CHAPTER 11

ELEMENTS OF DIGITAL FILTER DESIGN

11-1 INTRODUCTION

A filter is a network or system that operates on an input signal in a specified way to produce a desired output. The signals may be continuous-time entities and may be stated in time or frequency terms. On the other hand, the signals may be discrete time and may also be stated in time or frequency terms. Because digital filter design draws heavily upon the methods of continuous-frequency filter design, we will first review the basic features of the latter.

Depending upon the service to which the filter is to be applied, the design process may vary considerably. The simplest type of filters might be referred to as "brute force." This would include the simple shunt C or series L, RC, or LC type filters used for filtering the ac ripple produced when a rectifier is used to convert ac into dc. Here the requirement is to permit the dc to pass unimpeded but to attenuate the ac ripple component. Hence such filters are essentially low-pass devices with cutoff at 0 Hz. Actually, these filters are not very frequency selective, but multiple sections can be used in cascade if improved ac attenuation is desired. In addition to such passive filters, one may use an electronic regulator, a device that not only serves to maintain the output voltage constant but also greatly reduces the ripple. From this point of view, the electronic regulator may be considered an active filter.

Perhaps more typical are filters that are frequency-selective assemblies designed to pass signals of certain frequencies and block signals of other frequencies. There are many classes of analog domain (continuous) filters categorized according to their behavior in the frequency domain and specified in terms of their magnitude or phase characteristics. Based on their magnitude or transfer response, filters are classified as low-pass, high-pass, bandpass, or bandstop. In the ideal cases these response characteristics are as shown in Figure 11.1 (see also Chapter 5). However, these ideal characteristics are not physically realizable, and a number of different approaches to filter design have been developed over the years to achieve acceptable approximations to the ideal responses. This has led to the formulation of constant-K filter design, m-derived filter design, and variants of these approaches. These filter designs, which are usually called classical filter design, have resulted in very acceptable filters.

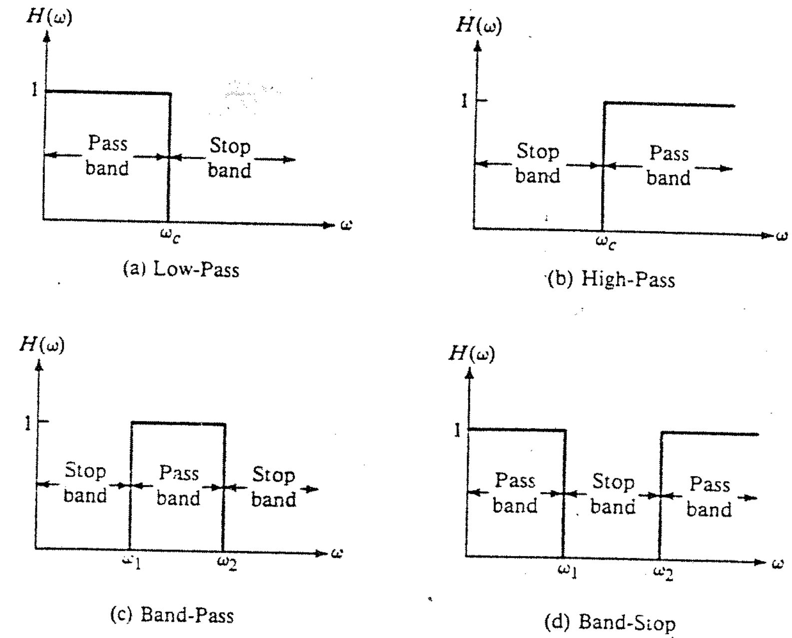

Figure 11.1 Ideal frequency response characteristics of analog filters: (a) Low-Pass. (b) High-Pass. (c) Bandpass. (d) Bandstop.

In modern filter design, which stems from the 1930s, the design problem is approached in a different manner. Essentially, the modern technique is first to find an analytic approximation to the specified filter characteristics as a transfer function, and then to develop a network that realizes this desired transfer function. For the filters shown in Figure 11.1 certain well-developed procedures exist, and these lead to such functions as Butterworth, Chebyshev, and elliptic. The use of these functions has the advantage that the formulas for them are well established and design tables are available. The second step in the design is the realization of the transfer function by passive or active networks. An extensive literature has been developed on active networks for use in such filter design problems.

An important feature of modern filter design is that the problems of approximation and of realization are solved separately to achieve optimum results. For our purposes we are mainly interested in the approximation problem. We will study the means for converting from the s-plane, in which the $H(s)$ approximation exists, to the z-plane and to a corresponding $H(z)$. $H(z)$ can then be realized by discrete systems either by transformation to difference equation form, which can then be adapted for computer calculations, or by direct hardware implementation. That is, the resulting difference equation can be considered to denote a digital filter approximation to the analog filter.

We will develop this approach to digital filter design. However, it is important to know that greater flexibility in filter specifications can be obtained using optimization techniques than by using classical analog-to-digital conversion. A number of different techniques have been developed, but we will not pursue them here.

From a practical point of view, the implementation of the filter on a general-purpose digital computer places accuracy constraints on the realization of the transfer function because the registers can only contain a finite number of binary bits at any one instant. Therefore, the filter coefficients, which can be defined with infinite precision over the real number field, can be represented only by a finite number of binary digits in a hardware register. As a result filter coefficients as well as the arithmetic performed within a digital filter are subject to approximation errors and uncertainty. These limitations can cause the frequency response of the filter to differ measurably from that of the design model. This is true because the filter that is specified is very sensitive to the polynomial coefficients of $H(s)$. These sources of error within a digital filter are referred to as **roundoff noise** and require attention in practical implementation practices.

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11-2 THE BUTTERWORTH FILTER

We will examine the use of the Butterworth function to approximate the low-pass filter shown in Figure 11.1a. The features of this function are illustrated graphically in Figure 11.2. Note that attention in this case is being given only to the amplitude function.

The amplitude response of the nth-order normalized Butterworth filter is given by

$$|H_n(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \qquad n = 1, 2, 3, \ldots \tag{11.1}$$

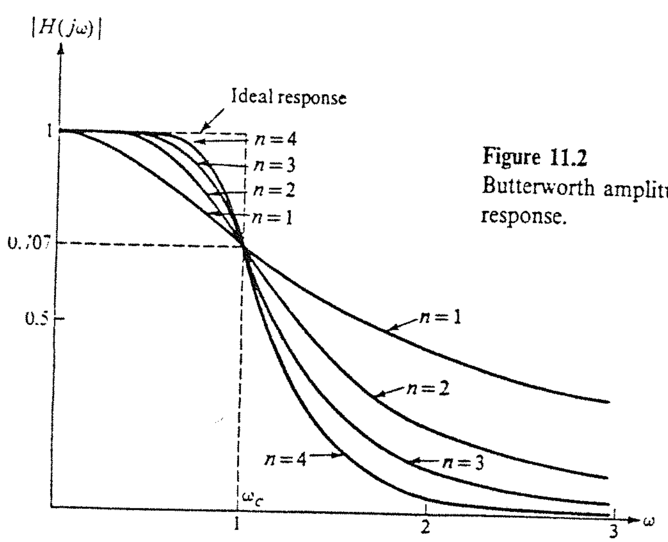

Figure 11.2 Butterworth amplitude response.

As shown in Figure 11.2, the response is monotonically decreasing, having its maximum value $|H_n(j\omega)|_{max} = 1$ at $\omega = 0$. Further, the cutoff point at the normalized $\omega_c = 1$ is

$$|H_n(j1)| = \frac{1}{\sqrt{2}} = 0.707|H_n(j\omega)|_{max}$$

for all orders. Because the functions all approach the value $H_n(j0) = 1$ smoothly, this function is called **maximally flat.** It is observed that the approximation to the square wave improves as n increases.

To obtain the transfer function form of the Butterworth function, we make use of the fact that $H_n(j\omega) = H_n(s)|_{s=j\omega}$ and rewrite (11.1) in squared form

$$H_n(s)H_n(-s) = \frac{1}{1+\left(\frac{(j\omega)^2}{j^2}\right)^n} = \frac{1}{1+(-s^2)^n} \tag{11.2}$$

Let us write the denominator polynomial in the form

$$D(s)D(-s) = 1 + (-s^2)^n \tag{11.3}$$

The roots of this function are obtained from

$$1 + (-s^2)^n = 0$$

Therefore

$$(-1)^n s^{2n} = -1 = e^{j(2k-1)\pi} \qquad k = 1, 2, \ldots, 2n \tag{11.4}$$

from which

$$s^{2n} = e^{j(2k-1)\pi} e^{j\pi n}$$

The kth root is

$$s_k \triangleq \sigma_k + j\omega_k = e^{j(2k+n-1)\pi/2n} = je^{j(2k-1)\pi/2n} \tag{11.5}$$

From this we write

$$s_k = -\sin\left[(2k-1)\frac{\pi}{2n}\right] + j\cos\left[(2k-1)\frac{\pi}{2n}\right] \qquad 1 \le k \le 2n \tag{11.6}$$

It is clear from (11.5) that the roots of s_k are on a unit circle and are spaced π/n radians apart. Moreover, no s_k can occur on the $j\omega$-axis since $(2k-1)$ cannot be an even integer. We thus see that there are n left half plane roots and n right half plane roots. The left half plane roots are associated with $H(s)$ since σ_k is negative for these. These results are shown in Figure 11.3 for the case $n = 4$:

$$s_1 = e^{j5\pi/8} \qquad s_2 = e^{j7\pi/8} \qquad s_3 = e^{j9\pi/8} \qquad s_4 = e^{j11\pi/8}$$

so that

$$D(s) = (s-s_1)(s-s_2)(s-s_3)(s-s_4)$$
$$= 1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4$$

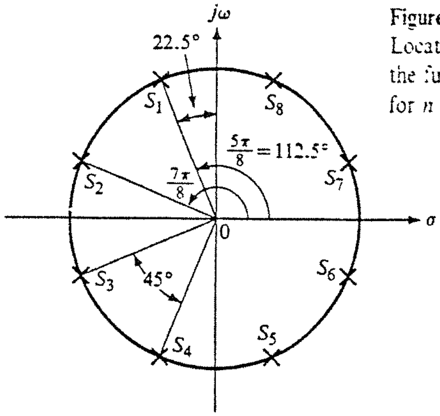

Figure 11.3
Location of the zeros of the function (11.5), shown for $n = 4$.

If we write $D(s)$ of the normalized Butterworth function of order n in the general form

$$D(s) = 1 + a_1 s + a_2 s^2 + \cdots + a_{n-1}s^{n-1} + s^n \tag{11.7}$$

then the coefficients can be computed in the manner shown above. Note that a_0 and a_n are always unity because the poles are all on the unit circle. Table 11.1 gives the functions for n to 8.

