Digital Controls & Digital Filters
Lectures 19 & 20

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Spring 2017

Stability Analysis (Chapter 7)

Definition 1: Bounded Input, Bounded Output (BIBO) Stability

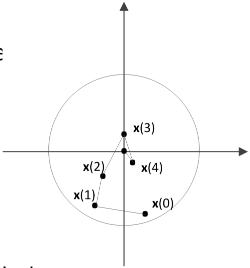
An LTI system is BIBO stable iff a bounded input yields a bounded output (for any IC), i.e.

If $|x(n)| < M < \infty \implies |y(n)| < N < \infty$

Theorem 1:

The necessary and sufficient condition for BIBO stability is that the impulse response $h(n)$ is absolutely summable

$$\sum_{n=0}^{\infty} |h(n)| < P < \infty$$



Definition 2: Asymptotic Stability

Consider autonomous (no input) LTI system:

$\mathbf{x}(n+1) = A\mathbf{x}(n)$

System is asymptotically stable iff for any IC $\mathbf{x}(0)$ such that

$$||\mathbf{x}(0)|| < \delta \implies \lim_{n\to\infty} ||\mathbf{x}(n)|| = 0$$

where $||\mathbf{x}||$ represents the Euclidean norm of vector $\mathbf{x}$.

Stability Analysis-Cont.

Theorem 2:
A discrete-time LTI system given before, is asymptotically stable iff all the eigenvalues of matrix A (poles of the system) lie strictly inside the unit circle.

Theorem 3:
Asymptotic stability implies BIBO stability and vice versa.

Example: Consider the following LTI system

$$\mathbf{x}(n+1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(n) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n)$$

$$y(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(n)$$

The roots of CE or eigenvalues of matrix A are obtained by solving,

$$\varrho(z) = |zI - A| = \begin{vmatrix} z & -1 \\ 1 & z \end{vmatrix} = (z^2+1) = 0 \implies z_{1,2} = \pm j = e^{\pm j\pi/2}$$

i.e. roots on the unit circle and hence not asymptotically stable.
On the other hand, we can test for BIBO stability by finding $h(n)$ and checking absolute summable requirement.

Stability Analysis-Cont.

State transition matrix,

$$A^n = \phi(n) = \mathcal{Z}^{-1}\left[(zI - A)^{-1}z\right]$$

$$(zI - A)^{-1} = \frac{\begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix}}{z^2+1}$$

$$A^n = \mathcal{Z}^{-1}\begin{bmatrix} \frac{z^2}{z^2+1} & \frac{z}{z^2+1} \\ \frac{-z}{z^2+1} & \frac{z^2}{z^2+1} \end{bmatrix} = \begin{bmatrix} \cos\frac{n\pi}{2} & \sin\frac{n\pi}{2} \\ -\sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \end{bmatrix}$$

Then the impulse response, $h(n) = CA^{n-1}B$, becomes,

$$h(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{(n-1)\pi}{2} & \sin\frac{(n-1)\pi}{2} \\ -\sin\frac{(n-1)\pi}{2} & \cos\frac{(n-1)\pi}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cos\frac{(n-1)\pi}{2}$$

$$\sum_{n=0}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |\cos\frac{(n-1)\pi}{2}| \to \infty$$

i.e. not BIBO stable.

Jury's Stability Test

Let the closed-loop CE be,

$\varrho(z) = a_N z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0 = 0$

where $a_i's$ are the coefficients with $a_N > 0$

Form the Jury's table:

$$\begin{array}{lcccccc}
Row\ 0 & z^0 & z^1 & \ldots & z^{N-2} & z^{N-1} & z^N \\
Row\ 1 & a_0 & a_1 & \ldots & a_{N-2} & a_{N-1} & a_N \\
Row\ 2 & a_N & a_{N-1} & \ldots & a_2 & a_1 & a_0 \\
Row\ 3 & b_0 & b_1 & \ldots & b_{N-2} & b_{N-1} & \\
Row\ 4 & b_{N-1} & b_{N-2} & \ldots & b_1 & b_0 & \\
Row\ 5 & c_0 & c_1 & \ldots & c_{N-2} & & \\
Row\ 6 & c_{N-2} & c_{N-3} & \ldots & c_0 & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
Row\ (2N-3) & m_0 & m_1 & m_2 & & &
\end{array}$$

where

$$b_k = \begin{vmatrix} a_0 & a_{N-k} \\ a_N & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{N-1-k} \\ b_{N-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{N-2-k} \\ c_{N-2} & c_k \end{vmatrix}$$

Jury's Stability Test-Cont.

