% or the flux, ϕ, flows in the core geometry. This leakage flux creates parasitic inductance termed the leakage inductance.



Actually there are two leakage inductance's, one for the primary coil and one for the secondary coil. ϕ_{l1} and ϕ_{l2} represent **outside the core fluxes,** which lead to leakage inductance's. $\Re$(leakage) is the magnetic reluctance to flux flow in the air and depends on core geometry, core μ_r and wire geometry. We can calculate these reluctance's individually to find:.

$L_{\ell 1} = N_1^2 / \Re_{\ell 1}$ (air near primary)

$L_{\ell 2} = N_2^2 / \Re_{\ell 2}$ (air near secondary)

if $\Re_{l1} = \Re_{l2}$ then $L_{l1}/L_{l2} \approx N_1^2/N_2^2$

Well designed transformers:L_{l1}/L_{l2} »N_1^2/N_2^2 and Â$_{l1}$/Â$_{l2}$ »N_1^2/N_2^2

Both magnetizing and leakage inductance causes voltage spikes during switching transitions, resulting in EMI and possible damage or destruction of switches and rectifiers. L_l causes undesired inductive kick in voltage @ primary/secondary windings when any rapidly changing current waveform, like a square wave, is employed. Leakage inductance also delays the transfer of current between switches and rectifiers during switching transitions. These delays, proportional to load current, are one main cause of regulation and cross regulation problems in feedback control circuits.

Protective snubbers and clamps are often then required and the stored energy then ends up as loss in the snubbers or clamps. Leakage and mutual inductance energy is sometimes put to good use in zero voltage transition (ZVT) circuits. This requires caution as leakage inductance energy disappears at light load, and mutual inductance energy is often unpredictable.

Previously we showed that $L_l \approx L_m/\mu$,but this indicated only the effect of the choice of core material. We also saw that primary-secondary coil interweaving helped reduce proximity effects, which raised coil $I^2 R$ losses. Now we show the effect of interleaved wire windings to reduce the leakage inductance. In particular we will show: $L_\ell \sim \frac{1}{P^2}$ Where P is the # of prim/sec interfaces in the winding of the transformer. We **actually get a double win from interleaving coil windings, decreased proximity effects in copper loss and reduced leakage inductance as we show below**.

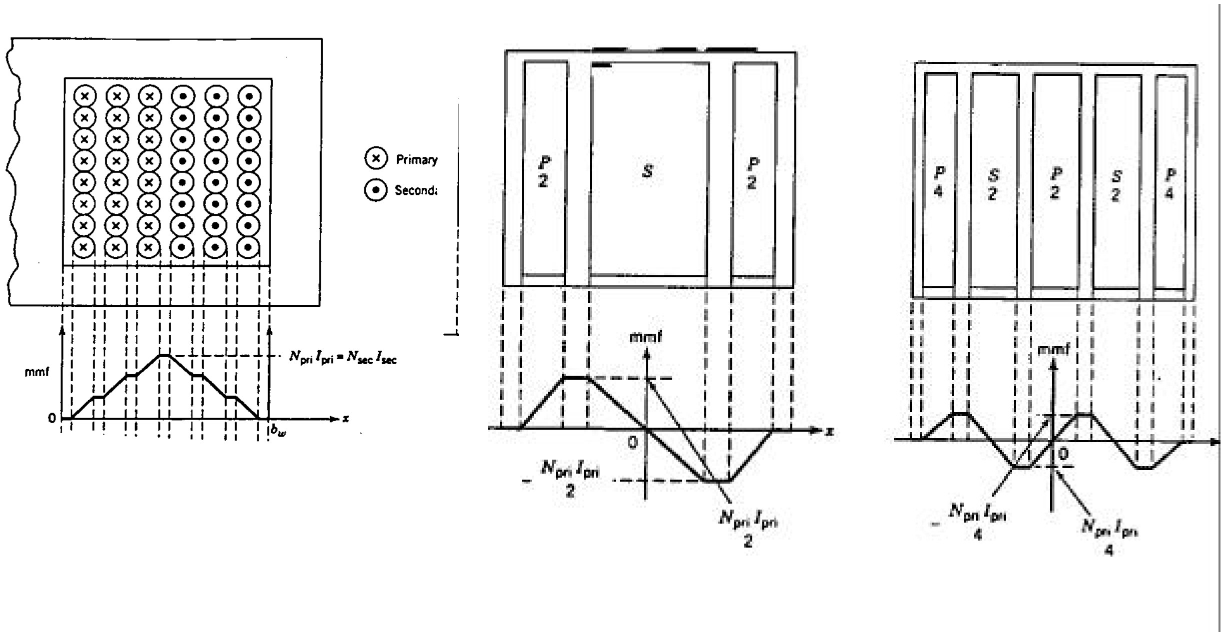
The energy stored in leakage inductance is undesired for a transformer which aims to transfer energy from one coil to another.