Table 11.2 gives the Butterworth polynomials [of (11.7)] in factored form.

EXAMPLE 11.1
Find the transfer function of a Butterworth filter that has an attenuation of at least 10 dB at twice the cutoff frequency $\omega_c = 2.5 \times 10^3$ rad/s.

Solution: We initially find the normalized Butterworth filter. At $\omega = 2\omega_c = 2 \times 1$ ($\omega_c = 1$, normalized)

$$|H_n(j2)|^2 = \frac{1}{1 + 2^{2n}} \tag{11.8}$$

The dB attenuation is given by

$$-10 \log_{10}|H_n(j2)|^2 = 10 \log_{10}(1 + 2^{2n}) \geq 10$$

from which $\text{antilog}_{10} 1 \leq 1 + 2^{2n}$. Hence we find that

$$1 + 2^{2n} \geq 10 \quad \text{or} \quad 2^{2n} \geq 9$$

The order of the filter must then be $n = (\ln_2 9)/2 = 1.584$; hence a second-order Butterworth filter will satisfy this requirement. The corresponding transfer function of the normalized filter is (see Table 11.2)

$$H_n(s) = \frac{1}{s^2 + 1.4142s + 1} \tag{11.9}$$

Table 11.1 Coefficients of Butterworth Polynomials

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7
2	1.4142						
3	2.0000	2.0000					
4	2.6131	3.4142	2.6131				
5	3.2361	5.2361	5.2361	3.2361			
6	3.8637	7.4641	9.1416	7.4641	3.8637		
7	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	
8	5.1528	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258

Table 11.2 Factors of Butterworth Polynomials

n	*Factored Polynomial*
1	$s + 1$
2	$s^2 + 1.4142s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6639s + 1)(s^2 + 1.9616s + 1)$
9	$(s + 1)(s^2 + 0.3473s + 1)(s^2 + s + 1)(s^2 + 1.5321s + 1)(s^2 + 1.8794s + 1)$

The amplitude and phase of this filter are shown in Figure 11.4. For a cutoff frequency $\omega_c = 2.5 \times 10^3$, the transfer function is

$$H(s) = H_n\left(\frac{s}{\omega_c}\right) = \frac{1}{1.6 \times 10^{-7}s^2 + 5.6569 \times 10^{-4}s + 1} \tag{11.10}$$

The denominator polynomial has conjugate complex roots with negative real parts. If we split the function $D(s)$ of (11.10) into two polynomials including, respectively, the even and odd powers of s

$$D(s) = e(s) + o(s) = (1.6 \times 10^{-7}s^2 + 1) + (5.6569 \times 10^{-4}s) \tag{11.11}$$

their roots are purely imaginary (zero included) and, additionally, alternate $-2.5 \times 10^3, 0, 2.5 \times 10^3$. Polynomials that possess these properties are known as **Hurwitz polynomials**.

Observe also that (11.10) is of the general form

$$H(s) = \frac{k}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{k}{D(s)} \tag{11.12}$$

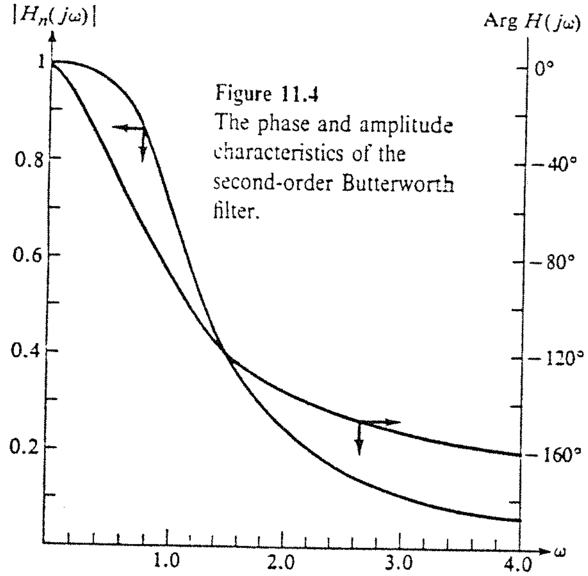

Figure 11.4
The phase and amplitude characteristics of the second-order Butterworth filter.

It can be shown that a transfer function of this form can be realized with passive elements if and only if $D(s)$ is a Hurwitz polynomial.

Suppose that we wish to find a two-port network of the terminated form shown in Figure 11.5a appropriate to (11.10). Since $D(s)$ is of second order, the lossless 2-port network shown in Figure 11.5b appears to be an appropriate form. The transfer function of this circuit is

$$\frac{V_0(s)}{V_i(s)} = H(s) = \frac{1}{CLs^2 + (C + L)s + 2} \tag{11.13}$$

This form differs from (11.10) by the factor 2 in the denominator. If we multiply the numerator and denominator of (11.10) by 2, we obtain an equivalent transfer function

$$\frac{H(s)}{2} = \frac{1}{3.2 \times 10^{-7}s^2 + 1.1314 \times 10^{-3}s + 2} \tag{11.14}$$

By comparing these two equations, we obtain the equations

$$CL = 3.2 \times 10^{-7}$$

$$C + L = 1.1314 \times 10^{-3}$$

These can be solved to yield $C = 5.7 \times 10^{-4}\ F$; $L = 5.6 \times 10^{-4}\ H$. ■

EXAMPLE 11.2

A Butterworth filter must have an attenuation of at least 20 dB at twice the cut-

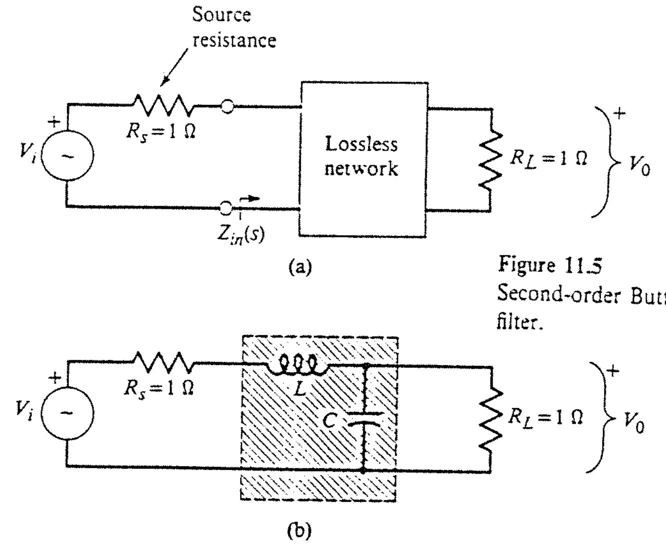

Figure 11.5
Second-order Butterworth filter.

off frequency of 3 kHz. Find the transfer function of the appropriate low-pass filter.

Solution: First determine the normalized Butterworth filter. This requires that at twice the cutoff frequency

$$|H_n(j2)|^2 = \frac{1}{1 + 2^{2n}} \tag{11.15}$$

The dB attenuation is then given by

$$-10 \log_{10}|H_n(j2)|^2 = 10 \log_{10}(1 + 2^{2n}) \geq 20$$

From this $\text{antilog}_{10} 2 \leq 1 + 2^{2n}$ or

$$1 + 2^{2n} \geq 10^2 = 100$$

and the order of the filter must be $n = 3.3$. Hence a fourth-order Butterworth will satisfy this requirement. The transfer function of the normalized filter is

$$H_n(s) = \frac{1}{1 + 2.6131s + 3.4142s^2 + 2.6131s^3 + s^4} \tag{11.16}$$

The amplitude and phase characteristics of this filter are shown in Figure 11.6. With a cutoff frequency of 3 kHz, the denormalizing factor is

$$\omega_c = 2\pi \times 3000 = 18{,}849.54$$

Hence the desired transfer function is

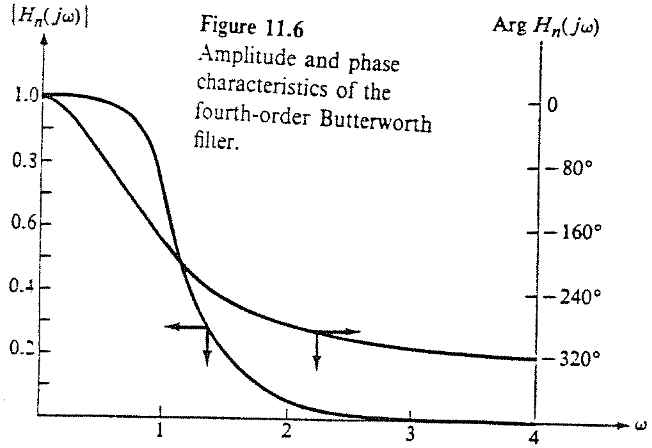

Figure 11.6
Amplitude and phase characteristics of the fourth-order Butterworth filter.

$$H(s) = H_n\left(\frac{s}{\omega_c}\right)$$

$$= \frac{1}{1 + 1.3863 \times 10^{-4}s + 9.609 \times 10^{-9}s^2 + 3.9017 \times 10^{-13}s^3 + 7.9213 \times 10^{-18}s^4} \qquad (11.17)$$

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11-3 THE CHEBYSHEV LOW-PASS FILTER

An examination of Figure 11.2 shows that the Butterworth low-pass amplitude response is very good in the region of small ω and also in the region of large ω but is not very good in the neighborhood of the cutoff frequency ($\omega = 1$). The Chebyshev low-pass filter possesses sharper cutoff response, but it does possess amplitude variations within the passband. The features of the Chebyshev response are shown in Figure 11.7 for n even ($= 4$) and n odd ($= 5$). Several general features are contained in these figures; the oscillations in the passband have equal amplitudes for a given value of ϵ; the curves for n even always start from the trough of the ripple whereas the curves for n odd always start from the peak; and at the normalized cutoff frequency of 1, all curves pass through the same point, shown in the figures as $1/(1 + \epsilon^2)$.

The amplitude response of the Chebyshev low-pass filter is defined by

$$|H_n(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2(\omega)}} \qquad n = 1, 2, 3, \ldots \qquad (11.18)$$

Figure 11.7
General shape of Chebyshev approximations. (a) n even ($= 4$). (b) n odd ($= 5$).

