Digital Controls & Digital Filters
Lectures 11 & 12

M.R. Azimi, Professor

Department of Electrical and Computer Engineering
Colorado State University

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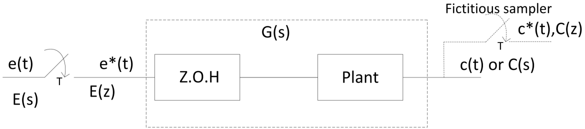
Pulse Transfer Function

Consider the following open-loop digital control system with a sampler and hold. To study the behaviour of this system using discrete-time methods, we assume there is a *fictitious sampler* at the output. Then,



$$C(s) = G(s)E^*(s)$$

Take starred transform

$$C^*(s) = [G(s)E^*(s)]^*$$

From ③, $C^*(s) = \frac{1}{T}\sum_{-\infty}^{\infty} C(s+jn\omega_s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jn\omega_s)E^*(s+jn\omega_s) =$

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jn\omega_s)E^*(s) = \underbrace{\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jn\omega_s)}_{G^*(s):\ pulse\ transfer\ function}E^*(s) = G^*(s)E^*(s)$$

Cascaded Systems

Since $E^*(s+jn\omega_s)=E^*(s)$ and $\frac{1}{T}\sum_{n=-\infty}^{\infty}G(s+jn\omega_s)=G^*(s)$. Thus

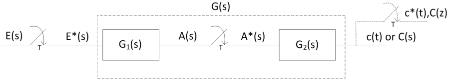
$C^*(s)=G^*(s)E^*(s)$ or $C(z)=G(z)E(z)$

where $G(z)$: Pulse Transfer Function

Cascaded Systems

Case 1: Systems Separated by Samplers

Both $G_1(s)$ and $G_2(s)$ contain hold devices.



$$
\begin{aligned}
C(s)=A^*(s)G_2(s) &\implies C^*(s)=G_2^*(s)A^*(s)\\
A(s)=G_1(s)E^*(s) &\implies A^*(s)=G_1^*(s)E^*(s)
\end{aligned}
$$

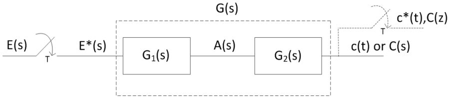
Combine:

$C^*(s)=G_1^*(s)G_2^*(s)E^*(s)$ or $C(z)=G_1(z)G_2(z)E(z)$

$G^*(s)=\frac{C^*(s)}{E^*(s)}=G_1^*(s)G_2^*(s)$ or $G(z)=\frac{C(z)}{E(z)}=G_1(z)G_2(z)$

Cascaded Systems-Cont.

Case 2: No Sampler in Between



$$\left.\begin{aligned} C(s) &= A(s)G_2(s) \\ A(s) &= G_1(s)E^*(s) \end{aligned}\right\} \Longrightarrow C(s) = G_1(s)G_2(s)E^*(s)$$

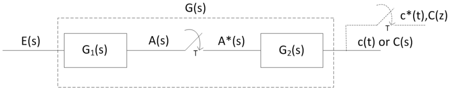
Thus

$$C^*(s) = \overline{G_1G_2}^*(s)E^*(s)$$
$$\overline{G_1G_s}^*(s) = [G_1(s)G_2(s)]^*$$

Overall transfer function

$$G^*(s) = \frac{C^*(s)}{E^*(s)} = \overline{G_1G_2} \quad \text{or} \quad G(z) = \overline{G_1G_2}(z)$$

Case 3: Sampler In Between But Not at Input



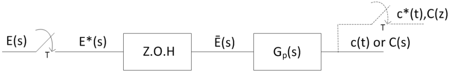
Cascaded Systems-Cont.

$$C(s) = A^*(s)G_2(s) \implies C^*(s) = A^*(s)G_2^*(s)$$
$$A(s) = G_1(s)E(s) \implies A^*(s) = \overline{G_1E}^*(s)$$
$$C^*(s) = G_2^*(s)\overline{G_1E}^*(s) \quad \text{or} \quad C(z) = \overline{G_1E}(z)G_2(z)$$

No overall transfer function in this case.

Remark:

ZOH is followed by a continuous-time system as shown.



$$\left.\begin{aligned} C(s) &= \bar{E}(s)G_p(s) \\ \bar{E}(s) &= \frac{(1-e^{-Ts})}{s}E^*(s) \end{aligned}\right\} \implies C(s) = \frac{(1-e^{-Ts})}{s}G_p(s)E^*(s)$$

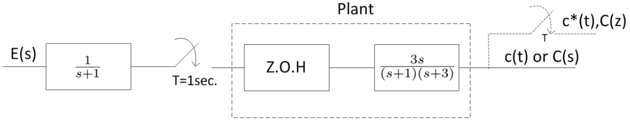
$$C^*(s) = (1-e^{-Ts})\left[\frac{G_p(s)}{s}\right]^* E^*(s)$$

$$C(z) = (1-z^{-1})\mathbf{Z}\left[\frac{G_p(s)}{s}\right]E(z)$$

where from now on $\mathbf{Z} \triangleq \{[\ ]^*\}|_{s=\frac{1}{T}\ln z}$

Cascaded Systems-Cont.