$$\frac{1}{2} L_{\ell}1 \; I^2_{prim} = \frac{1}{2} \int_{\text{window volume}} m_o H^2 (\text{window})\, d\, V_w$$

Note below how it varies with non-interleaved and interleaved wire windings as shown below.

P = 1 | P = 2 | P = 4

$L_l \approx$ full value = L_m/μ_r | L_l = ¼ full L_m/μ | L_l = 1/16full L_m/μ



One can tailor L_l (leakage) to be bigger or smaller by up to an order of magnitude from L_m/μ_r choices or by **by choosing winding arrangements as shown**.

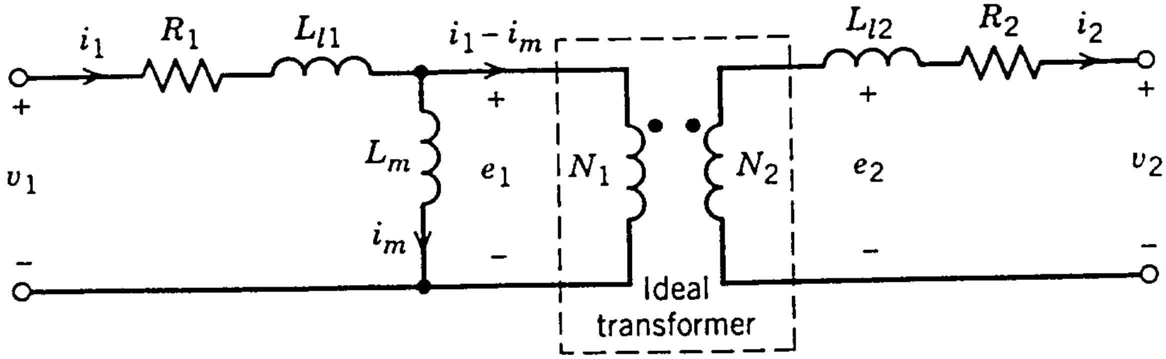
Full Winding $(L_l)_{max}$ | | Split Windings $L_l \downarrow$

$L_l \equiv 1.0$ | $L_l \rightarrow 1/4$ | $L_l \rightarrow 1/16$

Conclusion: Split or partition windings to reduce L_l *and to reduce the copper losses due to proximity effects.*

5. Transformer Equivalent Circuit

All of the above discussion results in a transformer model that incorporates both the ideal transformer and the two types of parasitic inductance's as shown on the top of page 12.



Ideal Transformer Properties

a. $R_1 = R_2 = 0$ only for superconducting wire, but will be larger than the expected DC wire resistance's for all AC currents due to both skin effects and proximity effects

b. For μ_c(core) $\rightarrow \infty$ then $\Re_c = 0 \Rightarrow L_m \rightarrow \infty$ The big magnetizing inductor draws no current from the input voltage. This means $\Rightarrow i_m$ is negligible compared to i_1. Moreover, $\Re_{l1}$ and $\Re_{l2} \rightarrow \infty \Rightarrow$ $L_l \rightarrow 0$ because leakage flux $\equiv 0$. This is not the case in practice. Consider the extreme case of the flyback "transformer", where we purposefully make L_m to be small so we can store lots of energy in the air gap.