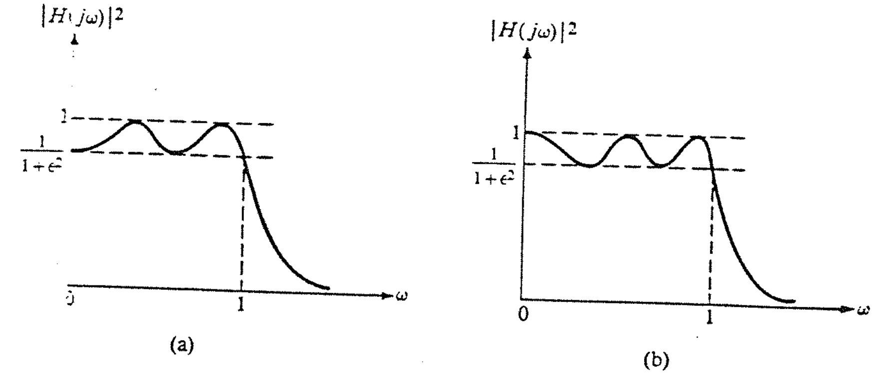


Table 11.3 Chebyshev Polynomials

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

where ϵ is a constant, and where $C_n(\omega)$ is the Chebyshev polynomial given by the equation

$$C_n(\omega) = \cos(n \cos^{-1} \omega) \qquad \text{for } |\omega| \le 1 \quad \text{(a)}$$
$$= \cosh(n \cosh^{-1} \omega) \qquad \text{for } |\omega| > 1 \quad \text{(b)} \qquad \textbf{(11.19)}$$

as already discussed in Section 1-6. The Chebyshev polynomials of orders up to 5 are shown in Figure 11.8 for the range $|\omega| \le 1$. The analytic form of the Chebyshev polynomials from orders 0 to 10 are tabulated in Table 11.3. Figure 11.8 shows the Chebyshev polynomials $C_n(\omega)$ of (11.19) for $\omega \ge 0$.

By taking into consideration (11.19) and the recurrence relationship [see (1.56)]

$$C_{n+1}(\omega) = 2\omega C_n(\omega) - C_{n-1}(\omega) \qquad n = 1, 2, \ldots$$

The nth-order Chebyshev polynomial has the following properties:

a. For any n $\quad 0 \leq |C_n(\omega)| \leq 1 \quad$ for $0 \leq |\omega| \leq 1$
$\quad |C_n(\omega)| > 1 \quad$ for $|\omega| \geq 1$

b. $C_n(\omega)$ is monotonically increasing for $\omega \geq 1$ for all n

c. $C_n(\omega)$ is an odd polynomial, if n is odd
$C_n(\omega)$ is an even polynomial, if n is even

d. $|C_n(0)| = 0 \quad$ for n odd
$|C_n(0)| = 1 \quad$ for n = even

The curves shown in Figure 11.8 together with (11.18) show that $|H_n(j\omega)|$ attains its maximum value of 1 at the zeros of $C_n(\omega)$ and for $|\omega| \leq 1$ attains its minimum value of $1/\sqrt{1+\epsilon^2}$ at the points where $C_n(\omega)$ attains its maximum

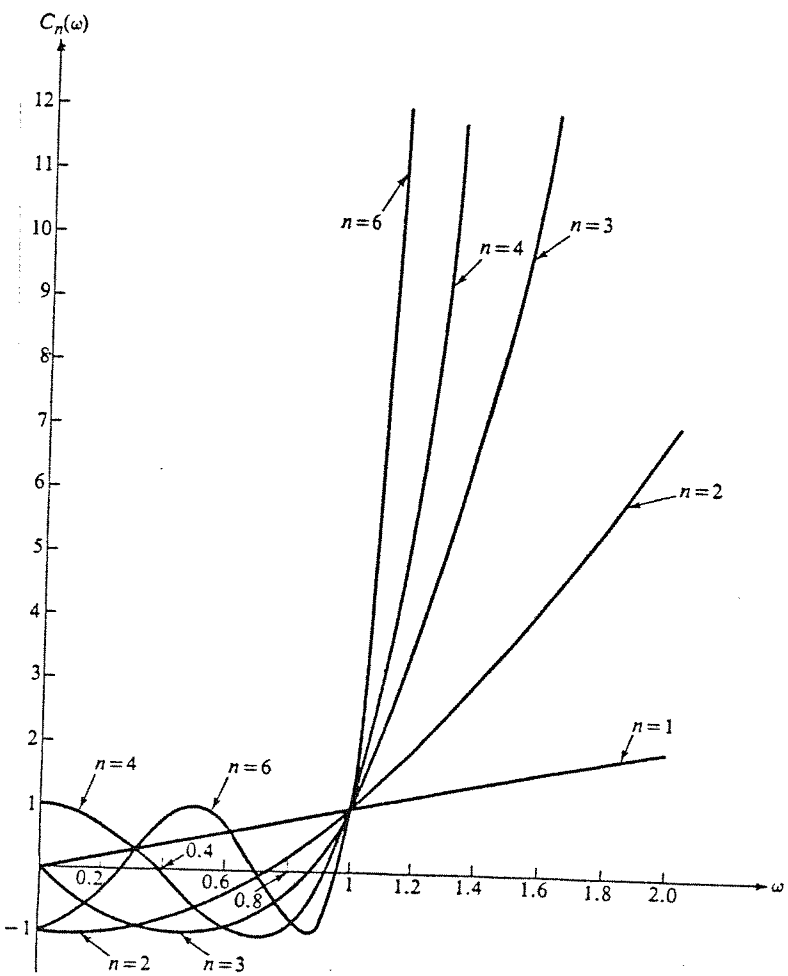


Figure 11.8
Chebyshev polynomials
$C_n(\omega)$ for $\omega \geq 0$.

values of 1. Thus the ripples in the passband $0 \leq \omega \leq 1$ have a peak-to-peak amplitude of

$$r = 1 - \frac{1}{\sqrt{1+\epsilon^2}} \tag{11.20}$$

The ripple in decibels is given by

$$r_{\text{dB}} = -20 \log_{10} \frac{1}{\sqrt{1+\epsilon^2}} = 10 \log_{10}(1+\epsilon^2) \tag{11.21}$$

Outside of the passband $\omega > 1$, $|H_n(j\omega)|$ is monotonically decreasing.

To find the pole locations of $H_n(s)$ where $s = j\omega$, we consider the denominator of the function

$$H_n(s)H_n(-s) = \frac{1}{1 + \epsilon^2 C_n^2\left(\frac{s}{j}\right)} \tag{11.22}$$

More specifically, the poles of interest occur when $C_n(s/j) = \pm\sqrt{-1/\epsilon^2}$ or when

$$\cos\left[n \cos^{-1}\left(\frac{s}{j}\right)\right] = \pm\frac{j}{\epsilon} \tag{11.23}$$

We proceed by defining

$$\cos^{-1}\left(\frac{s}{j}\right) = \alpha - j\beta \tag{11.24}$$

Combine this with (11.23) from which

$$\cos n\alpha \cosh n\beta + j \sin n\alpha \sinh n\beta = \pm\frac{j}{\epsilon} \tag{11.25}$$

Real and imaginary parts must be equal on the two sides, and so

$$\cos n\alpha \cosh n\beta = 0 \quad \text{(a)}$$
$$\sin n\alpha \sinh n\beta = \pm\frac{1}{\epsilon} \quad \text{(b)} \tag{11.26}$$

From (11.26a) it is found, since $\cosh n\beta \neq 0$, that

$$\alpha = (2k-1)\frac{\pi}{2n} \qquad k = 1, 2, 3, \ldots, 2n \tag{11.27}$$

From the second equation (11.26b) together with (11.27), β is found to be, because $\sin n\alpha = \pm 1$,

$$\beta = \pm\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{11.28}$$

Equation (11.24) can be used to give the poles

$$s_k = j\cos(\alpha - j\beta)$$
$$= -\sin\left[(2k-1)\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]$$
$$+ j\cos\left[(2k-1)\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \quad (11.29)$$

These points are located on an ellipse in the s-plane, as illustrated in Figure 11.9 for $n = 4$. To prove that the locus is an ellipse, let

$$s_k = \sigma_k + j\omega_k \quad \text{(a)} \quad (11.30)$$

so that

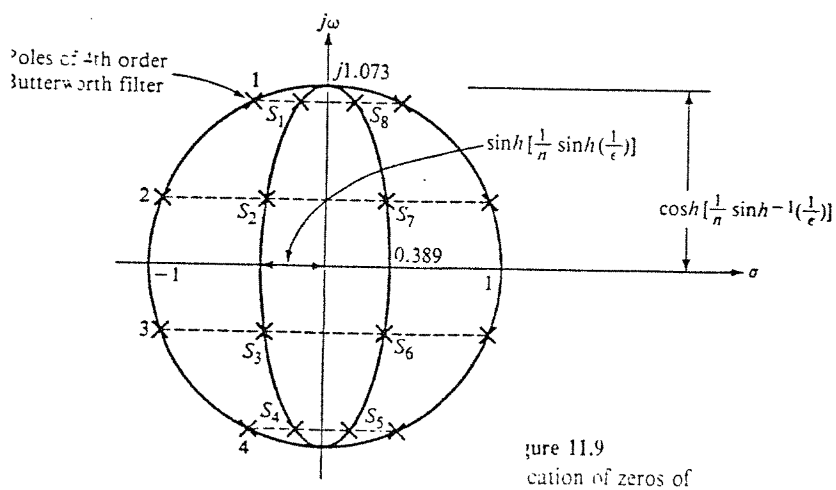
$$\sigma_k = -\sin\left[(2k-1)\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \quad \text{(b)}$$

$$\omega_k = \cos\left[(2k-1)\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right] \quad \text{(c)}$$

It follows from (11.30b) and (11.30c) that

$$\frac{\sigma_k^2}{\left\{\sinh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\right\}^2} + \frac{\omega_k^2}{\left\{\cosh\left[\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right]\right\}^2} = 1 \quad (11.31)$$

This is the equation of the ellipse shown in Figure 11.9. Equation 11.30c shows that the imaginary parts are the same as if the zeros had been uniformly spaced



gure 11.9
cation of zeros of
nction (11.23) on the
lane shown for $n = 4$
d $\epsilon = 0.458$.

Table 11.4 Coefficients of the Polynomial in (11.32)

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
			$r = 0.5$dB ($\epsilon = 0.3493$)					
1	2.8628							
2	1.5162	1.2456						
3	0.7157	1.5349	1.2529					
4	0.3791	1.0255	1.7169	1.1974				
5	0.1789	0.7525	1.3096	1.9374	1.1725			
6	0.0948	0.4324	1.1719	1.5898	2.1718	1.1592		
7	0.0447	0.2821	0.7557	1.6479	1.8694	2.4127	1.1512	
8	0.0237	0.1525	0.5736	1.1486	2.1840	2.1492	2.6567	1.1461
			$r = 1.0$dB ($\epsilon = 0.5088$)					
1	1.9652							
2	1.1025	1.0977						
3	0.4913	1.2384	0.9883					
4	0.2756	0.7426	1.4539	0.9528				
5	0.1228	0.5805	0.9744	1.6888	0.9368			
6	0.0689	0.3071	0.9393	1.2021	1.9308	0.9283		
7	0.0307	0.2137	0.5486	1.3575	1.4288	2.1761	0.9231	
8	0.0172	0.1073	0.4478	0.8468	1.8369	1.6552	2.4230	0.9198

on a circle of radius $\cosh[(1/n)\sinh^{-1}(1/\epsilon)]$. Therefore, the graphical construction indicated can be used to locate the roots.