Example 1: Consider the open-loop system shown below where $e(t) = u_s(t)$. Find $C(z)$ and $c(n)$.



From Case 3:

$$C(z) = G_2(z)\overline{G_1E}(z)$$

$$G_2(s) = \frac{1-e^{-Ts}}{s}G_p(z) = (1-e^{-Ts})\frac{3}{(s+1)(s+3)}$$

$$G_2^*(s) = (1-e^{-Ts})\left[\frac{3}{(s+1)(s+3)}\right]^* \Longrightarrow G_2(z) = (1-z^{-1})\mathbf{Z}\left[\frac{3}{(s+1)(s+3)}\right]$$

Using Table on page 513,

$$G_2(z) = \frac{3}{2}\frac{(e^{-1}-e^{-3})z}{(z-e^{-1})(z-e^{-3})}$$

And,

$$\overline{G_1E}(z) = \mathbf{Z}\left[\frac{1}{s(s+1)}\right] = \frac{(1-e^{-1})z}{(z-1)(z-e^{-1})}$$

Cascaded Systems-Cont.

$$C(z) = G_2(z)\overline{G_1E}(z) = \left[\frac{z-1}{z}\left(\frac{3}{2}\right)\frac{(e^{-1}-e^{-3})z}{(z-e^{-1})(z-e^{-3})}\right]\left[\frac{(1-e^{-1})z}{(z-1)(z-e^{-1})}\right] =$$

$$\frac{3/2(1-e^{-1})(e^{-1}-e^{-3})z}{(z-e^{-1})^2(z-e^{-3})}$$

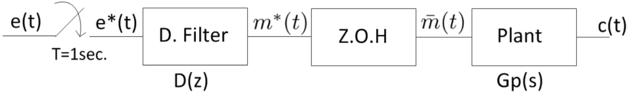
Using PFE,

$$\frac{C(z)}{z} = \frac{0.949}{(z-0.368)^2} - \frac{2.98}{z-0.368} + \frac{2.98}{z-0.0498}$$

$$c(nT) = c(n) = \frac{0.949}{0.368}n(0.368)^n - 2.98(0.368)^n + 2.98(0.0498)^n \ \forall n \geq 0$$

Open Loop Systems With Digital Filter/Controller

Consider,



$$C(s) = G_p(s)\bar{M}(s) = \left(\frac{1-e^{-Ts}}{s}\right) G_p(s)M^*(s) = \left(\frac{1-e^{-Ts}}{s}\right) G_p(s)D^*(s)E^*(s)$$

$$C^*(s) = (1-e^{-Ts})\left[\frac{G_p(s)}{s}\right]^* D^*(s)E^*(s)$$

$$C(z) = (1-z^{-1})\mathbf{Z}\left[\frac{G_p(s)}{s}\right] D(z)E(z)$$

Overall transfer function is:

$$G(z) = \frac{C(z)}{E(z)} = (1-z^{-1})\mathbf{Z}\left[\frac{G_p(s)}{s}\right] D(z)$$

Open Loop Systems With Digital Filter/Controller

Example 2:

Consider previous scenario where digital filter is given by:

$m(n) = -m(n-1) + e(n-1)$

and $e(t) = u_s(t)$, $T = 1Sec$, $G_p(s) = 1/s$

Find $c(nT)$.

$$M(z) = -z^{-1}M(z) + z^{-1}E(z) \implies D(z) = \frac{M(z)}{E(z)} = \frac{z^{-1}}{1+z^{-1}} = \frac{1}{z+1}$$

$$E(z) = \mathbf{Z}\left[\frac{1}{s}\right] = \frac{z}{z-1}$$

$$\mathbf{Z}\left[\frac{G_p(s)}{s}\right] = \mathbf{Z}\left[\frac{1}{s^2}\right] = \frac{Tz}{(z-1)^2} = \frac{z}{(z-1)^2}$$

$$C(z) = \frac{z-1}{z}\frac{z}{(z-1)^2}\frac{1}{z+1}\frac{z}{z-1} = \frac{z}{(z+1)(z-1)^2}$$

Apply PFE

$$\frac{C(z)}{z} = \frac{1/4}{z+1} + \frac{1/2}{(z-1)^2} - \frac{1/4}{(z-1)} \implies$$

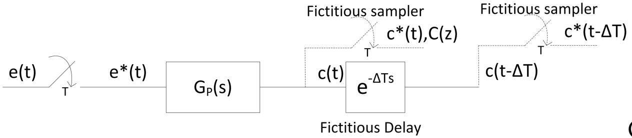
$$c(n) = (\tfrac{1}{4}(-1)^n + \tfrac{1}{2}n - \tfrac{1}{4})u_s(n).$$

Modified z-Transform

Applications

1. Determine system response in between the samples.
2. System (plant or digital filter) includes fractional delays (not exact multiple of sampling period).
3. Non synchronous samplers.

