Digital Controls & Digital Filters
Lectures 21 & 22

M.R. Azimi, Professor

Department of Electrical and Computer Engineering
Colorado State University

Spring 2015
Types of Analog Filters:

1. Butterworth
2. Chebyshev
3. Elliptic (Cauer)
4. Bessel

Narrower Transition: (1) Elliptic, (2) Chebyshev, (3) Butterworth, (4) Bessel

More linear phase over PB: (1) Bessel, (2) Butterworth, (3) Chebyshev, (4) Elliptic

The first three filters have a common form of magnitude-squared function:

\[ A_N(\tilde{\omega}^2) = \frac{1}{1+F(\tilde{\omega}^2)} \]

\[ \tilde{\omega} = \frac{\omega}{\omega_c} \]: Normalized frequency (\( \omega_c \): Cutoff frequency)

1. Low-pass Butterworth Filters
   - Maximally flat passband region
   - All-pole filter
   - For sharp cutoff, order must be large
Butterworth Filters

Here $F(\tilde{\omega}^2) = \tilde{\omega}^{2n}$ and $n$: order

or $A_N(\omega^2) = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$

$\omega = 0 \implies A_N(0) = 1$
$\omega = \omega_c \implies A_N(\omega_c^2) = \frac{1}{2}$
$\omega$ large $\implies A_N(\omega^2) \approx 0$

Changing $j\omega \to s$ in $A_N(\omega^2)$ gives,

$A_N(-s^2) = \frac{1}{1 + (\frac{s}{j\omega_c})^{2n}}$

Now, to find pole locations of this filter

$1 + (\frac{s}{j\omega_c})^{2n} = 0 \implies s_k = (-1)^{1/2n} j\omega_c$

But, using $(-1)^{1/N} = e^{j(2k-1)\pi/N}$, $k \in [0, N - 1]$, we get

$s_k = e^{j(2k-1)\pi/2n} j\omega_c = \{\sin \frac{(2k-1)\pi}{2n} + j \cos \frac{(2k-1)\pi}{2n}\} j\omega_c$, $k \in [0, 2n - 1]$

There are $n$ LH poles and $n$ RH poles. The LH poles are the ones chosen according to the spectral factorization (or use tables).
Butterworth Filters—Cont.

Disadvantages:

1. Wide transition region (for low order filters).
2. Accuracy of approximation cannot be uniformly distributed in PB and SB regions.

Example:
Design a Butterworth LPF, $H_A(s)$, that has attenuation of at least 10dB at $\omega = 2\omega_c$ where $\omega_c = 2.5 \times 10^3$

\[
A_N(\tilde{\omega}^2) = \frac{1}{1 + \tilde{\omega}^{2n}}
\]

\[
A_N(2^2) = \frac{1}{1 + 2^{2n}}
\]

\[-10 \log_{10} |A_N(2^2)| \geq 10 \implies (1 + 2^{2n}) \geq 10 \implies n = 1.58 \implies n = 2
\]

From table: $H_N(s) = \frac{1}{s^2 + 1.414s + 1}$

Denormalize, $s \rightarrow s/\omega_c$

$H_A(s) = H_N\left(\frac{s}{\omega_c}\right) = \frac{1}{1.6 \times 10^{-7}s^2 + 5.657 \times 10^{-4}s + 1}$
Short-Cut Method:

The required filter order for providing a magnitude of $1/A$ (not in dB) at frequency $\omega$ can be obtained using,

$$n = \frac{\log_{10}(A^2 - 1)}{2 \log_{10}\left(\frac{\omega}{\omega_c}\right)}$$

For previous example,

$$-20 \log_{10} \frac{1}{A} = 10 \text{ dB} \implies A = 3.3$$

$$n = \frac{\log_{10}(3.3^2 - 1)}{2 \log_{10} 2} = 1.653 \implies n = 2$$
2. Lowpass Chebyshev Filters

- Sharper cutoff or narrower transition region.
- Magnitude of error is equiripple over the passband (Type I) or over the stopband (Type II).