We can use the same reasoning as in Section 11-2 to show that the desired response can be obtained by limiting consideration to the zeros on the left half plane. The transfer function is thus

$$H(s) = \frac{K}{a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1}s^{n-1} + s^n} \quad (11.32)$$

For a specified dB ripple, ϵ can be found from (11.21), and the poles of an nth-order Chebyshev filter can then be found from (11.30). The constant K must be selected to meet the specified dc gain level. Table 11.4 gives the coefficients in the denominator of (11.32) for two values of r.

EXAMPLE 11.3

Repeat Example 11.2 using a Chebyshev filter with a 1 dB ripple in the passband.

Table 11.5 Factors of the Polynomial in (11.32)

n	Factored Polynomial
	$r = 0.5$dB $\quad \epsilon = 0.3493$
1	$s + 2.8628$
2	$s^2 + 1.4256s + 1.5162$
3	$(s^2 + 0.6265)(s^2 + 0.6265s + 1.1424)$
4	$(s^2 + 0.3507s + 1.0635)(s^2 + 0.8467s + 0.3564)$
5	$(s + 0.3623)(s^2 + 0.2239s + 1.0358)(s^2 + 0.5862s + 0.4768)$
6	$(s^2 + 0.1553s + 1.0230)(s^2 + 0.4243s + 0.5900)(s^2 + 0.5796s + 0.1570)$
	$r = 1$ dB $\quad \epsilon = 0.5088$
1	$s + 1.9652$
2	$s^2 + 1.0978s + 1.1025$
3	$(s + 0.4942)(s^2 + 0.4941s + 0.9942)$
4	$(s^2 + 0.2791s + 0.9865)(s^2 + 0.6737s + 0.2794)$
5	$(s + 0.2895)(s^2 + 0.1789s + 0.9883)(s^2 + 0.4684s + 0.4293)$
6	$(s^2 + 0.1244s + 0.9907)(s^2 + 0.3398s + 0.5577)(s^2 + 0.4641s + 0.1247)$

Solution: By (11.21) for the 1 dB ripple

$$\epsilon^2 = 10^{1/10} - 1 = 0.2589$$

from which

$$\epsilon = 0.5088$$

The attenuation can be found from (11.18), beginning with

$$|H_n(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n^2(\omega)}$$

The dB attenuation for high frequencies is given by $[\epsilon^2 C_n^2(\omega) \gg 1]$

$$-10 \log_{10}|H_n(j\omega)|^2 = 10 \log_{10} \epsilon^2 + 10 \log_{10} C_n^2(\omega) \geq 20$$

Therefore

$$\log_{10} \epsilon + \log_{10}[\cosh(n \cosh^{-1} 2)] \geq 1$$

so that

$$\log_{10}[\cosh(n \cosh^{-1} 2)] \geq 1 - \log_{10} \epsilon = 1 - (-.2935) = 1.2935$$

$$\cosh[(n \cosh^{-1} 2)] \geq 19.65$$

$$n \cosh^{-1} 2 \geq 3.670$$

from which we have

$$n \geq \frac{3.670}{1.317} = 2.8$$

Hence the conditions of the problem can be met with a third-order Chebyshev filter, and the normalized transfer function is

$$H_n(s) = \frac{0.4913}{0.4913 + 1.2384s + 0.9883s^2 + s^3} \tag{11.33}$$

With the denormalizing factor specified by

$$\omega_c = 2\pi \times 3000 = 18{,}849.54$$

the desired transfer function is

$$H(s) = H_n\left(\frac{s}{\omega_c}\right) \tag{11.34}$$

$$= \frac{0.4913}{0.4913 + 6.5699 \times 10^{-5}s + 2.7816 \times 10^{-9}s^2 + 1.4931 \times 10^{-13}s^3}$$

We could also proceed in the following manner. Using the data $\epsilon = 0.5088$ and $n = 3$, we obtain

$$\frac{1}{3}\sinh^{-1}\left(\frac{1}{\epsilon}\right) = \frac{1.4280}{3} = 0.4760$$

Therefore,

$$\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{0.5088}\right)\right] = 0.4942$$

$$\cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{0.5088}\right)\right] = 1.1154$$

and by (11.30b) and (11.30c) we obtain the following poles

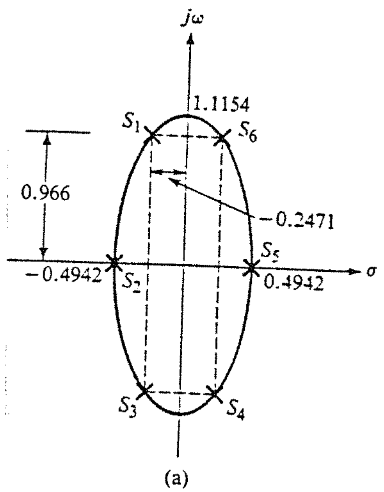
$$s_1 = \sigma_1 + j\omega_1 = -0.2471 + j0.9660$$

$$s_2 = \sigma_2 + j\omega_2 = -0.4942$$

$$s_3 = \sigma_3 + j\omega_3 = -0.2471 - j0.9660$$

When these pole positions are substituted into the transfer function expression

$$H_n(s) = \frac{0.4913}{(s - s_1)(s - s_2)(s - s_3)} \tag{11.35}$$



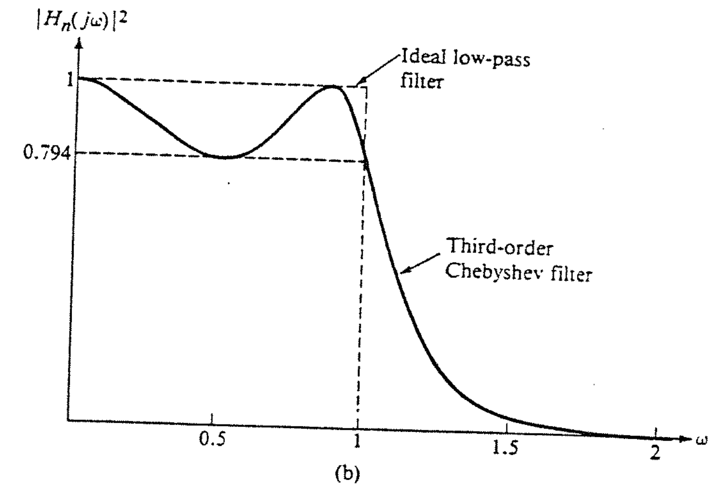


Figure 11.10
Illustrating Example 11.3.

the resulting form for $H_n(s)$ is precisely that given by (11.33). Figure 11.10a indicates graphically the location of the three poles, and Figure 11.10b shows a plot of the normalized third-order Chebyshev filter of (11.33).

$$|H_n(j\omega)|^2 = \frac{1}{1 + 0.2589(4\omega^3 - 3\omega)^2} \tag{11.36}$$

■

■ ■ ■

11-4 ELLIPTIC FILTERS

We have found that the Butterworth approximation possesses a monotonic characteristic in both the passband and the stopband, while the Chebyshev approximation has a magnitude response that varies between equal maximum and equal minimum values in the passband and decreases monotonically in the stopband. Moreover, because of the willingness to accept a ripple in the passband, the Chebyshev filter possesses sharper cutoff characteristics in the stopband.

Another type of approximation is characterized by a magnitude response that is equiripple in both the passband and the stopband, as shown in Figure 11.11. This approximation is given by the amplitude function

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2(\omega)}} \tag{11.37}$$

where $R_n(\omega)$ is a Chebyshev rational function. The roots of the rational function ω_k are related to the Jacobi elliptic sine functions and the resulting filter is called the **elliptic filter**.

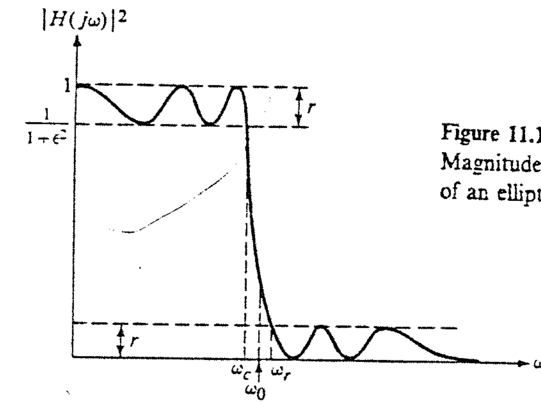


Figure 11.11
Magnitude-squared response of an elliptic filter.

We will not undertake a discussion of the elliptic function filter, but note that the response curve is an improvement over the Chebyshev and Butterworth filters.

■ ■ ■

11-5 PHASE CHARACTERISTICS

Our prior discussion has focused on the amplitude response of low-pass filters. These results show that in most respects the Chebyshev filter is superior to the Butterworth and in some cases the elliptic filter is superior to both. However, in these discussions, we have totally ignored the phase characteristics, which become progressively worse (less linear) as the amplitude response is improved. Often the phase characteristic is an important factor, since a linear phase response is necessary if one wishes to transmit a pulse through a network without distortion (although a time delay will ensue).

Instead of beginning with considerations of the amplitude function, it is possible to consider obtaining a realizable approximation to an ideal constant delay function e^{-s}. One method for obtaining a realizable approximation leads to an $H(s)$ specified in terms of Bessel polynomials. Pursuing this matter is beyond our present concerns.

■ ■ ■

11-6 LOW-PASS TO HIGH-PASS TRANSFORMATION

By an appropriate frequency transformation, the $H_n(s)$ for a normalized low-pass filter can be used to obtain $H_n(s)$ for a normalized high-pass filter. If we use p to denote the low-pass case and s to denote the high-pass case, then $H(s)$ can be obtained from $H(p)$ through the frequency transformation

$$p = \frac{\omega_{oh}}{s} \tag{11.38}$$

where ω_{oh} is a constant chosen to meet the specifications of the high-pass filter. Evidently $p = j0$ maps into $s = \pm j\infty$, and $p = \pm j1$ maps into $s = \mp \omega_{oh}$. For a Butterworth or Chebyshev filter, ω_{oh} is chosen as the passband cutoff frequency and will correspond to the cutoff frequency of 1 rad/s associated with these low-pass filters. For the elliptic filter, the normalized frequency ω_o is the geometric mean of the two edge frequencies shown in Figure 11.11.