Application 1: Response in between samples: Insert a fictitious delay, ΔT, at output of plant where Δ is swept in $0 \leq \Delta < 1$.



Output of the delay after sampling,

$$c^*(t-\Delta T) = c(t-\Delta T)\delta_T(t) = c(t-\Delta T)\sum_{k=0}^{\infty}\delta(t-kT)$$

$$C^*(s,\Delta) = \mathscr{L}\{c^*(t-\Delta T)\} = \frac{1}{2\pi j}\left[\mathscr{L}\{c(t-\Delta T)\} * \mathscr{L}\{\delta_T(t)\}\right]$$

$$= \frac{1}{2\pi j}\left[C(s)e^{-\Delta T s} * \frac{1}{1-e^{-Ts}}\right]$$

Modified z-Transform-Application 1

Using residue theorem:

$$C^*(s,\Delta)|_{s=\frac{1}{T}\ln z} = \sum_{poles\ of\ C(\xi)} Residues\ C(\xi)\frac{e^{-\Delta T\xi}}{1-e^{-T(s-\xi)}}|_{s=\frac{1}{T}\ln z}$$

Let $m = 1-\Delta, \quad 0 < m \leq 1$

① $C(z,m) = z^{-1} \sum_{poles\ of\ C(\xi)} Res\ C(\xi)\frac{e^{mT\xi}}{1-z^{-1}e^{T\xi}}$:Modified z-transform of $c(t)$

or $\mathbf{Z}_m[C(s)] = C(z,m) = \mathbf{Z}[C(s)e^{-\Delta Ts}]|_{\Delta=1-m}$

As before,

② $C(z,m) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jn\omega_s)e^{-(1-m)T(s+jn\omega_s)}|_{s=\frac{1}{T}\ln z}$

$$C(z,m) = \mathbf{Z}[c(t-\Delta T)] = \sum_{k=0}^{\infty} c(kT-\Delta T)z^{-k} = \sum_{k=0}^{\infty} c(kT+mT-T)z^{-k}$$

③ $C(z,m) = z^{-1}\sum_{k=0}^{\infty} c((k+m)T)z^{-k}$

Note: These equations are valid assuming $c(0)=0$ otherwise add $c(0)/2$ to each.

Modified z-Transform–Application 1

Now, using the input-output relationship of the system, the expression for the modified z-transform $C(z,m)$ can be obtained as follows:

$$C(s) = G_p(s)E^*(s)$$

Using this equation in ② we get,

$$C(z,m) = \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jn\omega_s)e^{-(1-m)T(s+jn\omega_s)}\Big|_{s=\frac{1}{T}\ln z}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} G_p(s+jn\omega_s)E^*(s+jn\omega_s)e^{-(1-m)T(s+jn\omega_s)}\Big|_{s=\frac{1}{T}\ln z}$$

But since $E^*(s+jn\omega_s) = E^*(s)$

$$= E(z)\frac{1}{T}\sum_{n=-\infty}^{\infty} G_p(s+jn\omega_s)e^{-(1-m)T(s+jn\omega_s)}\Big|_{s=\frac{1}{T}\ln z}$$

Or,

$$C(z,m) = G_p(z,m)E(z)$$

Modified z-Transform–Application 1

Example: Consider $G_p(s) = \frac{1}{s+1}$ and $e(t) = u_s(t)$. Find expression for output in between the samples.

We use $C(z,m) = G_p(z,m)E(z)$

Here $E(z) = E(s)^*|_{s=\frac{1}{T}\ln z} = \frac{z}{z-1}$

$G_p(z,m) = \mathbf{Z}_m\left[\frac{1}{s+1}\right]$

From the Table on pages 513-514,

$G_p(z,m) = \frac{e^{-mT}}{z-e^{-T}}$

Thus, $C(z,m) = \frac{e^{-mT}}{z-e^{-T}}\frac{z}{z-1}$

Using long-division $C(z,m)$ can be expanded in power series in z^{-1}. It can be shown that the k^{th} coefficient for term z^{-k} is:

$C_m(k) = \frac{e^{-mT}(1-e^{-kT})}{1-e^{-T}} \ \forall k > 0$

which gives the output response for the time duration $(k-1)T < t \leq kT$ when m is varied between 0 and 1.