Theorem:

Necessary and sufficient conditions for asymptotic stability are:

1. $\varrho(1) > 0$
2. $(-1)^N \varrho(-1) > 0$
3. $|a_0| < a_N$
4. $|b_0| > |b_{N-1}|$
5. $|c_0| > |c_{N-2}|$
6. $|d_0| > |d_{N-3}|$

$$\vdots$$

$$|m_0| > |m_2|$$

Remarks:

1. For a second order system $(N = 2)$ Jury's table contains only one row and the conditions for stability are:

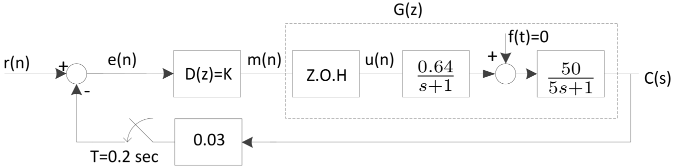
1. $\varrho(1) > 0$
2. $\varrho(-1) > 0$
3. $|a_0| < a_2$

Jury's Stability Test-Cont.

2. For $N = 3$ conditions are:

1. $\varrho(1) > 0$
2. $-\varrho(-1) > 0$
3. $|a_0| < a_3$
4. $|b_0| > |b_2|$

Example 1: Given the following digital control system, find the range of gain K for stability.



$$G(z) = (1 - z^{-1})\mathbf{Z}\left[\frac{6.4}{s(s+1)(s+0.2)}\right] = \frac{0.16z+0.064}{z^2-1.78z+0.78}$$

Closed-loop transfer function,

$$T(z) = \frac{C(z)}{R(z)} = \frac{KG(z)}{1+0.03KG(z)}$$

Jury's Stability Test-Cont.

Thus, the CE of the closed-loop system,

$$\varrho(z) = 1 + 0.03KG(z) = z^2 - 1.78z + 0.78 + 0.03K(0.16z + 0.064) = 0$$

or,

$$\varrho(z) = z^2 - (1.78 - 0.0048K)z + (0.78 + 0.0019K) = 0$$

For stability,

1. $\varrho(1) > 0 \implies 0.007 + 0.0067K > 0 \implies K > -1.04$
2. $\varrho(-1) > 0 \implies 3.57 - 0.0028K > 0 \implies K < 1.24 \times 10^3$
3. $a_0 < a_2 \implies 0.78 + 0.0019K < 1 \implies K < 111$

Therefore, the range of K for stability is $0 < K < 111$.

Example 2: The CE of a digital control system is given by,

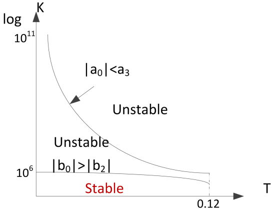
$$\varrho(z) = z^3 + (111.6T^2 + 16.74T - 3)z^2 + (3 - 33.48T + 1.395 \times 10^{-4}KT^3)z + (1.395 \times 10^{-4}KT^3 + 16.74T - 111.6T^2 - 1) = 0$$

find the range of T (sampling period) and gain K for stability.

Jury's Stability Test-Cont.

1. $\varrho(1) > 0 \implies KT^3 > 0 \implies K > 0$
2. $\varrho(-1) < 0 \implies -8 + 66.96T < 0 \implies T < 0.12\ sec$
3. $|a_0| < a_3 \implies |1.395 \times 10^{-4}KT^3 + 16.74T - 111.6T^2 - 1| < 1$
4. $|b_0| > |b_2| \implies (1.395 \times 10^{-4}KT^3 + 16.74T - 111.6T^2 - 1)^2 - 1 > |(1.395 \times 10^{-4}KT^3 + 16.74T - 111.6T^2 - 1)(111.6T^2 + 16.74T - 3) - (3 - 33.48T + 1.395 \times 10^{-4}KT^3)|$

The stable region can be identified by plotting.



This example shows the utility of the Jury's test for design purposes as well.

Jury's Stability Test-Cont.

Remark:

When some or all elements of a row in Jury's table become zero, tabulation terminates.

Remedy:

Expand or contract the unit circle by $z \to (1 \pm \epsilon)z$ where $\epsilon > 0$
$z^N \to (1 \pm \epsilon)^N z^N$
But since $(1 \pm \epsilon)^N \approx 1 \pm N\epsilon$ we use $z^N \to (1 \pm N\epsilon)z^N$.
i.e. we use $(1 \pm N\epsilon)z^N$ instead of z^N in table and proceed.

Example:

Let $\varrho(z) = z^3 + z^2 + z + 1$

Row	z^0	z^1	z^2	z^3
1	1	1	1	1
2	1	1	1	1
3	0	0	0	

Substitute

$z^3 \to (1 + 3\epsilon)z^3$, $\quad z^2 \to (1 + 2\epsilon)z^2$, $\quad$ and $z \to (1 + \epsilon)z$

Jury's Stability Test-Cont.