■ ■ ■

11-7 LOW-PASS TO BANDPASS TRANSFORMATION

In parallel with the discussion in Section 11-6, $H(s)$ for a normalized low-pass filter can be used to obtain the $H(s)$ for a normalized bandpass filter when a proper low-pass to bandpass transformation is effected. The required transformation is

$$p = \frac{\omega_{ob}}{B}\left(\frac{s}{\omega_{ob}} + \frac{\omega_{ob}}{s}\right) \tag{11.39}$$

where ω_{ob} and B are constants to be chosen to satisfy certain frequency specifications of the bandpass filter. Suppose that we let $s = j\omega$ be the point that corresponds to $p = j1$, the cutoff point of the low-pass filter. Under this condition, (11.39) yields the two values corresponding to the cutoff values of the bandpass filter. These are, from solving (11.39) in the standard quadratic form,

$$\omega_1 = \frac{B}{2} - \sqrt{\left(\frac{B}{2}\right)^2 + \omega_{ob}^2} \quad \text{(a)}$$

$$\omega_2 = \frac{B}{2} + \sqrt{\left(\frac{B}{2}\right)^2 + \omega_{ob}^2} \quad \text{(b)} \tag{11.40}$$

The negative sign that appears in (11.40a) is ignored. Taking account of this negative value, we find from (11.40) that

$$B = \omega_2 - \omega_1 \tag{11.41}$$

which specifies the bandwidth of the filter. Also we find

$$\omega_{ob} = \sqrt{\omega_1 \omega_2} \tag{11.42}$$

which shows that ω_{ob} is the geometric mean of ω_1 and ω_2, the two cutoff frequencies.

It is important to realize that frequency transformations do not necessarily preserve the stability of filters. Unstable filters can be stabilized, but a better approach is to seek frequency transformations that will preserve the stability in the first place. We will not pursue this matter.

■ ■ ■

11-8 DIGITAL FILTERS

We have already noted that any device or process that will transform an input sequence of numbers into an output sequence of numbers might be called a digital filter. If we consider the digital filter to be a computational algorithm for carrying out this transformation process according to some prescribed rule, this rule is either a difference equation (as discussed in Chapter 8) or the convolution summation (discussed in Chapter 9). Digital filter design is concerned with the selection of the coefficients of the difference equation or with the unit sample response $h(k)$ used in the convolution summation.

As already noted, digital filter design often stems from analog filters of the low-pass and high-pass class by the use of a transformation that yields an equivalent z-plane expression for a given analog description in the s-plane or in the time domain. In essence this means that we establish a roughly equivalent sampled form for the given analog function. We will discuss the impulse invariant response method and also the use of a bilinear transformation for effecting this analog-digital transformation.

We discussed in Section 8-2 that the difference equation description of a discrete time system (filter) was of two types—FIR and IIR. The general form of the difference equation for the IIR system, given by (8.4), relates the present output value with immediate past values of the output and the present and past values of the input. This involves a recursive process using these present and past values to update the output. Similarly, if the output depends only upon the present and past values of the input, the system is nonrecursive or finite duration impulse response (FIR). More precisely, however, FIR and IIR describe digital filters relative to the length of their unit sample response sequence; it is possible to implement an FIR digital filter in a recursive fashion, and an IIR digital filter can be implemented in a nonrecursive fashion.

■ ■ ■

11-9 THE IMPULSE INVARIANT RESPONSE METHOD

Suppose that the system function of an analog filter that has certain desired properties is specified by

$$H(s) = \sum_{i=1}^{m} \frac{A_i}{(s + s_i)} \quad \text{(a)} \tag{11.43}$$

Assume that all poles are distinct, with the impulse response function being

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \sum_{i=1}^{m} A_i e^{-s_i t} \quad \text{(b)}$$

If $h(kT)$ is the corresponding sampled version of $h(t)$, then we can write

$$H(z) = \sum_{k=0}^{\infty} h(kT) z^{-k}$$

Therefore,

$$H(z) = \sum_{k=0}^{\infty} z^{-k} \sum_{i=1}^{m} A_i e^{-s_i kT} = \sum_{i=1}^{m} A_i \sum_{k=0}^{\infty} z^{-k} e^{-s_i kT}$$

which is, using the well-known formula of geometric series,

$$\boxed{H(z) = \sum_{i=1}^{m} \frac{A_i}{1 - e^{-s_i T} z^{-1}}} \tag{11.44}$$

A comparison of (11.43) and (11.44) shows that a continuous-time filter specified by the system function $H(s)$ transforms, via **impulse invariant techniques,** by setting

$$s + s_i = 1 - e^{-s_i T} z^{-1} \tag{11.45}$$

into a digital filter specified by $H(z)$. As already noted, some degree of approximation exists in this transformation because the digital filter is necessarily band-limited whereas $H(s)$, being a rational function of s, is not band-limited.

Let us examine how well the frequency response of the digital filter corresponds to the original analog filter frequency response. We use the fact that [see (4.57)]

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \qquad \omega_s = \frac{2\pi}{T}$$

The Laplace transform is

$$F_s(s) = \mathscr{L}\{f(nT)\} = \mathscr{L}\left[\sum_{n=-\infty}^{\infty} f(t)\,\delta(t - nT)\right]$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_0^{\infty} f(t) e^{-(s - jn\omega_s)t}\, dt = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(s - jn\omega_s) \tag{11.46}$$

where ω_s is the radian sampling frequency. Also, the Z-transform of $f(nT)$ is

$$\mathscr{Z}\{f(nT)\} = F(z)$$

But we know that $z = e^{sT}$ from Section 9-11, and thus

$$F(z) = F_s\left(s = \frac{1}{T} \ln z\right) \qquad z = e^{j\omega_d}, \quad \ln z = j\omega_d \tag{11.47}$$

where ω_d is the frequency in the discrete domain. We apply these results to (11.44) to write

$$H(z) = \sum_{i=1}^{m} \frac{A_i}{1 - e^{-s_i T} z^{-1}} = H_s\left(s = \frac{1}{T} \ln z\right) \triangleq \mathscr{L}\left\{\sum_{n=-\infty}^{\infty} h(t)\,\delta(t - nT)\right\}$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} H_s\left(s = \frac{1}{T} \ln z - jk\omega_s\right)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} H_s\left(\frac{j\omega_d}{T} - jk\omega_s\right) \tag{11.48}$$

From this equation it follows that in the base band

$$\frac{-\omega_s}{2} \le \omega_d \le \frac{\omega_s}{2} \qquad k = 0$$

the frequency response characteristic of the digital filter $H(z)$ will differ from that of the analog filter $H(s)$, the difference being the amount "added" or "folded in" from the additional terms of the form

$$H_S\left(\left[s = \frac{1}{T} \ln z\right] - jk\omega_s\right)$$

which make up the summation in (11.48). As the discussion of the sampling theorem in Chapter 7 demonstrates, no folding error exists if

$$|H_s(s)| \triangleq |H_s(j\omega)| = 0 \qquad |\omega| > \frac{\omega_s}{2}$$

and in this case the frequency response of the digital filter is identical with that of the continuous filter, when

$$H(e^{j\omega_d}) = \frac{1}{T} H_s(s) = \frac{1}{T} H_s\left(j\frac{\omega_d}{T}\right) \qquad |\omega_d| \le \pi \tag{11.49}$$

If $H_s(s)$ includes a repeated root, then, in addition to terms such as those in (11.43), we would have such terms as (see Table 6.1)

$$h(t) = \mathscr{L}^{-1}\left\{\frac{A}{(s + s_i)^r}\right\} = \frac{A}{(r-1)!} t^{r-1} e^{-s_i t} u(t) \tag{11.50}$$

The sampled version of this is

$$h(kT) = \begin{cases} \dfrac{A}{(r-1)!} (kT)^{r-1} e^{-s_i kT} & k \ge 0 \\ 0 & k < 0 \end{cases} \tag{11.51}$$

The corresponding Z-transform is

$$H(z) = \frac{A}{(r-1)!} T^{r-1} \sum_{k=0}^{\infty} k^{r-1} (e^{-s_i T} z^{-1})^k \tag{11.52}$$

In accordance with our discussion above, we can follow the following steps in carrying out a digital filter design:

1. A set of filter specifications are given.
2. Create an analog transfer function $H(s)$ that meets the specifications of Step 1.
3. Determine the impulse response of the analog filter by means of the Laplace inversion technique, $h(t) = \mathscr{L}^{-1}\{H(s)\}$.

4. Sample $h(t)$ at T second intervals, thus creating a sequence $\{h(kT)\}$
5. Deduce $H(z)$ of the resulting digital filter by taking the Z-transform of the discrete function $h(kT)$, $H(z) = \sum_{k=0}^{\infty} h(kT)z^{-k}$.

EXAMPLE 11.4

Determine the digital equivalent of the first-order Butterworth filter. The cut-off frequency is $20/2\pi$ Hz.

Solution: The normalized analog transfer function is found in Table 11.2 for the designated filter

$$H_n(s) = \frac{1}{s+1}$$

The system transfer function is given by

$$H\left(\frac{s}{\omega_c}\right) = H\left(\frac{s}{2\pi \times \frac{20}{2\pi}}\right) = \frac{1}{\frac{s}{20}+1} = \frac{20}{s+20}$$

The impulse response of this filter is given by

$$h(t) = \mathcal{L}^{-1}\left\{H\left(\frac{s}{20}\right)\right\} = \mathcal{L}^{-1}\left\{\frac{20}{s+20}\right\} = 20e^{-20t} \qquad t \geq 0$$

The Z-transform of the discrete function $h(kT)$ is [see (9.3) and (11.44)]

$$H(z) = 20\sum_{k=0}^{\infty} e^{-20kT}z^{-k} = 20\sum_{k=0}^{\infty}(e^{-20T}z^{-1})^k = \frac{20}{1-e^{-20T}z^{-1}}$$

To proceed, for the specific sampling time $T = 0.005$ s we deduce the absolute value of the analog transfer function

$$\left|H\left(\frac{j\omega}{20}\right)\right| = \frac{1}{\left|\frac{j\omega}{20}+1\right|} = \frac{1}{\sqrt{1+\left(\frac{j\omega}{20}\right)^2}}$$

t is convenient to plot this function versus ω_d, where $\omega = \omega_d/T$. The equivalent xpression becomes

$$|H(j\omega_d)| = \frac{1}{\sqrt{1+(10\omega_d)^2}}$$

his is shown plotted as the lower curve in Figure 11.12.