The magnetizing inductance is solely a magnetic property of the core and is not at all effected by the load current drawn by the transformer secondaries. Its simply given by:

$$L_m = \frac{N_1^2}{\Re_c}$$

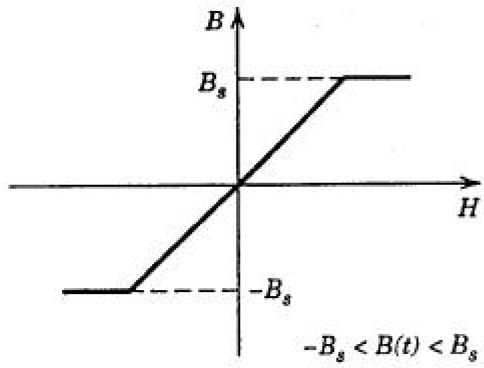
That is for operating transformers **the net mmf from the copper windings** $N_1 i_1 + N_2 i_2$ **is supposed to be small, nearly zero and there are no amp-turn constraints**.

However, the voltage placed across the coil of the transformer is subject to volt-sec constraints. That is above a critical current level, the above magnetizing inductance will change subject to the level of the induced magnetizing current. I_M levels are driven only by input volt-seconds conditions, which determine the magnetizing current under AC drive conditions.

H(core) l(core) = n i_m.

If n i_m is too high, then H exceeds H(critical) or B_{SAT} . Then the core saturates causing $\mu_r \to \mu_o$.

To determine B(max) for a transformer driven by AC signals we employ the flux linkage, $\lambda = N\phi$, and $v_L = d\lambda/dt$. This gives a volt-sec limit.



Find $\int v_L dt = \lambda = N\phi$ = NB(core)A(core).

As long as **$\int v_L dt$ < NB(saturation)A(core)** no core saturation occurs in the volt-sec limit. For a sinusoidal voltage $v_o \sin\omega t$ this means**:** **$v_o/\omega NA < B_{sat}$** Thus the core saturation parameter sets the **maximum volts per turn allowed on the windings**.

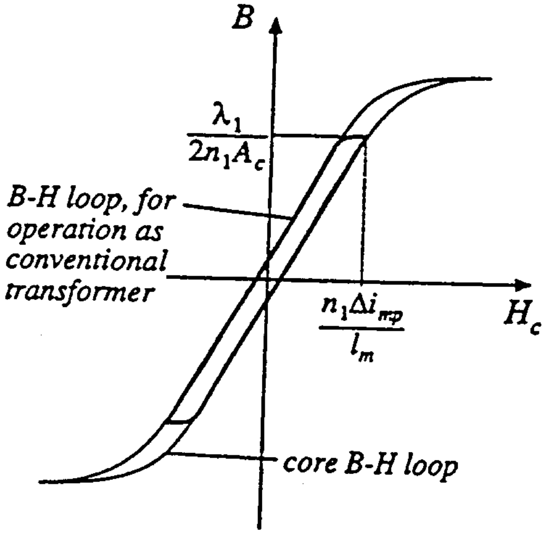
Any even small dc current in a transformer winding creates a **different route to core saturation via amp-turn limits**. The full core anti-saturation criterion then becomes $\int \frac{vdt}{NA} + \frac{Ni_{DC}}{\Re A} < B_{sat}$. The core saturation parameter is limiting the AC voltages across the primary and the DC currents in the wire coils.

$$L_m = \frac{N_1^2}{\frac{\ell_c}{\mu_c A_c}} = \frac{N_1^2}{\ell_c}\mu_c A_c \to$$

Above B_{sat} $\mu_c \to \mu_o$ causing a factor 100-1000 change in L_m. In the transformer model this shorts out the primary of the ideal transformer.

