Digital Controls & Digital Filters
Lectures 13 & 14

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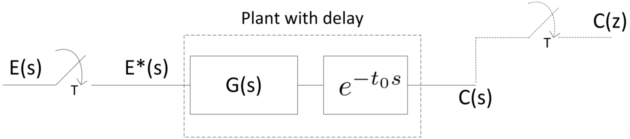
Systems with Actual Time Delays-Application 2

Case 1: Delay in the Plant



Let $t_0 = kT + \Delta T, \quad K \in I, \quad 0 < \Delta \leq 1$

$$C(s) = G(s)e^{-t_0 s}E^*(s)$$

$$C(z) = \mathbf{Z}\left[G(s)e^{-t_0 s}\right]E(z) = \mathbf{Z}\left[G(s)e^{-kTs}e^{-\Delta T s}\right]E(z) =$$

$$z^{-k}\mathbf{Z}\left[G(s)e^{-\Delta T s}\right]|_{\Delta=1-m}E(z) = z^{-k}\mathbf{Z}_m\left[G(s)\right]E(z)$$

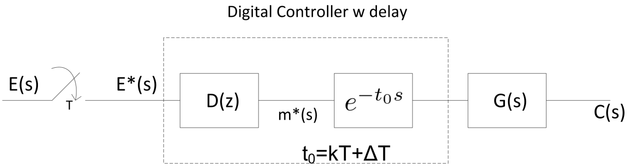
Thus, for this case we have

$$\boxed{C(z) = z^{-k}G(z,m)E(z)}$$

Systems with Actual Time Delays-Application 2

Case 2: Delay in Digital Controller

Delay corresponds to the time required for digital controller to compute the response (software or hardware).



$$C(s) = G(s)e^{-t_0 s}D^*(s)E^*(s) \implies \boxed{C(z) = z^{-k}G(z,m)D(z)E(z)}$$

Example: A plant with delay is shown. Find: (a) response to a unit step when $t_0 = 2\ sec$ and $T = 0.6\ sec$; (b) $c(t)$ and show $c(t)|_{t=nT}$ yields $c(n)$ in (a).

$$t_0 = 2 = 3T + 0.2 \implies \Delta = 1/3,\ m = 2/3$$

(a) $C(s) = \frac{1-e^{-Ts}}{s}\frac{2e^{-2s}}{s+0.5}E^*(s)$

Systems with Actual Time Delays-Cont.

$$C(z) = (1 - z^{-1})\mathbf{Z}\left[\frac{2e^{-2s}}{s(s+0.5)}\right]E(z) = (1 - z^{-1})z^{-3}\mathbf{Z}\left[\frac{2e^{-\frac{T}{3}s}}{s(s+0.5)}\right]\frac{z}{z-1}$$

$$C(z) = z^{-3}\mathbf{Z}\left[\frac{2e^{-(1-2/3)Ts}}{s(s+0.5)}\right] = z^{-3}\mathbf{Z}_m\left[\frac{2}{s(s+0.5)}\right]\Big|_{m=2/3} = \frac{(0.7256z-0.3112)}{z^3(z-1)(z-0.7408)}$$

Expand $\frac{C(z)}{z} = \frac{(0.7256z-0.3112)}{z^4(z-1)(z-0.7408)}$ and take IZT,

$$c(n) = [4 - 3.27(0.7408)^{n-4}]u_s(n-4)$$

(b) Using shifting property $C(s) = \frac{2e^{-2s}}{s(s+0.5)} \implies c(t) = \mathcal{L}^{-1}\{\frac{2}{s(s+0.5)}\}|_{t=t-2}$

But $\frac{2}{s(s+0.5)} = \frac{4}{s} - \frac{4}{s+0.5}$

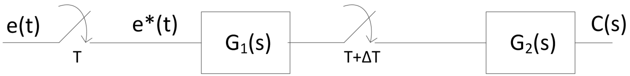
Hence $c(t) = 4(1 - e^{-0.5(t-2)})u_s(t-2)$

Now, $c(n) = 4(1 - e^{-0.5(t-2)})u_s(t-2)|_{t=nT} = 4(1 - e^{1}e^{-0.5nT})u_s(nT-2)$

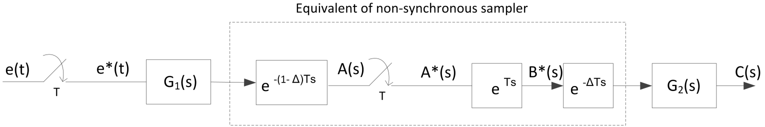
$$= (4 - 4e^{1}e^{-1.2}e^{-0.5(n-4)})u_s(n-4) = [4 - 3.27(0.7408)^{n-4}]u_s(n-4)$$

Non-synchronous Sampling-Application 3

Second sampler samples at $\Delta T, T+\Delta T, 2T+\Delta T, \ldots$.