Then, Jury's table becomes

Row	z^0	z^1	z^2	z^3
1	1	$1+\epsilon$	$1+2\epsilon$	$1+3\epsilon$
2	$1+3\epsilon$	$1+2\epsilon$	$1+\epsilon$	1
3	-6ϵ	-4ϵ	-2ϵ	
4	-2ϵ	-4ϵ	-6ϵ	

1. $\varrho(1) > 0 \implies \varrho(1) = 4 > 0$
2. $-\varrho(-1) > 0 \implies \varrho(1) = 0$
3. $|a_0| < |a_3| \implies 1 < 1 + 3\epsilon \implies \epsilon > 0$
4. $|b_0| > |b_2| \implies 6\epsilon > 2\epsilon$

Stable for $\epsilon > 0$ and unstable for $\epsilon < 0 \implies$ poles must be on the unit circle.

Design of Digital Filters

Comprises of 4 general steps:

1. Approximation: Process of generating a transfer function satisfying a set of desired specs (frequency domain).
2. Realization: Process of converting a transfer function into a filter structure.
3. Quantization effects: Process of studying finite word length effects in digital systems.
    - i Input Quantization (at A/D).
    - ii Coefficient Quantization (at multipliers).
    - iii Product Quantization (output of multipliers).
4. Implementation: Process of implementing the designed filter either in
    - i Software, or
    - ii Hardware.

Approximation

1. Approximation Methods for Recursive or IIR Digital Filters

Consider a recursive digital filter described by difference equation (or transfer function):

$$y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{j=0}^{N} a_j y(n-j)$$

$x(n)$: input, $y(n)$ output, N: order, $a_j's, b_i's$: filter coefficients

Advantages of IIR filters:

1. Low order (small N) gives sharper cutoff frequencies than FIR filters of the same order.
2. Can be implemented easily in time-domain (recursive).

Disadvantages:

1. Nonlinear phase
2. Stability issues
3. Quantization issues

Approximation Methods

Indirect Methods: Based on the design of analog prototype.

Idea: Start with the design of an analog filter, $H_A(s)$, and map it to a digital transfer function, $H_D(z)$, i.e. $H_A(s) \rightarrow H_D(z)$ using an appropriate mapping:

1. Impulse Invariant ✓
2. Bilinear z-Mapping ✓
3. Matched z-Transform

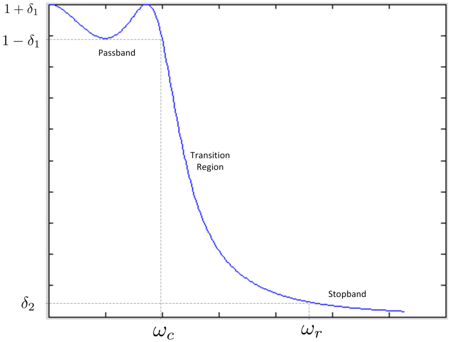
Direct Methods: Based on using optimization packages:

1. Linear programming.
2. Non-linear programming.

Indirect Method

Conditions for Mapping

1. Should preserve all essential characteristics of analog filter (BW, PB loss, SB loss).
2. A stable analog filter should be mapped to a stable digital filter.
3. The transfer function $H_D(z)$ should be rational and proper with real coefficients.



ω_c : Cutoff frequency
ω_r : Lower stopband frequency
δ_1 : Passband error
δ_2 : Stopband error

Review of Analog Filters-Preliminaries

Recall: $H_A(s) = \frac{Y(s)}{U(s)} = \int\limits_0^\infty h(t)e^{-st}dt$

Define *Magnitude-Squared Function*:

$A(-s^2) = H_A(s)H_A(-s)$ (Spectral Factorization)

Alternatively by letting $s = j\omega$

$A(\omega^2) = H_A(j\omega)H_A(-j\omega) = H_A(j\omega)H_A^*(j\omega) = |H_A(j\omega)|^2$

Properties of $A(-s^2)$:

1. Even with real coefficients.
2. Poles and zeros of $A(-s^2)$ occur in quadruples i.e. if p_i is a pole/zero then $-p_i$, p_i^*, and $-p_i^*$ are also poles/zeros.

Goal: Given $A(-s^2)$ (or $A(\omega^2)$), $H_A(s)$ should be chosen such that it contains all the poles and zeros of $A(-s^2)$ that lie in the LH of the s-plane (spectral factorization).

Review of Analog Filters-Preliminaries

Example:

Given $A(\omega^2) = \frac{2+\omega^2}{1+\omega^4}$, determine $H_A(s)$.

$$A(-s^2) = \frac{2-s^2}{1+s^4} = \frac{(\sqrt{2}-s)(\sqrt{2}+s)}{(s+\alpha)(s+\alpha^*)(s-\alpha)(s+\alpha^*)} = H_A(s)H_A(-s)$$

$$\alpha = \frac{1+j}{\sqrt{2}}$$

$$H_A(s) = \frac{s+\sqrt{2}}{(s+\alpha)(s+\alpha^*)}$$

i.e. $H_A(s)$ captures all poles/zeros that are in LH of s-plane.