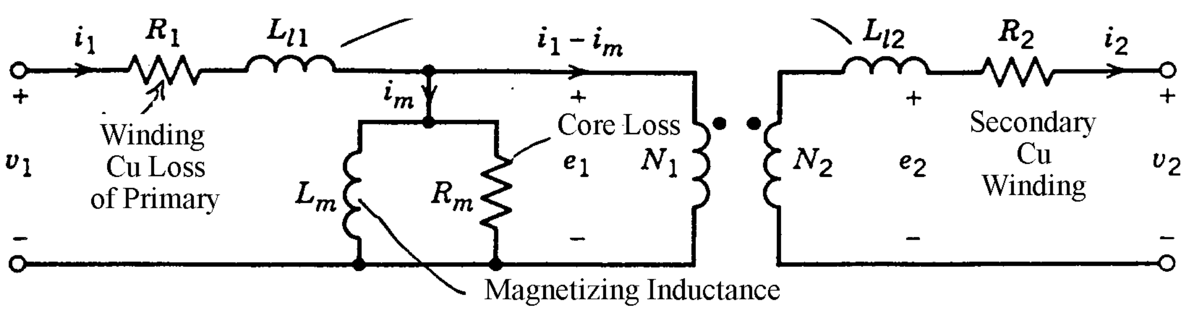
a. **Total Core Loss: R_m**

For $i_L < i(sat)$ we get lots of core loss and copper loss occurring before core saturation occurs and L= 0. Over the past twenty years the core loss due to hysteresis and eddy currents in cores has improved primarily by the use of new core materials. Low loss cores are now available up to 5 Mhz. To briefly review the origin of the hysteresis core loss we plot below the B-H curve that the transformer typically operates under.



For $i_m << i_{sat}$ we don't have a catastrophe but we still create core loss via Δi_m swings in the core. We model this combination of hysteresis and eddy current loss by an equivalent R_m in parallel | | with L_m

The equivalent transformer circuit model then has two currents i_{L_m} for reactive current in the core and i_{Rm} which models all core losses



6. How is the magnetizing current i_m created?

This is important enough to consider twice. The ideal transformer model will cause a primary current to flow that is soley related to the secondary current(s) drawn.

$$i_{primary}\text{(from a load)} \approx \frac{N_2 i_2(load)}{N_1}$$ due to transformer action. The

magnetizing inductance draws current from the input voltage also, so the total current flowing is:

$i_1 = i_{primary} + i_m$

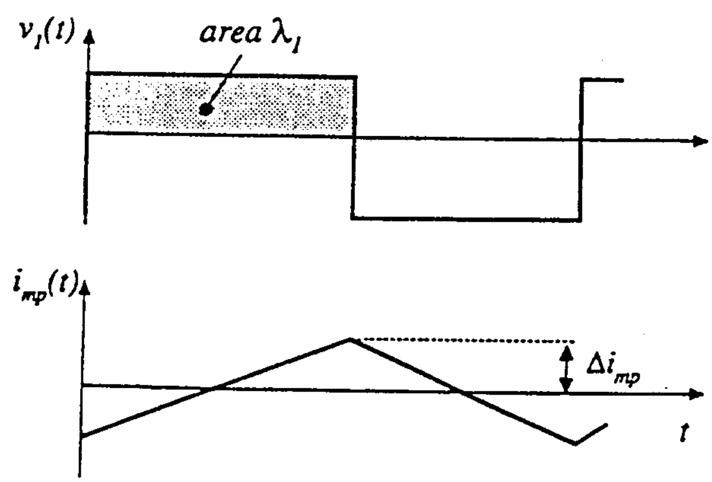
$i_1 - \frac{N_2 i_2}{N_1} \approx i_m$ Even though i_1 and i_2 are both large, i_m can still be small, even zero. The magnetizing current causes a corresponding core flux which is $f = \frac{[N_1 i_1 - N_2 i_2]}{R_c} = \frac{N_i i_m}{R_c}$.

