Digital Image Processing
Lectures 5 & 6

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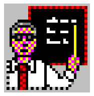
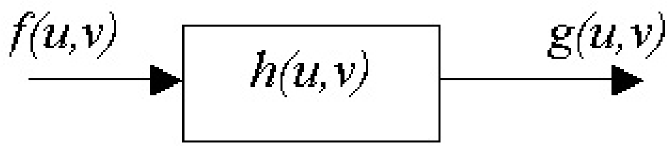
wherever it appears to load the next sound file 

1. Parseval's Theorem and Inner Product Preservation
   Another important property of FT is that the inner product of two functions is equal to the inner product of their FT's.
$$\int\int_{-\infty}^{\infty} x(v,u)y^*(u,v)dudv$$
$$= \frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} X(\omega_1,\omega_2)Y^*(\omega_1,\omega_2)d\omega_1 d\omega_2$$
   where $*$ stands for complex conjugate operation. When $x = y$ we obtain the well-known Parseval energy conservation formula i.e.
$$\int\int_{-\infty}^{\infty} |x(u,v)|^2 dudv = \frac{1}{4\pi^2}\int\int_{-\infty}^{\infty} |X(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2$$
   i.e. the total energy in the function is the same as in its FT.
2. Frequency Response and Eigenfunctions of 2-D LSI Systems
   An eigen-function of a system is defined as an input function

that is reproduced at the output with a possible change in the amplitude. For an LSI system eigen-functions are given by

$$f(u,v) = \exp(j\omega_1 u + j\omega_2 v)$$



Using the 2-D convolution integral

$$g(u,v) = \int\int_{-\infty}^{\infty} h(u-u', v-v') \exp[j\omega_1 u' + j\omega_2 v'] du' dv'$$

change $\tilde{u} = u - u', \tilde{v} = v - v'$, then

$$g(u,v) = H(\omega_1, \omega_2) \exp[j\omega_1 u + j\omega_2 v]$$

where

$$\begin{aligned} H(\omega_1, \omega_2) &= \int\int_{-\infty}^{\infty} h(\tilde{u}, \tilde{v}) \exp[j\omega_1 \tilde{u} + j\omega_2 \tilde{v}] d\tilde{u} d\tilde{v} \\ &= \mathcal{F}\{h(u,v)\} \end{aligned}$$

is the frequency response of the 2-D system. The output is a complex exponential function with the same frequency as the input signal but its amplitude and phase are changed by the complex gain $H(\omega_1, \omega_2)$ of the 2-D system.

Example:

Determine the frequency response of a 2-D system whose impulse response is

$$h(u, v) = \begin{cases} 1 & |u| \leq X, |v| \leq Y \\ 0 & \text{elsewhere} \end{cases}$$

Find $H(\omega_1, \omega_2)$.

$$H(\omega_1, \omega_2) \;=\; \int\int_{-\infty}^{\infty} h(u', v') e^{-j(\omega_1 u' + \omega_2 v')} du' dv'$$

$$
\begin{aligned}
&= \int_{-X}^{X}\int_{-Y}^{Y} e^{-j(\omega_1 u' + \omega_2 v')} du' dv' \\
&= \left(\int_{-X}^{X} e^{-j\omega_1 u'} du'\right)\left(\int_{-Y}^{Y} e^{-j\omega_2 v'} dv'\right) \\
&= (\frac{e^{j\omega_1 X} - e^{-j\omega_1 X}}{j\omega_1})(\frac{e^{j\omega_2 Y} - e^{-j\omega_2 Y}}{j\omega_2}) \\
&= 2X(\frac{\sin \omega_1 X}{\omega_1 X})2Y(\frac{\sin \omega_2 Y}{\omega_2 Y}) \\
&= 4XY \text{sinc}(\omega_1 X)\text{sinc}(\omega_2 Y)
\end{aligned}
$$

See Figure 4.3 in your textbook for the plot of 2D Sinc function. Any interesting observation?