**Type I Chebyshev:** Magnitude-squared function is given by

\[ A_N(\tilde{\omega}^2) = |H_N(j\tilde{\omega})|^2 = \frac{1}{1 + F(\tilde{\omega}^2)} \]

where \( F(\tilde{\omega}^2) = \epsilon^2 C_n^2(\tilde{\omega}) \)

\( \epsilon \): Ripple factor (determines ripple in passband)

\( C_n(\tilde{\omega}) \): \( n^{th} \) order Chebyshev polynomial

\( C_n(\tilde{\omega}) = \cos \left( n \cos^{-1} \tilde{\omega} \right) \) for \( 0 \leq \tilde{\omega} \leq 1 \) (passband)

\( C_n(\tilde{\omega}) = \cosh \left( n \cosh^{-1} \tilde{\omega} \right) \) for \( \tilde{\omega} > 1 \) (stopband)

For any \( n \), \( 0 \leq |C_N(\tilde{\omega})| \leq 1 \) for \( 0 \leq \tilde{\omega} \leq 1 \)

\[ |C_N(\tilde{\omega})| > 1 \] for \( \tilde{\omega} > 1 \)

\( n = 1 \implies C_1(\tilde{\omega}) = \tilde{\omega} \)

\( n = 2 \implies C_2(\tilde{\omega}) = 2\tilde{\omega}^2 - 1 \)

\( n = 3 \implies C_3(\tilde{\omega}) = 4\tilde{\omega}^3 - 3\tilde{\omega} \)

\[ \vdots \]

i.e. \( C_n(\tilde{\omega}) \) is an odd/even polynomial when \( n \) is odd/even.
Chebyshev Filters-Cont.

Chebyshev polynomials can be generated recursively using,

\[ C_{n+1}(\tilde{\omega}) + C_{n-1}(\tilde{\omega}) = 2\tilde{\omega}C_n(\tilde{\omega}) \]

For \( n \): even, \( C_n^2(0) = 1 \implies A(0) = \frac{1}{1+\epsilon^2} \)

For \( n \): odd, \( C_n^2(0) = 0 \implies A(0) = 1 \)

\[
\text{max } |H_N(j\omega)| = A_{max} = 1: \text{ Max gain in passband}
\]
\[
\text{min } |H_N(j\omega)| = A_{min} = \frac{1}{\sqrt{1+\epsilon^2}}: \text{ Min gain in passband}
\]

Define:
\[
r_{dB} = 20 \log_{10} \frac{A_{max}}{A_{min}} = 10 \log_{10} (1 + \epsilon^2) : \text{ Loss in passband}
\]
\[
r(\text{peak-to-peak}) = A_{max} - A_{min} = 1 - \frac{1}{\sqrt{1+\epsilon^2}}
\]
To find pole locations of the Chebyshev filter, change $\omega = s / j$ and set the denominator to zero,

$$1 + \epsilon^2 C_n^2(s/j) = 0 \implies \cos (n \cos^{-1}(s/j)) = \pm \frac{j}{\epsilon}$$

But, using $s_k = \sigma_k + j\omega_k$, we get

$$\sigma_k = -\sinh \varphi \sin \left[ \frac{(2k-1)\pi}{2n} \right] \quad k \in [0, 2n - 1]$$

$$\omega_k = \cosh \varphi \cos \left[ \frac{(2k-1)\pi}{2n} \right] \quad k \in [0, 2n - 1]$$

where

$$\sinh \varphi = \frac{\gamma - \gamma^{-1}}{2}, \quad \cosh \varphi = \frac{\gamma + \gamma^{-1}}{2}, \quad \text{and}$$

$$\gamma = \left( \frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon} \right)^{1/n}.$$ 

It is interesting to note that from equations for $\sigma_k$ and $\omega_k$, we have,

$$\frac{\sigma_k^2}{\sinh^2 \varphi} + \frac{\omega_k^2}{\cosh^2 \varphi} = 1$$

i.e. the poles are located on an ellipse in the $s$-plane.
Chebyshev Filters-Cont.