In general, in a well designed transformer the V_1(across the primary winding) waveform alone drives i_m. If we assume R_1 and $L_{\ell 1}$ are small so that the full input voltage appears across L_M. Then we can say:

$$i_m = \frac{\int V_1 dt\ \}\ l_1}{L_m}$$

Using $L_m i_m = N_1 \phi_c$

$$B_c = \frac{l_1 = \int V_1 dt}{Z\ N_1\ A_c}$$



We are always conscious that B(core) should never exceed B(saturation), **which constrains the absolute number of copper turns possible on the primary.**

To reduce B_c the number of wire turns in the primary can be increased. This does not effect N_2/N_1 provided N_2 is changed proportionally to maintain the desired turns ratio. As $N_1 \uparrow$, then $B_c \sim \int V_1 dt$ decreases However, we do have another constraint from the wire winding area of the chosen core. That is we cannot increase N arbitrarily as we must fit the primary and other wires in the finite wire winding area. Design compromises must be made between the allowed number of turns and the core size.

In summary, neglecting the core winding area constraint, to keep

B(core) < B(sat)

$$\int v\,dt/NA + Ni_{DC}/\Re A < B(saturation).$$

7. Transformer Heating Limits

Transformer losses are limited by a maximum "hot spot" temperature at the core surface or inside the center of the wire windings. As we have shown to a first approximation, temperature rise (°C) equals core thermal resistance (°C/Watt) times total power loss (Watts). P_{total} = P(core) + P(windings)

$$\Delta T = R_C \times P_T(\text{core plus copper})$$

Ultimately, the appropriate core size for the application is the smallest core that will handle the required power with losses that are acceptable in terms of transformer temperature rise or power supply efficiency. We usually cannot exceed a core temperature of 100°. Thermal radiation and convection both allow heat to escape from the core.

$$R_q(\text{total}) = \frac{R(\text{conv})\,R(\text{rad})}{R(\text{conv})\,R(\text{rad})} = \text{Parallel Combination}$$

Typically we find in practice for cores the thermal resistance varies over a range: $1 \leq R_Q(\text{total}) \leq 10°\,C/W$

R_Q(total) depends both on core size/shape and thermal constants.

T(core) - T(ambient) = R_T (W/°C) P_T (total power loss)
Practically, T(core), is never to exceed 100°C since core μ(T) and wire insulation degrades.
Typically, T(ambient) = 40°C, R_T = 10°C/W, and P_T = Typically 2-20W

T(core) = $R_T P_T$ + 40°C = 90°C

8. Compare an Inductor versus a Transformer for the Same sinusoidal excitation: f, Max B, specific core

FOR HW #1 GO THROUGH THIS SECTION AND VERIFY THE ARGUMENTS STEP BY STEP

Inductor

1. Power rating in V-A

$$L I_{rms} I_{peak} \approx \frac{S_T}{2.2 f}$$

$S_T = 2.2f\, L I_{rms} I_{peak}$

Given L and I_{peak} & I_{rms} we can then say for a sinusoid current

$$I_{peak} = \sqrt{2}\; I_{rms}$$

The V-A rating

$S_T(\text{inductor}) = \pi f L\, I^2_{rms}$

Transformer

1. Power rating in V-A

$$S_T = V_p I_p \text{ rating}$$

For a sinusoidal excitation

$$V_p = N_p A_c \frac{d}{dt}[B_c(\max)\sin wt]$$

$$V_p = N_p A_c\, w B_c(\max)$$

$I_p = J_p\, A(\text{Cu wire})$

$$S_T = \frac{N_p A_c}{\sqrt{2}}\, w\, B_c(\max) * J\, H_c$$

$$A_{cu} = \frac{K_{cu} A_w}{2 N_p}$$

The V-A rating

$S_T(\text{transformer}) = 2.2 K_{cu} f A_c A_w J_p B(\max)$

This S_T(V-A) rating we determine the required core size.

We next relate L size to that of a transformer rated at a particular value of S_T.