This system can be represented by the following (see your text) equivalent synchronous system.



Now, we can write

$$A(s) = e^{-(1-\Delta)Ts} G_1(s) E^*(s) \implies A(z) = G_1(z,m)|_{m=\Delta} E(z)$$

Also, $B^*(s) = e^{Ts} A^*(s) \implies B(z) = zA(z)$

The output is

$$C(s) = e^{-\Delta Ts} G_2(s) B^*(s) \implies C(z) = G_2(z,m)|_{m=1-\Delta} B(z)$$

Combining these results yields,

$$C(z) = zG_1(z,m)|_{m=\Delta} G_2(z,m)|_{m=1-\Delta} E(z)$$

Sampling Of A Continuous-Time State Space System

Plant is described by continuous-time state-space equations. We would like to arrive at discrete state equations for the plant to preserve the natural states of the system and also to be able to analyse high order systems.

Plant state-space equations,

$$\begin{cases} \dot{\mathbf{v}}(t) &= A_c\mathbf{v}(t) + B_c\bar{m}(t) \\ c(t) &= C_c\mathbf{v}(t) + d_c\bar{m}(t) \end{cases}$$

$\mathbf{v}(t)$: State vector of plant $\quad \bar{m}(t)$: Input of plant $\quad c(t)$: Output of plant

Assuming IC at t_0,

$$\mathbf{v}(t) = \phi_c(t-t_0)\mathbf{v}(t_0) + \int_{t_0}^{t} \phi_c(t-\tau)B_c\bar{m}(\tau)d\tau$$

Thus, the response at time t is,

$$c(t) = C_c \underbrace{\phi_c(t-t_0)\mathbf{v}(t_0)}_{zero-input\ response} + \underbrace{\int_{t_0}^{t} C_c\phi_c(t-\tau)B_c\bar{m}(\tau)d\tau + d_c\bar{m}(t)}_{zero-state\ response}$$

Sampling Of A Continuous-Time State Space System

$\phi_c(t)$: State transition matrix of continuous-time plant

$$\phi_c(t) = e^{A_c t} = I + A_c t + \frac{(A_c t)^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{A_c^k t^k}{k!}$$

Let $t = (n+1)T, \quad t_0 = nT$

Then,

$$\mathbf{v}((n+1)T) = \phi_c(T)\mathbf{v}(nT) + \int_{nT}^{(n+1)T} \phi_c(nT + T - \tau)B_c \bar{m}(\tau)d\tau$$

Note that when $nT \leq t < (n+1)T, \; \bar{m}(t) = \bar{m}(nT)$, then:

$$\mathbf{v}((n+1)T) = \phi_c(T)\mathbf{v}(nT) + \bar{m}(nT) \int_{nT}^{(n+1)T} \phi_c(nT + T - \tau)B_c d\tau$$

Define $\mathbf{x}(n) = \mathbf{v}(nT), \quad u(n) = \bar{m}(nT)$

$A = e^{A_c T} = \phi_c(T)$

$B = \int_{nT}^{nT+T} \phi_c(nT + T - \tau)d\tau B_c$

$C = C_c$

$d = d_c$

Sampling Of A Continuous-Time State Space System

Then we get the following discrete-time state-space mode for the plant:

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + Bu(k)$$
$$c(k) = C\mathbf{x}(k) + du(k)$$
$$A = \phi_c(T) = e^{A_cT} = \sum_{i=0\infty} \frac{A_c^iT^i}{i!} = I + A_cT + \frac{(A_cT)^2}{2!} + \ldots$$

For B, let $nT + T - \tau = \sigma$, then we have

$$B = \left(\int_0^T \phi_c(\sigma)d\sigma\right) B_c$$

But,

$$\int_0^T \phi_c(\sigma)d\sigma = \int_0^T (I + A_c\sigma + \frac{A_C^2\sigma^2}{2} + \ldots)d\sigma = IT + \frac{A_cT^2}{2} + \frac{A_c^2T^3}{3!} + \ldots$$

Note that $A - I = A_cT + \frac{A_c^2T^2}{2!} + \ldots$

Thus, $A_c^{-1}(A - I) = IT + \frac{A_cT^2}{2!} + \ldots$ (if A_c^{-1} exists) $\Longrightarrow$

which yields an alternative equation for B,

$$B = A_C^{-1}(A - I)B_c$$

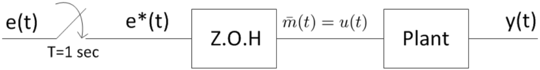
Sampling Of A Continuous-Time State Space System