The equivalent expression for the discrete function is [see (11.44)]

$$|H(e^{j\omega_d})| = \frac{20}{|1-0.9048e^{-j\omega_d}|}$$

ote, however, that $H(e^{j0}) = 210.084$ is the value of $H(e^{j\omega_d})$ for $\omega_d = 0$ (or

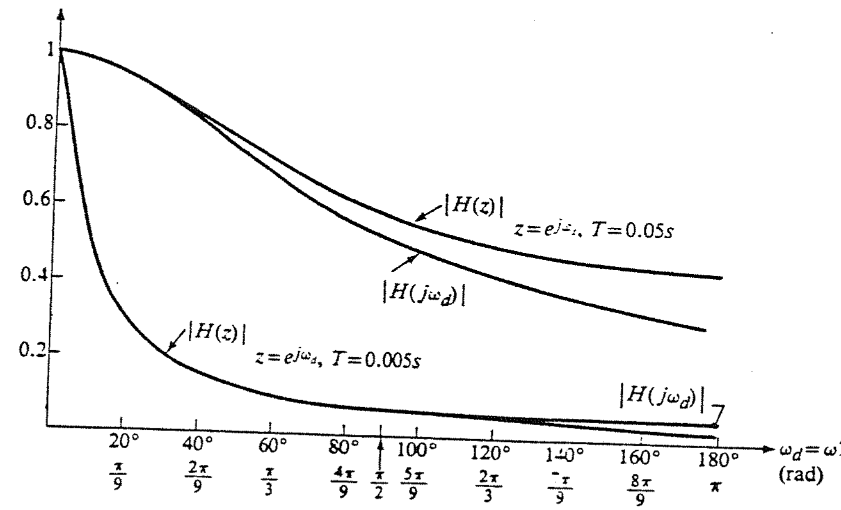


Figure 11.12 Comparison between the analog and its corresponding digital filter for two different sampling times.

$z = 1$). We thus consider the normalized function

$$\frac{|H(e^{j\omega_d})|}{210.084} = \frac{20}{210.084}\frac{1}{[(1-0.9048\cos\omega_d)^2+(0.9048\sin\omega_d^2)]^{1/2}}$$

which is also plotted in Figure 11.12 (lower curve).

For the case when $T = .05$ s

$$H(e^{j\omega_d}) = \frac{20}{1-0.3679e^{-j\omega_d}}$$

The value of $H(e^{j0}) = 31.641$ at $\omega_d = 0$, and the normalized relation becomes

$$\frac{|H(e^{j\omega_d})|}{31.641} = \frac{20}{31.641}\frac{1}{[(1-0.3679\cos\omega_d)^2+(0.3679\sin\omega_d)^2]^{1/2}}$$

This relation is plotted in Figure 11.12 (upper curve) together with $|H(j\omega_d)| = 1/\sqrt{1+\omega_d^2}$ since $\omega = \omega_d/0.05$.

Figure 11.12 clearly shows the effect of sampling times T. The upper curves start to separate at about $\omega_d = 30°$, which is equal to $30/57.2958 = 0.5236$ radians; hence at $T = 0.05$ s the error begins to be pronounced at a frequency roughly equal to $f = \omega_d/2\pi T = 0.5236/(2\pi \times .05) = 1.666$ Hz. Furthermore, the error for the lower curves begins at about 90° or 1.571 radians; hence for $T = .005$ s the error starts at the frequency $f = 1.571/(2\pi \times .005) = 50$ Hz, which is above the cutoff frequency. Thus an aliasing error is present in one case but not in the other. This indicates that good accuracy is achieved with our digital filter if we choose the sampling time $T \ll 0.55$ s.

Had we continued the plots of the above curves from $\omega_d = 180° = \pi$ to $\omega_d = 2\pi$ the curves of $H(z)$ would have repeated themselves, being symmetric about $\omega_d = \pi$. However, the curves for $|H(j\omega)|$ would have continued to decrease, as expected from the form of $H(j\omega)$. ■

EXAMPLE 11.5

Determine the digital equivalent of a Butterworth third-order low-pass filter, and find T if the sampling frequency is 15 times the cutoff frequency.

Solution: The normalized system function of this filter is deduced from Table 11.1 or Table 11.2. It is

$$H_n(s) = \frac{1}{1 + 2s + 2s^2 + s^3} = \frac{1}{(s+1)(s^2 + s + 1)} = \frac{A_1}{s + s_1} + \frac{A_2}{s + s_2} + \frac{A_3}{s + s_3}$$

where

$$s_1 = 1 \qquad s_2 = \frac{1}{2}(1 - j\sqrt{3}) \qquad s_3 = \frac{1}{2}(1 + j\sqrt{3})$$

Evaluating the constants A in this expression, we find that

$$A_1 = 1 \qquad A_2 = \frac{2}{-3 + j\sqrt{3}} \qquad A_3 = A_2^* = \frac{2}{-3 - j\sqrt{3}}$$

The impulse response of this system is given by

$$h(t) = A_1 e^{-t} + A_2 e^{-s_2 t} + A_3 e^{-s_3 t}$$

Further, to find the sampling time $\omega_s = 2\pi/T = 15 \cdot \omega_c = 15$, from which the sampling time is $T = 0.419$. The sampled impulse response function is

$$h(0.419k) = A_1 e^{-0.419k} + A_2 e^{-0.419 s_2 k} + A_3 e^{-0.419 s_3 k}$$

The Z-transform of this function is

$$H(z) = \frac{z}{z - e^{-0.419}} + A_2 \frac{z}{z - e^{-s_2 0.419}} + A_3 \frac{z}{z - e^{-0.419 s_3}}$$

$$= \frac{z}{z - 0.658} + \frac{z(-z + 0.924)}{z^2 - 1.516z + 0.658}$$

$$= \frac{0.066z^{-1} + 0.0497z^{-2}}{1 - 2.174z^{-1} + 1.656z^{-2} - 0.433z^{-3}}$$

realization of this filter is shown in Figure 11.13. ■

In general, a digital filter designed using the invariant impulse method results a transfer function in the form of the ratio of two polynomials. The difference equation written from this sytem function is a recursive expression and the filter so realized is an infinite impulse response (IIR) filter. But note that the invariant impulse response method is equivalent to analog filtering of an impulse-sampled input signal. We have already discussed the fact that for a sampled signal approximate the continuous signal, the sampling rate (Nyquist rate) must be least twice the highest frequency component contained in the signal. However,

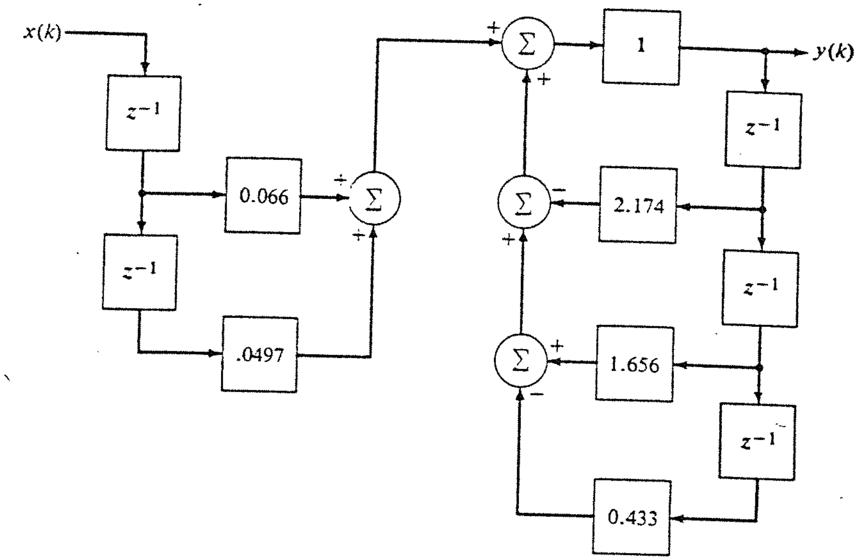


Figure 11.13 Digital filter realization of a third-order Butterworth low-pass filter.

a practical analog filter $H(s)$ is never strictly band-limited. Therefore, an aliasing error will occur when this design method is used. As a practical matter, if the sampling frequency is 5 or more times the cutoff frequency of the low-pass analog filter, the aliasing effect on the frequency response is extremely small.

If $H(z)$ is used to obtain the appropriate $h(k)$, and if this is used in a convolution summation expansion, the result will be a finite-duration impulse response (FIR) filter, if $h(k)$ is truncated to N terms. The FIR filter obtained this way suffers from the fact that a large number of samples will be required to approximate $h(k)$. This same limitation exists if $h(k)$ is to approximate the $h(t)$ of an analog filter directly. The disadvantage of this procedure is that the computation time required to generate an output sample is longer than with a recursive filter.

Another problem with an FIR realization developed this way results from the abrupt truncation of the unit response function, which introduces the problem of the Gibbs phenomenon. In such cases one may use a window function to alleviate the problem by smoothing the sampled data in the neighborhood of the truncation region. An advantage of the FIR approach is that any errors in the computations are not recycled into subsequent calculations.

EXAMPLE 11.6

Find the digital filter equivalent of the RC network in Figure 11.14 where the 3 dB cutoff frequency is 1/5 of the sampling frequency.

Solution: By a simple determination, the transfer function is

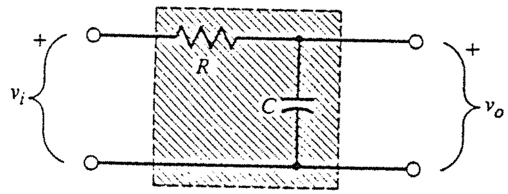


Figure 11.14
The RC network under survey.

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{\omega_c}{s + \omega_c}$$

where

$$\omega_c = \frac{1}{RC} = \frac{1}{5}\frac{2\pi}{T}$$

We then write $H(s)$ in the form

$$H(s) = \frac{\frac{0.4\pi}{T}}{s + \frac{0.4\pi}{T}}$$

By the invariant impulse method, the digital equivalent $H(z)$ is (for numerical convenience it was multiplied by T)

$$\frac{Y(z)}{X(z)} \triangleq H(z) = \frac{T \times \frac{0.4\pi}{T}}{1 - e^{-0.4\pi}z^{-1}} = \frac{1.2566}{1 - 0.285z^{-1}}$$

The equivalent difference equation is then found from

$$(1 - 0.285z^{-1})Y(z) = 1.2566X(z)$$

from which we can write

$$y(k) = 1.2566x(k) + 0.285y(k-1)$$

■

EXAMPLE 11.7
The normalized transfer function (11.33) of the third-order Chebyshev filter of Example 11.3 is

$$H_n(s) = \frac{0.4913}{0.4913 + 1.2384s + 0.9883s^2 + s^3} \tag{11.53}$$

Determine the following:

a. The corresponding impulse-invariant digital filter.

b. The amplitude characteristics of the digital filter.

c. The impulse response $h(t)$ of $H_n(t)$.

d. The unit sample response $h(kT)$ of the digital filter for $T = 1$.