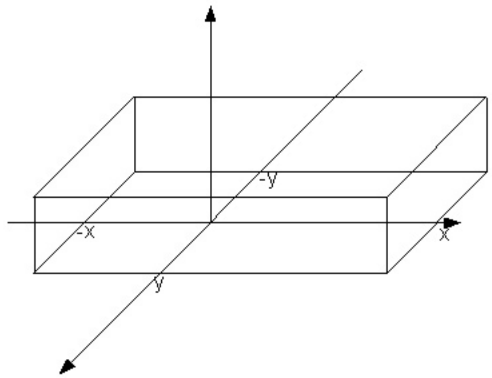
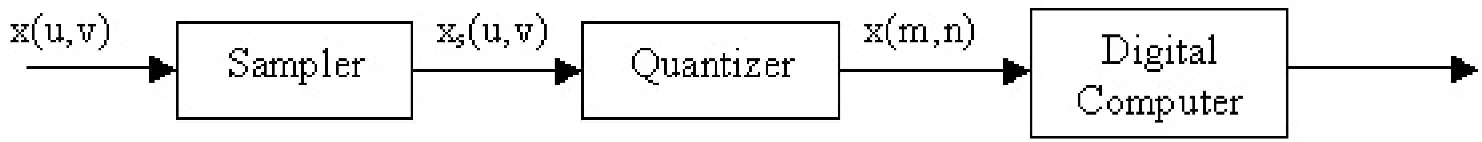


Image Sampling and Quantization

The most basic requirement for computer processing of images is that the images must be available in digital form i.e. arrays of integer numbers. For digitization the given image is sampled on a discrete grid and each sample or pixel is quantized to an integer value representing a gray level. The digitized image can then be processed by the computer.



2-D Sampling Theorem

Definition: An image $x(u, v)$ is called "bandlimited" if its FT $X(\omega_1, \omega_2)$ is zero outside a bounded region in the frequency plane i.e.

$$X(\omega_1, \omega_2) = 0 \quad |\omega_1| > \omega_{10}, |\omega_2| > \omega_{20}$$

ω_{10}, ω_{20}: Bandlimits of the image. If the spectrum is circularly symmetric then the single spatial frequency $[\omega_0 \stackrel{\triangle}{=} \omega_{10} = \omega_{20}]$ is the bandwidth.

Sampling Vs. Replication

The FT of a sampled image is a scaled, periodic replication of the FT of the original image. To show this result, we sample $x(u,v)$ using and ideal image sampling function, $s_a(u,v)$ that is defined as

$$s_a(u,v;\Delta u,\Delta v) \triangleq \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(u-m\Delta u, v-n\Delta v)$$

with the "sampling intervals" $\Delta u, \Delta v$. The sampled image is

$$\begin{aligned} x_s(u,v) &= x(u,v)s_a(u,v;\Delta u,\Delta v) \\ &= \sum\sum_{m,n=-\infty}^{\infty} x(m\Delta u, n\Delta v)\delta(u-m\Delta u, v-n\Delta v) \end{aligned}$$