$i_L \uparrow B_{max}(L) \uparrow$	i_p and i_{sec} do not affect i_m
for an inductor only!	for a transformer.
	$i_m = \dfrac{\int V dt}{L}$
$B_{max} < B(sat)$	$i_m(\max) < i_m(\text{max for saturation})$

For both inductors and transformers $\int v dt / N(Cu) A(core) < B(sat)$

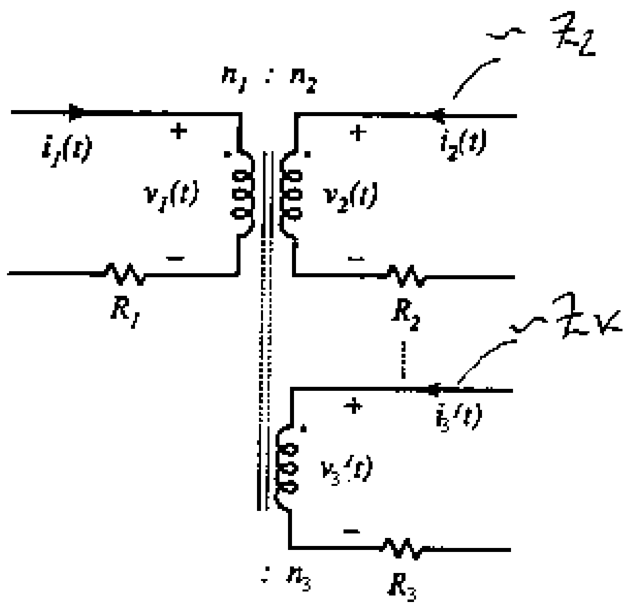
dc: $V_{dc}\Delta t < B_{sat} N(Cu) A(core)$

ac: $V(peak)/N < B_{sat} w A(core)$

B. Copper Winding Loss in Transformers

1. Overview

Consider a three winding transformer wound on a common magnetic core. There are two secondaries with N_2 and N_3 turns. There is one primary with N_1 turns



Voltages scale as turns ratios

$$\frac{V_1}{n_1} = \frac{V_2}{n_2} = \frac{V_3}{n_3}$$

Secondary currents, however, scale only as loads on each set of secondary turns.

As an aside, if we assume the magnetic core $\mu \to \infty$, then $N_1 i_1 + N_2 i_2 + N_3 i_3 = 0$ for a three winding transformer, giving rise to the dot convention for transformer currents in the various coils. Primary current due to two secondary load currents follow the dot convention where i flow into the dot is positive n_i.

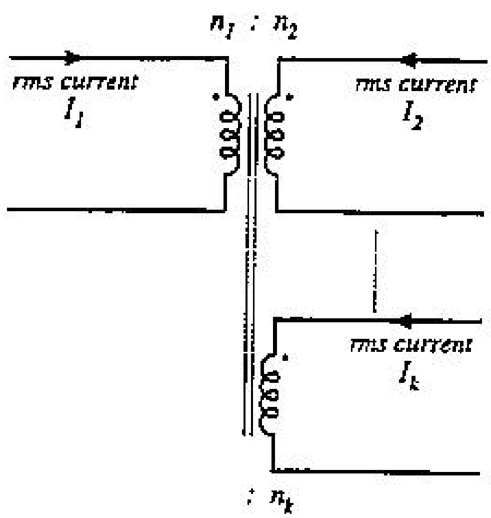
$$n_p i_p + n_{s1} i_{s1} + n_{s2} i_{s2} = 0$$

primary secondaries

(1) all currents are into the dotted side of the transformer.
(2) All currents are RMS
(3) Currents are not related by wiring turns ratios.

2. Optimum Area for Primary and Secondary Windings

It would appear at first blush that the two secondary windings operate independent of each other and of the primary. However, all windings must be wrapped around the same core in the one available wire winding window. Each winding has a resistance, which could be minimized by using the biggest diameter wire. But this increases the resistance of the other windings. The resistance of the j'th winding will be proportional to N^2 (for the j'th winding) rather than N because the optimum area of the wire in the j'th winding will go as $1`/N_J$ as we will show below.