Solution: **a.** Begin with the Chebyshev function in factored form

$$H_n(s) = \frac{0.4913}{(s + 0.2471 - j0.9660)(s + 0.2471 + j0.9660)(s + 0.4942)}$$

This is written in partial fraction form

$$H_n(s) = \frac{A_1}{s + 0.2471 - j0.9660} + \frac{A_2}{s + 0.2471 + j0.9660} + \frac{A_3}{s + 0.4942} \tag{11.54}$$

where

$$A_1 = \frac{0.4913}{2j \times 0.9660(0.2471 + j0.9660)}$$

$$A_2 = \frac{0.4913}{-2j \times 0.9660(0.2471 - j0.9660)}$$

$$A_3 = \frac{0.4913}{(0.2471 + j0.9660)(0.2471 - j0.9660)} = 0.4942$$

Now set $s + s_i = 1 - e^{-s_iT}z^{-1}$ in (11.54), which gives the impulse invariant digital filter representation,

$$H_n(z) = \frac{A_1}{1 - e^{-0.2471T + j0.9660T}z^{-1}} + \frac{A_2}{1 - e^{-0.2471T - j0.9660T}z^{-1}} + \frac{A_3}{1 - e^{-0.4942T}z^{-1}} \tag{11.55}$$

b. The corresponding frequency response is obtained by substituting $z = \exp(j\omega_d) = \exp(j\omega T)$ in (11.55). The resulting expression is

$$H_n(e^{j\omega T}) = \frac{A_1}{1 - e^{-0.2471T + j0.9660T}e^{-j\omega T}} + \frac{A_2}{1 - e^{-0.2471T - j0.9660T}e^{-j\omega T}} + \frac{A_3}{1 - e^{-0.4942T}e^{-j\omega T}} \tag{11.56}$$

This expression is plotted in Figure 11.15 for $T = 1$.

c. Apply the inverse Laplace transform to (11.54) to find the impulse response. This is given by

$$\begin{aligned} h(t) &= A_1e^{-0.2471t}e^{j0.9660t} + A_2e^{-0.2471t}e^{-j0.9660t} + A_3e^{-0.4942t} \\ &= e^{-0.2471t} \times [A_1e^{j0.9660t} + A_1^*(e^{j0.9660t})^*] + 0.4942e^{-0.4942t} \\ &= e^{-0.2471t} \times 0.51\cos(0.966t - 165.65^\circ) + 0.4942e^{-0.4942t} \end{aligned} \tag{11.57}$$

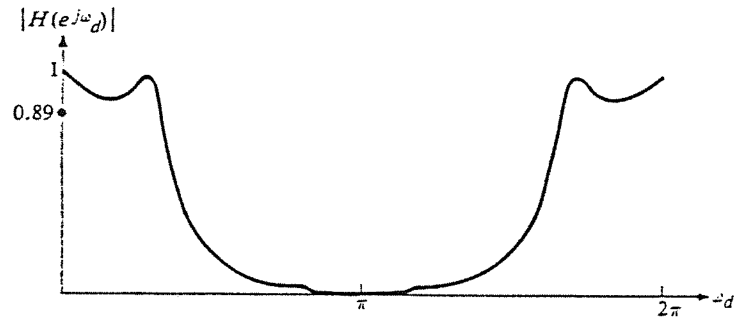


Figure 11.15 The frequency response of the Chebyshev filter under review.

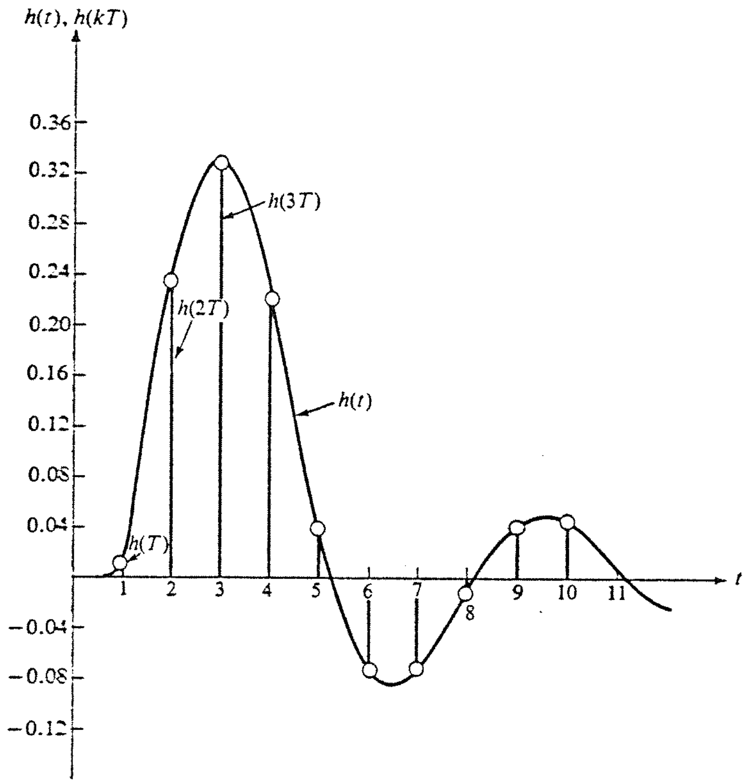


Figure 11.16 The impulse responses $h(t)$ and $h(kT)$, $T = 1$.

Figure 11.16 shows the impulse response corresponding to (11.57). ■

■ ■ ■

11-10 THE BILINEAR TRANSFORMATION

To circumvent the "folding" problem of the impulse invariant response transformation noted in the previous section, a transformation from the s-plane to

the p-plane can be employed that will map the entire s-plane into a horizontal strip in the p-plane bounded by the lines $p = -j\omega_s/2$ and $p = +j\omega_s/2$. Moreover, since $H(z)$ is also periodic in ω (with period ω_s), this transformation may also cause $H(s)$ to be mapped identically into each of the other horizontal strips bounded by the lines $p = j(n - \frac{1}{2})\omega_s$ and $p = j(n + \frac{1}{2})\omega_s$, where n is an integer (see Figure 11.17). A transform having the requisite properties is the bilinear transformation, which is defined by

$$s = \frac{2}{T}\tanh\frac{pT}{2} \tag{11.58}$$

Using the identity

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

we have

$$s = \frac{2}{T}\frac{e^{pT/2} - e^{-pT/2}}{e^{pT/2} + e^{-pT/2}}$$

Upon substituting the quantity $z = e^{pT}$ in this expression, we have

$$\boxed{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)} \quad \text{(a)} \qquad \text{(11.59)}$$

and

$$\boxed{z = \frac{1 + Ts/2}{1 - Ts/2}} \quad \text{(b)}$$

In terms of the z-plane, this algebraic transformation uniquely maps the left half of the s-plane into the interior of the unit-circle in the z-plane, as shown in Figure 11.17. Because no folding occurs, no folding errors will arise. However, a shortcoming of this transformation is that the frequency response is nonlinear (that is, warped) in the digital domain.

If we insert $z = \exp(j\omega_d)$ into (11.59a), we obtain a relationship between the frequency ω of the analog filter and ω_d of the digital filter. We find that

$$s \triangleq \sigma + j\omega = \frac{2(e^{j\omega_d} - 1)}{T(e^{j\omega_d} + 1)} = \frac{2e^{j\omega_d/2}(e^{j\omega_d/2} - e^{-j\omega_d/2})}{Te^{j\omega_d/2}(e^{j\omega_d/2} + e^{-j\omega_d/2})}$$
$$= j\frac{2}{T}\tan\frac{\omega_d}{2} \tag{11.60}$$

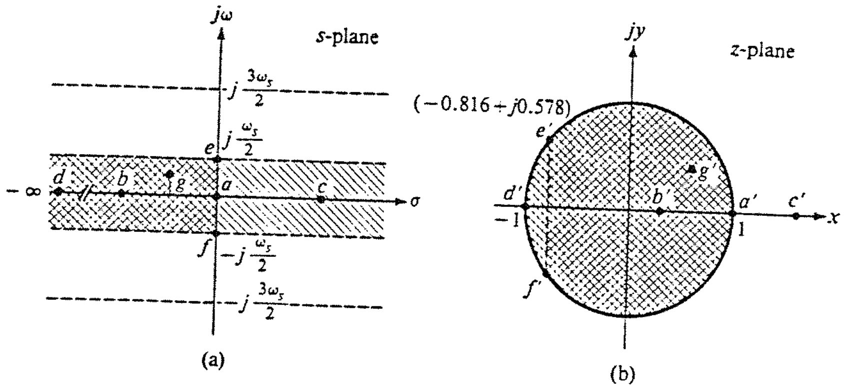


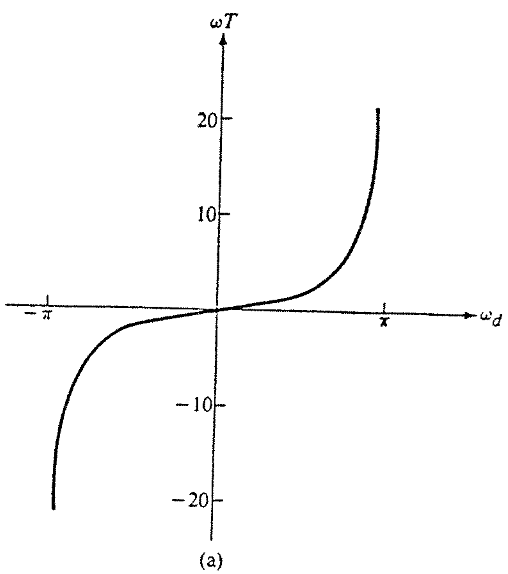
Figure 11.17
(a) A step in finding the inverse-Z transform. (b) Mapping of the s-plane onto the z-plane.

From this equation, by separately equating real and imaginary parts, we obtain

$$\sigma = 0 \qquad \text{(a)}$$

$$\omega = \frac{2}{T}\tan\frac{\omega_d}{2} \qquad \text{(b)} \qquad (11.61)$$

Observe that the relationship between the two frequencies ω and ω_d is a nonlinear one. This is the **warping** effect. Figure 11.18a shows the warping relationship graphically. It is evident from the plot that the sampling time T changes



gure 11.18
raphic illustration of ω d ω_d in bilinear transrmation.

the ωT axis by stretching or compressing it. Figure 11.18b below clearly shows the warping effect. Note that the shapes are similar, but the higher frequency bands are reduced disproportionately.