It can be shown that the FT of the sampling function with spacing

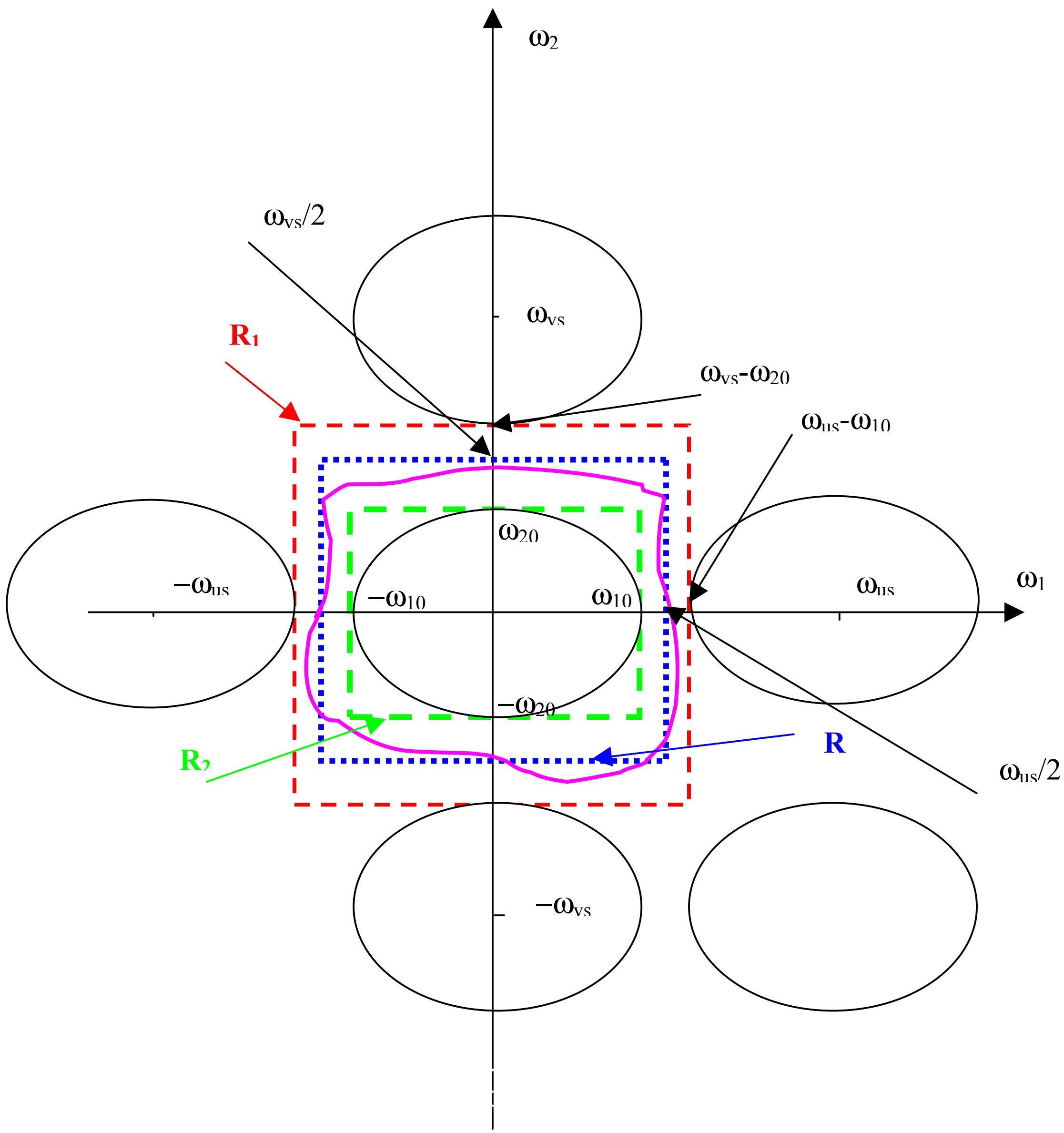
$\Delta u, \Delta v$ is another sampling function with spacing $\frac{2\pi}{\Delta u}, \frac{2\pi}{\Delta v}$. Then, using the convolution in frequency domain we get

$$X_s(\omega_1, \omega_2) = \frac{1}{\Delta u \Delta v} \sum \sum_{k,l=-\infty}^{\infty} X(\omega_1 - \frac{2\pi k}{\Delta u}, \omega_2 - \frac{2\pi l}{\Delta v})$$

Let us define discrete frequency variables (i.e. in discrete domain) $\Omega_1 \triangleq \omega_1 \Delta u, \Omega_2 \triangleq \omega_2 \Delta v$, then we get 2-D Discrete Space Fourier Transform (DSFT),

$$X_s(\Omega_1, \Omega_2) = \frac{1}{\Delta u \Delta v} \sum \sum_{k,l=-\infty}^{\infty} X(\frac{\Omega_1 - 2\pi k}{\Delta u}, \frac{\Omega_2 - 2\pi l}{\Delta v})$$

This result shows sampling in the spatial domain causes periodicity in the frequency domain (See Figure).



ω2
ωvs/2
ωvs
R1
ωvs-ω20
ωus-ω10
ω20
−ωus
−ω10
ω10
ωus
ω1
−ω20
R2
R
ωus/2
−ωvs

Let $\omega_{us} \triangleq \frac{2\pi}{\Delta u}$ and $\omega_{vs} \triangleq \frac{2\pi}{\Delta v}$ (i.e. sampling frequencies), then alternatively

$$X_s(\omega_1, \omega_2) = \frac{1}{4\pi^2}\omega_{us}\omega_{vs} \sum \sum_{k,l=-\infty}^{\infty} X(\omega_1 - k\omega_{us}, \omega_2 - l\omega_{vs})$$