Example:

Design a Chebyshev Type I filter with 1dB ripple in passband and attenuation of at least 20dB at \( \omega = 2\omega_c, f_c = 3kHz \).

\[
\begin{align*}
    r &= 10 \log_{10} (1 + \epsilon^2) = 1 \implies \epsilon = 0.05088 \\
    -10 \log_{10} \frac{1}{1 + \epsilon^2 C_n^2(2)} &\geq 20
\end{align*}
\]

Since in stopband, \( \epsilon^2 C_n^2(2) \gg 1 \)

Thus,

\[
10 \log_{10} \epsilon^2 + 10 \log_{10} C_n^2(2) \geq 20
\]

\[
\log_{10} [\cosh(n \cosh^{-1} 2)] \geq 1 - \log_{10} \epsilon = 1.2935 \implies n \cosh^{-1} 2 \geq 3.67
\]

or \( n \geq 2.8 \implies n = 3 \)

Use Table 11.4 (for \( r = 1dB \))

\[
H_N(s) = \frac{0.4913}{s^3 + 0.9883s^2 + 1.2384s + 0.4913}
\]

Denormalize: \( \omega_c = 2\pi \times 3 \times 10^3 = 6\pi \times 10^3 \text{Rad/Sec} \)

\[
s \rightarrow \frac{s}{\omega_c}
\]

\[
H_A(s) = \frac{0.4913}{1.493 \times 10^{-13}s^3 + 2.78 \times 10^{-9}s^2 + 6.569 \times 10^{-5}s + 0.4913}
\]
Review of Analog Filters-Cont.

Short-Cut Method:
If the magnitude of minimum stopband loss is $1/A$ at frequency $\omega_r$, then the order can be found using

$$n = \frac{\log_{10}(g + \sqrt{g^2 - 1})}{\log_{10}(\omega_r + \sqrt{\omega_r^2 - 1})}$$

where $g = \sqrt{\frac{A^2 - 1}{\epsilon^2}}$

3. Elliptic Filters:
Characteristics are:

1. Equiripple both in passband and stopband.
2. Excellent transition region.

The magnitude-squared function for elliptic filters is:

$$A(\tilde{\omega}^2) = \frac{1}{1 + \epsilon^2 R_n^2(\tilde{\omega}, \kappa_1)}$$

$R_n(\tilde{\omega}, \kappa_1)$ : Chebyshev rational function with roots that are related to Jacobi elliptic functions hence called elliptic filters.

$\kappa_1 \triangleq \epsilon / \sqrt{A^2 - 1}$: selectivity factor
Elliptic Filters

Define transition ratio $p = \frac{\omega_c}{\omega_r}$. Then the order of the elliptic filter to meet $\epsilon, A$ (as defined before) and $\omega_c, \omega_r$ specs is:

$$n = \frac{K(p)K(\sqrt{1-\kappa_1^2})}{K(\kappa_1)K(\sqrt{1-p^2})}$$

$K(.)$: Complete elliptic integral of the 1st kind,

$$K(x) = \int_0^1 \frac{1}{(1-t^2)^{1/2}(1-xt^2)^{1/2}} \, dt$$

Remarks:

1. In this filter, $\epsilon$ (ripple factor) determines the passband ripple, while the combination of $\epsilon$ and $\kappa_1$ specify the stopband ripple.
2. To obtain equiripple in both passband and stopband elliptic filters must have poles on $j\omega$ axis.
3. When $\kappa_1 \to \infty$ the elliptic rational function becomes a Chebyshev polynomial, and therefore the filter becomes a Chebyshev Type I filter, with ripple factor $\epsilon$.