Example: In open-loop system below, plant is given by:

$$\frac{d^2y(t)}{dt^2} - 0.7\frac{dy(t)}{dt} + 0.1y(t) = \bar{m}(t)$$

(a) Convert to continuous-time state equations

(b) Find discrete-time state equations for the system



Taking the Laplace transform (ICs=0), we get

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s^2-0.7s+0.1} = \frac{1}{(s-0.2)(s-0.5)}$$

Since distinct and real poles, we can convert to parallel form. Using PFE

$$\frac{Y(s)}{U(s)} = \frac{-10/3}{s-0.2} + \frac{10/3}{s-0.5}$$

Then, the continuous-time state space equation is parallel form is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -10/3 & 10/3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Sampling Of A Continuous-Time State Space System

For parallel form, state transition matrix can easily be found,

$$\phi_c(t) = e^{A_c t} = \begin{bmatrix} e^{0.2t} & 0 \\ 0 & e^{0.5t} \end{bmatrix}$$

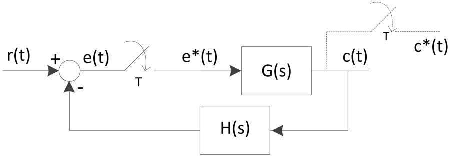
$$A = \phi_c(T) = \begin{bmatrix} e^{0.2} & 0 \\ 0 & e^{0.5} \end{bmatrix}$$

$$B = A_c^{-1}(A - I)B_c = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^{0.2} - 1 & 0 \\ 0 & e^{0.5} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 5(e^{0.2} - 1 \\ 2(e^{0.5} - 1) \end{bmatrix}, \quad C = C_c$$

Closed-Loop Systems

Case 1: Sampler before plant



① $C(s) = G(s)E^*(s)$

② $E(s) = R(s) - H(s)C(s)$

sub ① into ② gives ③ $E(s) = R(s) - G(s)H(s)E^*(s)$

star ③ $\Rightarrow E^*(s) = R^*(s) - \overline{GH}^*(s)E^*(s)$

or $E^*(s) = \frac{R^*(s)}{1+\overline{GH}^*(s)} \Longrightarrow \boxed{E(z) = \frac{R(z)}{1+\overline{GH}(z)}}$

From ① $C^*(s) = G^*(s)E^*(s)$ or

$C(z) = \frac{G(z)R(z)}{1+\overline{GH}(z)}$

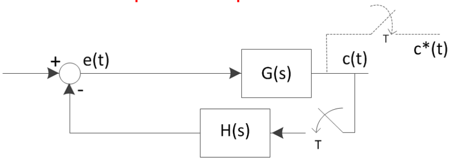
Thus, in this case closed-loop transfer function is

$$\boxed{T(z) = \frac{C(z)}{R(z)} = \frac{G(z)}{1+\overline{GH}(z)}}$$

Closed-Loop Systems-Cont.

Important Note: Care must be taken when starring equations. For instance starring ② would have led to $C(s)$ getting lost in the operation, hence no closed loop transfer function.

Case 2: Sampler after plant



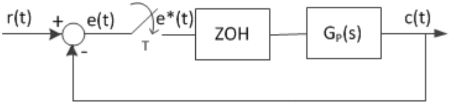
① $C(s) = G(s)E(s)$
② $E(s) = R(s) - H(s)C^*(s)$
$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$
$C^*(s) = \overline{GR}(s) - \overline{GH}^*(s)C^*(s)$
$C^*(s) = \frac{\overline{GR}^*(s)}{1+\overline{GH}^*(s)}$

$$\boxed{C(z) = \frac{\overline{GR}(z)}{1+\overline{GH}(z)}}$$

In this case the closed loop transfer function cannon be defined.

Closed-Loop Systems-Cont.

Example: In the closed-loop system shown below $G_P(s) = \frac{1}{s+1}$. Find the closed-loop transfer function.



From Case 1, closed-loop transfer function
$T(z) = \frac{G(z)}{1+\overline{GH}(z)}$

In this case, $H(s) = 1$, thus, $T(z) = \frac{G(z)}{1+G(z)}$

Also, $G(z)$ is given by

$$G(z) = (1 - z^{-1})\mathbf{Z}\left[\frac{G_p(s)}{s}\right] = (1 - z^{-1})\mathbf{Z}\left[\frac{1}{s(s+1)}\right].$$

From the table,

$$G(z) = \frac{(1-e^{-T})z^{-1}}{(1-e^{-T}z^{-1})}$$

Thus,

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{(1-e^{-T})z^{-1}}{1+(1-2e^{-T})z^{-1}}$$