As can be seen from the Figure, for no overlapping between the replica the lower bounds on the sampling frequencies are $2\omega_{10}$ and $2\omega_{20}$. These frequencies are referred to as "Nyquist frequencies". Their reciprocals are called "Nyquist intervals". *The sampling frequency must be equal to or greater than twice the frequency associated with the finest detail in the image (edges).*

Reconstruction

If the sampling frequencies ω_{us} and ω_{vs} are greater than twice of the bandlimits i.e. $\omega_{us} \geq 2\omega_{10}, \omega_{vs} \geq 2\omega_{20}$ then $X(\omega_1, \omega_2)$ can be recovered from $X_s(\omega_1, \omega_2)$ by a 2-D ideal low-pass filter (interpolation filter) with frequency response

$$H(\omega_1, \omega_2) = \begin{cases} \Delta u \Delta v & \omega_1, \omega_2 \in R \\ 0 & \text{otherwise} \end{cases}$$

where R is any region whose boundary ∂R is contained within the region between R_1 and R_2 as shown in the Figure. Then

$$\tilde{X}(\omega_1, \omega_2) = H(\omega_1, \omega_2) X_s(\omega_1, \omega_2) = X(\omega_1, \omega_2)$$

i.e. exact reconstruction.

A typical choice for R is a rectangular region, e.g. $R = \left[-\frac{\omega_{us}}{2}, \frac{\omega_{us}}{2}\right] \times \left[-\frac{\omega_{vs}}{2}, \frac{\omega_{vs}}{2}\right]$ for which the impulse response is the ideal 2-D interpolation function i.e.

$$h(u,v) = \text{Sinc}(u\omega_{us})\text{sinc}(v\omega_{vs})$$

That is, $X(\omega_1,\omega_2) = H(\omega_1,\omega_2) \cdot X_s(\omega_1,\omega_2)$, gives

$$\begin{aligned}\tilde{x}(u,v) &= h(u,v) * * x_s(u,v) \\ &= \sum\sum_{m,n=-\infty}^{\infty} x(m\Delta u, n\Delta v)\text{sinc}(\omega_{us}(u - m\Delta u)) \\ &\quad \text{sinc}(\omega_{vs}(v - n\Delta v))\end{aligned}$$

Remarks

1. The above equation is an infinite order interpolation that reconstructs the continuous function $x(u, v)$ from its samples $x(m\Delta u, n\Delta v)$.
2. For a rectangular passband with size $W_1 \times W_2$, the impulse response of the interpolation filter is $h(u, v) = \mathrm{sinc}(uW_1)\mathrm{sinc}(vW_2)$.

Undersampling and Aliasing Effects

If the sampling frequencies are below the Nyquist rates i.e. $\omega_{us} < 2\omega_{10}$ and $\omega_{vs} < 2\omega_{20}$, then the periodic replications of $X(\omega_1, \omega_2)$ will overlap, resulting in a distorted spectrum $X_s(\omega_1, \omega_2)$ from which $X(\omega_1, \omega_2)$ cannot be recovered. The frequencies above half the sampling frequencies, that is, above $\frac{\omega_{us}}{2}, \frac{\omega_{vs}}{2}$ are called the "fold-over frequencies". This overlapping causes some of the high frequencies (or fold-over frequencies) in the original image to appear as low frequencies (below $\frac{\omega_{us}}{2}, \frac{\omega_{vs}}{2}$) in the sampled image. This phenomenon is called "Aliasing". Aliasing cannot be removed by post filtering but can be avoided by pre low pass filtering the image so that its bandlimits are less than one-half of the sampling frequencies.

In images aliasing causes edge smearing and loss of details. The spectrum of an undersampled image can be written as

$$X_s(\omega_1, \omega_2) = \frac{1}{\Delta u \Delta v}[X(\omega_1, \omega_2) + E(\omega_1, \omega_2)]$$

where

$$E(\omega_1, \omega_2) = \sum \sum_{\substack{k,l=-\infty \\ (k,l)\neq(0,0)}}^{\infty} X(\omega_1 - k\omega_{us}, \omega_2 - l\omega_{vs})$$