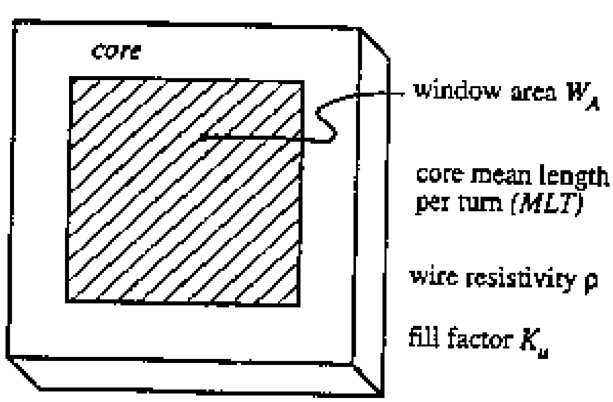


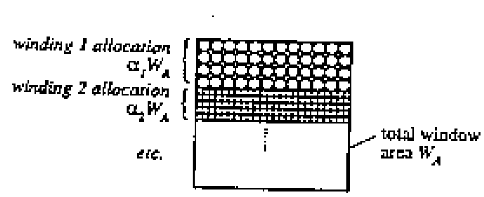
Each winding takes a fraction of the available window winding area of the core.

$$a_k = \frac{A_k}{W_A} \text{ for } K^{th} \text{ winding}$$

A_K is the winding area for the K'th winding only.

W_A is the full core window area for all windings as shown below.





The total resistance of the K^{th} winding which employs wire of length K and area A_k. Clearly, the length of the wire winding depends on the core it is wound around but also the number of prior wire turns that were wound the core before this winding was started. For a single turn in the k'th winding we can say:

$$R_k = r \frac{l_k}{A_{wk}} \text{ for } K^{th} \text{ winding}$$

What about N_k turns in the k'th winding? Will the total wire resistance of this winding vary as ~ N_k or N_k^2? Can you guess why one rather than the other? There is a hidden N_K variable here as the choice of A_{wk} for the wire in the k'th winding was set by its fractional area of the total winding area or the parameter α_k. That is $A_{wk} = W_A K_{cu}\alpha_k / N_k$. Later we will calculate the optimum values of α for each winding. For now **just realize that A_{wk}~1/ N_k . The length of wire is then:**

$$l_k \equiv N_k * (MLT)_k$$

Where MLT = mean length per turn and N_K = total # of turns in K^{th} winding

$$A_{wk}\begin{pmatrix}\text{area of wire}\\ \text{employed}\\ \text{in k winding}\end{pmatrix}\text{- from AWG \# gauge tables}$$

The total area of copper in the K^{th} winding is given by # turns times area per each wire. $A_{wk} N_k = W_A a_k K_u$ (wire fill factor)

As always K_u or K_{Cu} depends on wire type chosen for windings: K_u = 0.3 for Litz wire and K_u = 0.9 for foil. This varies R_K by a factor three. The real surprise however is in the dependence on the number of turns N_K.

$$R_k = r \frac{N_k^2 (MLT)_k}{W_A K_u a_k}$$

Note: $R_k \sim N_k^2$ not N_k.

Copper loss (not accounting for proximity loss) is

$$P_{cu,j} = I_j^2 R_j$$

Resistance of winding j is

$$R_j = \rho \frac{l_j}{A_{W,j}}$$

with

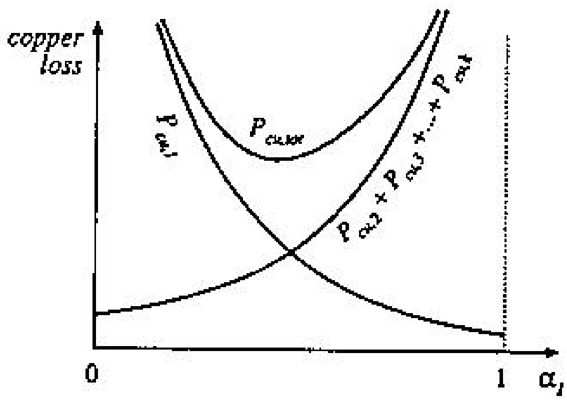
$$l_j = n_j (MLT) \quad \text{length of wire, winding } j$$

$$A_{W,j} = \frac{W_A K_u \alpha_j}{n_j} \quad \text{wire area, winding } j$$

Hence

$$P_{cu,j} = \frac{n_j^2 i_j^2 \rho (MLT)}{W_A K_u \alpha_j}$$

Is there an optimum area for each winding to minimize the total wire losses for all windings? We will try to answer this. For now we recap our astonishing conclusion that in a multi-winding transformer, the P(copper) for each winding, K, varies as $\sim N^2$(for the k'th winding).