Disadvantages:

1. Nonlinear phase characteristics.
2. Delay varies with frequency.
4. Bessel Filters:

- Excellent phase characteristic (nearly linear).
- Wide transition region (disadvantage).
- Cutoff Frequency changes as function of $n$ (disadvantage).

All-pole filter with transfer function,

$$H_A(s) = \frac{d_0}{B_n(s)}$$

where $B_n(s)$: $n^{th}$ order Bessel Polynomial

$$B_n(s) = \sum_{i=0}^{n} d_i s^i$$

$$d_i = \frac{(2n-i)!}{2^{n-i}i!(n-i)!}, \quad i \in [0, n]$$

Bessel polynomials satisfy,

$$B_n(s) = (2n - 1)B_{n-1}(s) + s^2B_{n-2}(s)$$

with $B_0(s) = 0$, $B_1(s) = s + 1$

Cutoff Frequency varies as function of order $n$, $\omega_c \approx d_0^{1/n}$
Idea: Analog systems are implemented using integrators, multipliers, and adders. Map every integrator $H_I(s) = 1/s$ (or $h_I(t) = u_s(t)$) to the digital domain.

Convolution integral: $y(t) = \int_0^t x(\tau)h_I(t - \tau)d\tau$

Let $0 < t_1 < t_2$, then

$y(t_2) = \int_0^{t_2} x(\tau)h_I(t - \tau)d\tau$

$y(t_1) = \int_0^{t_1} x(\tau)h_I(t - \tau)d\tau$

$y(t_2) - y(t_1) = \int_{t_1}^{t_2} x(\tau)d\tau = \text{Area}$

$\approx \frac{(t_2-t_1)}{2}[x(t_1) + x(t_2)]$  \text{ Trapezoidal rule of integration.}

Let $t_1 = (n - 1)T$, $t_2 = nT$

$y(nT) - y((n - 1)T) = \frac{T}{2}[x((n - 1)T') + x(nT')]$
Take $z$-transform:

$$Y(z) - z^{-1}Y(z) = \frac{T}{2} [z^{-1}X(z) + X(z)]$$

$$H_I(z) = \frac{Y(z)}{X(z)} = \frac{T}{2} \frac{1+z^{-1}}{1-z^{-1}} = \frac{T}{2} \frac{z+1}{z-1}$$

Thus the mapping from $s$-domain to $z$-domain is:

$$\frac{1}{s} \rightarrow \frac{T}{2} \frac{z+1}{z-1}$$

\[ s \rightarrow \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \quad \text{: Bilinear } z\text{-mapping} \]

Once $H_A(s)$ prototype is described:

$$H_D(z) = H_A(s) \bigg|_{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

Properties of Bilinear $z$-Mapping:

Recall: $s = \frac{2}{T} \frac{z-1}{z+1}$ or $z = \frac{2}{T+s}$

Also $z = re^{j\Omega}$, and $s = \sigma + j\omega$ which gives
Bilinear $z$-Mapping-Properties

\[ r e^{j\Omega} = \frac{(2/T + \sigma) + j\omega}{(2/T - \sigma) - j\omega} \]

Then equations for $r$ and $\Omega$ become,

\[
\begin{align*}
    r &= \left[ \frac{(2/T + \sigma)^2 + \omega^2}{(2/T - \sigma)^2 + \omega^2} \right]^{1/2} \\
    \Omega &= \tan^{-1} \frac{\omega}{\frac{2}{T} + \sigma} + \tan^{-1} \frac{\omega}{\frac{2}{T} - \sigma} 
\end{align*}
\]

Right half of $s$-plane: $\sigma > 0 \implies r > 1$: outside unit circle

On $j\omega$ axis: $\sigma = 0 \implies r = 1$: on unit circle

Left half of $s$-plane: $\sigma < 0 \implies r < 1$: inside the unit circle

$\implies$ Stable $H_A(s)$ is mapped to stable $H_D(z)$.