After the filtering with a 2-D LPF

$$H(\omega_1, \omega_2) = \begin{cases} \Delta u \Delta v & |\omega_1| \leq \frac{\omega_{us}}{2}, |\omega_2| \leq \frac{\omega_{vs}}{2} \\ 0 & \text{otherwise} \end{cases}$$

we get $\tilde{X}(\omega_1, \omega_2) = H(\omega_1, \omega_2) X_s(\omega_1, \omega_2)$. In the spatial domain, we get

$$\tilde{x}(u, v) = x(u, v) + e(u, v)$$

where

$$e(u,v) = \frac{1}{4\pi^2} \int_{-\frac{\omega_{us}}{2}}^{\frac{\omega_{us}}{2}} \int_{-\frac{\omega_{vs}}{2}}^{\frac{\omega_{vs}}{2}} E(\omega_1, \omega_2) \exp[j(\omega_1 u + \omega_2 v)] d\omega_1 d\omega_2$$

represents the aliasing error artifact in the reconstructed image after the filtering.

Example: An image described by the function

$$x(u,v) = 2\cos(3u + 4v)$$

is sampled at $\Delta u = \Delta v = 0.4\pi$. Find the reconstructed image $\tilde{x}(u,v)$.

Expand

$$cos(3u+4v) = \{e^{j(3u+4v)} + e^{-j(3u+4v)}\}/2$$

and use the property $\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0)$, to find the 2D spectrum of $x(u,v)$ as

$$X(\omega_1,\omega_2) = (2\pi)^2[\delta(\omega_1-3,\omega_2-4)+\delta(\omega_1+3,\omega_2+4)]$$

which is bandlimited since $X(\omega_1,\omega_2) = 0$ for $|\omega_1| > 3$,$|\omega_2| > 4$ Thus $\omega_{10} = 3$ and $\omega_{20} = 4$. Also $\omega_{us} = \frac{2\pi}{\Delta u} = 5$ and $\omega_{vs} = \frac{2\pi}{\Delta v} = 5$ which are less than the Nyquist frequencies $2\omega_{10} = 6$ and $2\omega_{20} = 8$. Thus aliasing is inevitable. The spectrum of the sampled image is

$$X_s(\omega_1,\omega_2) \;=\; \frac{1}{\Delta u \Delta v}\sum\sum_{k,l=-\infty}^{\infty} X(\omega_1 - \frac{2\pi}{\Delta u}k, \omega_2 - \frac{2\pi}{\Delta v}l)$$

$$= \ 25 \sum \sum_{k,l=-\infty}^{\infty} [\delta(\omega_1 - 3 - 5k, \omega_2 - 4 - 5l) + \delta(\omega_1 + 3 - 5k, \omega_2 + 4 - 5l)]$$

The LPF has a rectangular passband with cutoff frequencies at half the Nyquist frequencies i.e.

$$H(\omega_1, \omega_2) = \begin{cases} (0 \cdot 4\pi)^2 & |\omega_1| \leq 2.5, |\omega_2| \leq 2.5 \\ 0 & \text{otherwise} \end{cases}$$

After filtering we obtain

$$\tilde{X}(\omega_1, \omega_2) = (2\pi)^2[\delta(\omega_1 - 2, \omega_2 - 1) + \delta(\omega_1 + 2, \omega_2 + 1)]$$

which gives

$$\tilde{x}(u,v) = 2\cos(2u + v)$$

Remarks

1. Any frequency component in the original image which is above $[\frac{\omega_{us}}{2}, \frac{\omega_{vs}}{2}]$ by $(\Delta\omega_u, \Delta\omega_v)$ is reproduced (or aliased) as a lower frequency component at $[\frac{\omega_{us}}{2} - \Delta\omega_u, \frac{\omega_{vs}}{2} - \Delta\omega_v]$. In the previous example, the frequency components 3,4 are above $\frac{\omega_{us}}{2} = 2.5$ and $\frac{\omega_{vs}}{2} = 2.5$ by $\Delta\omega_u = 0.5$ and $\Delta\omega_v = 1.5$, respectively. Thus, these frequencies will be aliased at $2.5 - 0.5$ and $2.5 - 1.5$, which give $\tilde{x}(u,v) = 2\cos(2u + v)$.
2. Images corrupted by additive wideband noise have spectra with long tails. Thus, sampling based upon the bandlimits of the

original image will result in aliasing effects as the tail of the spectrum will fold-over into the bandlimits of the image. This obviously causes additional noise in the reconstructed image. To prevent this problem the image must be prefiltered prior to sampling.