Intuitively there is an optimum α_k to achieve $P_{total}(I_x^2 R_x)$ a minimum

$$a_k = \frac{V_k I_k}{\sum_{l=1}^{\infty} V_j I_j P_T} = \frac{P_K}{P_{total}}$$

For $\alpha_1 = 0$ the wire of winding one has zero area allotted and P_1 tends to infinity, whereas for $\alpha_1 = 1$ wires of the remaining windings have zero area allotted and their copper losses go to infinity. Clearly, there is an optimum choice of α_1 that minimizes the total copper losses.
As a guide we expect the required area to be proportional to the power

$$a_k = \frac{V_k I_k \,(\text{Power in } K^{th} \text{ winding})}{\text{total power in all windings}}$$

V_k varies as the turns ratio from primary to K^{th} secondary
I_k varies as the k'th load

Later we will deal with harmonics of waveforms and power factor considerations.

Next we show only the flow of the calculations required to find the optimum relative wire winding area factors, α_K. The calculation will involve the method of Lagrange multipliers and is only outlined on page 22-23.

Sum previous expression over all windings:

$$P_{cu,tot} = P_{cu,1} + P_{cu,2} + \cdots + P_{cu,k} = \frac{\rho\,(MLT)}{W_A K_u} \sum_{j=1}^{k} \left(\frac{n_j^2 I_j^2}{\alpha_j} \right)$$

Need to select values for $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the total copper loss is minimized

We next minimize the total power with respect to the α parameters.

Minimize the function

$$P_{cu,tot} = P_{cu,1} + P_{cu,2} + \cdots + P_{cu,k} = \frac{\rho\,(MLT)}{W_A K_u} \sum_{j=1}^{k} \left(\frac{n_j^2 I_j^2}{\alpha_j} \right)$$

subject to the constraint

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$$

Define the function

$$f(\alpha_1, \alpha_2, \cdots, \alpha_k, \xi) = P_{cu,tot}(\alpha_1, \alpha_2, \cdots, \alpha_k) + \xi\, g(\alpha_1, \alpha_2, \cdots, \alpha_k)$$

where

$$g(\alpha_1, \alpha_2, \cdots, \alpha_k) = 1 - \sum_{j=1}^{k} \alpha_j$$

is the constraint that must equal zero

and ξ is the Lagrange multiplier

Optimum point is solution of the system of equations

$$\frac{\partial f(\alpha_1, \alpha_2, \cdots, \alpha_k, \xi)}{\partial \alpha_1} = 0$$

$$\frac{\partial f(\alpha_1, \alpha_2, \cdots, \alpha_k, \xi)}{\partial \alpha_2} = 0$$

$$\vdots$$

$$\frac{\partial f(\alpha_1, \alpha_2, \cdots, \alpha_k, \xi)}{\partial \alpha_k} = 0$$

$$\frac{\partial f(\alpha_1, \alpha_2, \cdots, \alpha_k, \xi)}{\partial \xi} = 0$$

Result:

$$\xi = \frac{\rho\,(MLT)}{W_A K_u} \left(\sum_{j=1}^{k} n_j I_j \right)^2 = P_{cu,tot}$$

$$\alpha_m = \frac{n_m I_m}{\sum_{n=1}^{\infty} n_j I_j}$$

An alternate form:

$$\alpha_m = \frac{V_m I_m}{\sum_{n=1}^{\infty} V_j I_j}$$

We then have to interpret the mathematics in terms of the windings.

Interpretation of result

$$\alpha_m = \frac{V_m I_m}{\sum_{n=1}^{\infty} V_j I_j}$$

Apparent power in winding j is

$$V_j I_j$$

where V_j is the rms or peak applied voltage

I_j is the rms current

Window area should be allocated according to the apparent powers of the windings