Additionally, it is obvious that $H_D(z)$ will have real coefficients. However, there is an issue with this mapping called warping.
For simplicity, let $\sigma = 0$ in equation of $\Omega$,
\[ \Omega = 2 \tan^{-1} \frac{\omega}{2/T} = 2 \tan^{-1} \frac{\omega T}{2} \]
or $\omega = \frac{2}{T} \tan \frac{\Omega}{2}$: Nonlinear relation between $\Omega$, $\omega$

Let $\tilde{\Omega} = \frac{\Omega}{T}$ (Normalized $\Omega$)
\[ \omega = \frac{2}{T} \tan \frac{\tilde{\Omega} T}{2} \]

When $\frac{\tilde{\Omega} T}{2} < 0.15$, $\tan \frac{\tilde{\Omega} T}{2} \approx \frac{\tilde{\Omega} T}{2} \implies \omega \approx \tilde{\Omega}$
i.e. approximately linear portion of tangent curve.

Thus, If all essential frequencies ($\omega_c, \omega_r$) are below $\tilde{\Omega} = \frac{0.3}{T}$, no prewarping is needed since minimal warping effects and $\omega \approx \tilde{\Omega}$.

If not, prewarp the analog filter before bilinear $z$-mapping.
Bilinear $\mathcal{z}$-Mapping-Warping Effects

**Note:** Warping affects both magnitude and phase responses. Although, the amplitude distortion caused by warping can be remedied by prewarping, little can be done to linearize the phase except maybe using equalization methods.
Bilinear $z$-Mapping-Prewarpping

Procedure:
If all essential frequencies are above $\tilde{\Omega} = \frac{0.3}{T}$, then

1. Map all essential digital frequencies $\tilde{\Omega}_i$ to corresponding analog frequencies using
   \[ \omega_i = \frac{2}{T} \tan \left( \frac{\tilde{\Omega}_i T}{2} \right) \]
2. Design $H_A(s)$ using these analog frequencies $\omega_i$'s.
3. Map $H_A(s)$ to $H_D(z)$ using Bilinear $z$-mapping.

Example:
Design a digital Butterworth filter that meets:

1. $T = 50 \ \mu \text{sec.}$
2. $f_c = 4 \ \text{kHz}$ or $\tilde{\Omega}_c = 25.13 \times 10^3 \ \text{Rad/sec}$
3. At $2\tilde{\Omega}_c$ attenuation is $\geq 15 dB$

Since $\tilde{\Omega}_c \gg \frac{0.3}{T} \approx 6000 \implies$ Prewarping is needed

We prewarp both $\tilde{\Omega}_c$ and $2\tilde{\Omega}_c$, using
Bilinear $\mathbf{z}$-Mapping-Cont.

\[ \omega_c = \frac{2}{T} \tan \frac{\tilde{\Omega}_c T}{2} \approx 29.1 \times 10^3 \text{ Rad/sec} \]

\[ \omega_{15dB} = \frac{2}{T} \tan \frac{2\tilde{\Omega}_c T}{2} = 123.1 \times 10^3 \text{ Rad/sec} \]

We now use these frequencies to design $H_A(s)$

\[ |H_A(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}} \]

\[ -10 \log_{10} \left|\frac{1}{1 + \left(\frac{123.1}{29.1}\right)^{2n}}\right| \geq 15dB \implies n \geq 1.2 \implies n = 2 \]

From Table 11.2:

\[ H_N(s) = \frac{1}{s^2 + 1.4142s + 1} \]

Denormalize $s \rightarrow s/\omega_c$

\[ H_A(s) = \frac{(29 \times 10^3)^2}{s^2 + 41.15 \times 10^3 s + (29 \times 10^3)^2} \]

\[ H_D(z) = H_A(s)\bigg|_{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{2.117+4.234z^{-1}+2.117z^{-2}}{10.232-3.766z^{-1}+2.002z^{-2}} \]