EXAMPLE 11.8

Determine the characteristics of a digital filter if the corresponding analog filter has the transfer function

$$H(s) = \frac{H_0}{s - s_1} \qquad (11.62)$$

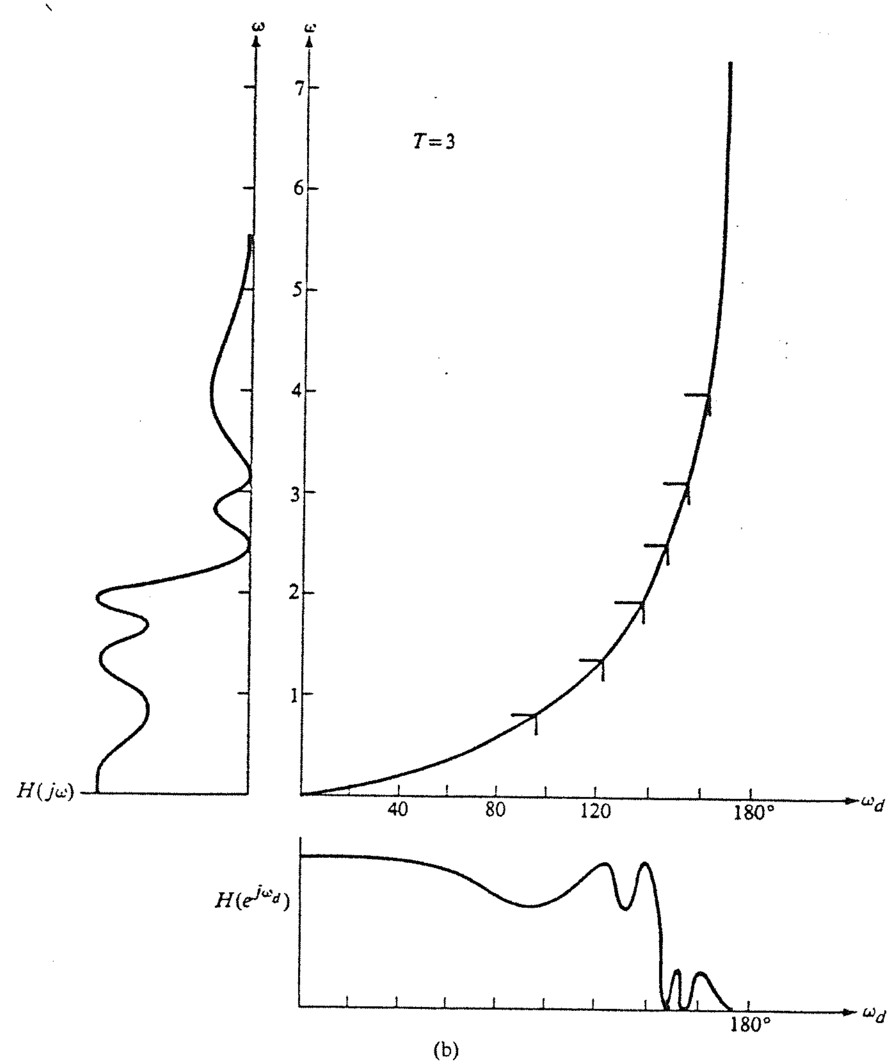


Figure 11.18 (continued)

Solution: By use of (11.59) we obtain

$$H(z) = H(s)|_{s=2(z-1)/T(z+1)} = \frac{H_0}{\frac{2}{T}\frac{z-1}{z+1} - s_1}$$

$$= H_0 T \frac{z+1}{(2 - s_1 T)z - (2 + s_1 T)} \qquad (11.63)$$

To illustrate the variation graphically, we set $H_0 = 0.8$, $s_1 = 0.8$, and $T = 1$. Figures 11.19a and 11.19b give the frequency response of the analog filter and the digital filter, respectively. The two transfer functions are given by

$$|H(s)| = \frac{0.8}{\sqrt{\omega^2 + (0.8)^2}} \qquad \text{(a)}$$

$$|H(z)| = 0.8 \frac{\sqrt{(\cos \omega_d + 1)^2 + \sin^2 \omega_d}}{\sqrt{(1.2 \cos \omega_d - 2.8)^2 + (1.2 \sin \omega_d)^2}} \qquad \text{(b)} \qquad (11.64)$$

The figures show the corresponding frequencies of the two filters, including the cutoff value of the digital filter. The nonlinear relation between the two curves is clearly evident. ■

EXAMPLE 11.9

We wish to design a digital Butterworth filter that will meet the following conditions:

a. The 3 dB cutoff point ω_d is to occur at 0.4π rad/s.

b. $T = 50\ \mu s$.

c. At $2\omega_d$ the attenuation is to be 15 dB.

Solution: First we find the analog equivalent criteria for the requisite digital filter. Thus we have, using (11.61)

$$\omega_c = \frac{2}{T}\tan\frac{\omega_d}{2} = \frac{2}{50 \times 10^{-6}}\tan\frac{0.4\pi}{2} = 29.1 \times 10^3 \text{ rad/s}$$

$$\omega|_{15\text{ dB}} = \frac{2}{T}\tan\frac{2\omega_d}{2} = \frac{2}{50 \times 10^{-6}}\tan\frac{2 \times 0.4\pi}{2} = 123.1 \times 10^3 \text{ rad/s}$$

Now using (11.1) with $\omega = \omega/\omega_c$, we obtain

$$-10\log|H(j123.1 \times 10^3)|^2 = -10\log\left|\frac{1}{1 \div \left(\frac{123.1 \times 10^3}{29.1 \times 10^3}\right)^{2n}}\right| \geq 15$$

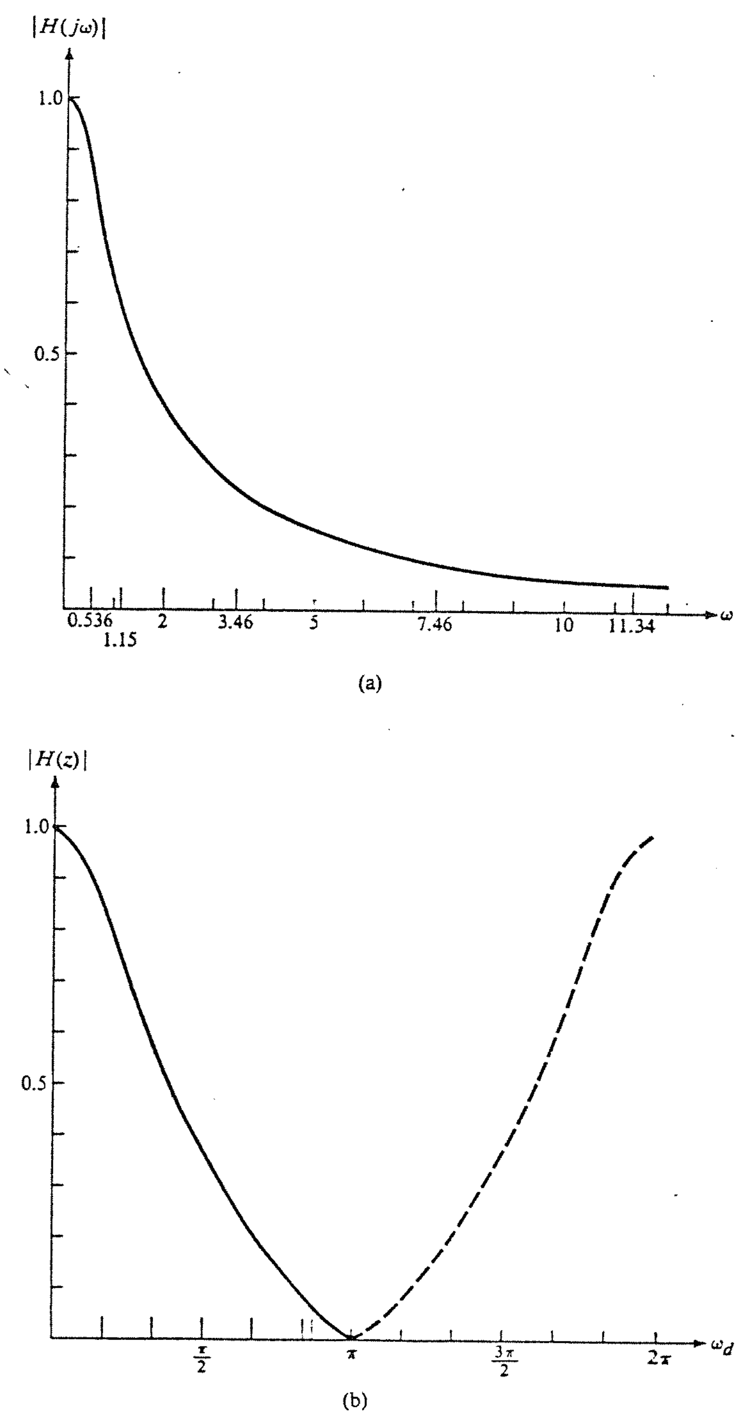


Figure 11.19 Frequency response of a first-order analog filter and its corresponding digital filter.

from which

$$1+\left(\frac{123.1\times 10^3}{29.1\times 10^3}\right)^{2n}\geq 31.62$$

From this we find that

$$n \geq 1.18$$

hence the minimal order of the Butterworth filter to meet the specifications is $n = 2$. The normalized form of the filter is, from Table 11.1,

$$H_n(s)=\frac{1}{s^2+1.4142s+1}$$

The analog filter satisfying the specifications is

$$H(s)=H_n\left(\frac{s}{\omega_c}\right)=H_n\left(\frac{s}{29.1\times 10^3}\right)=\frac{1}{\left(\frac{s}{29.1\times 10^3}\right)^2+\left(\frac{1.4142}{29.1\times 10^3}\right)s+1}$$

$$H(s)=\frac{(29.1\times 10^3)^2}{s^2+41.1532\times 10^3 s+(29.1\times 10^3)^2}$$

Introduce (11.59) into this equation, which leads to

$$H(z)=H(s)|_{s=2(z-1)/T(z+1)}$$

$$=\frac{29.1^2\times 10^6}{\left(\frac{2}{T}\right)^2\frac{(z-1)^2}{(z+1)^2}+41.1532\times 10^3\left(\frac{2}{T}\right)\frac{(z-1)}{(z+1)}+29.1^2\times 10^6}$$

$$=\frac{2.117+4.234z^{-1}+2.117z^{-2}}{10.232-3.766z^{-1}+2.002z^{-2}}$$

This is the final digital Butterworth design. ■

EXAMPLE 11.10

Determine the characteristics of a digital Butterworth filter to meet the specifications of Example 11.2. The sampling frequency is 10 times the cutoff frequency.

Solution: We find that we must choose a filter with $n = 4$ to meet the conditions of the problem. The normalized function is then known to be

$$H(s)=\frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$$

which can also be written

$$H(s)=\frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$

Now apply the bilinear z-transformation for the normalized function

$$s=\frac{2}{10}\left[\frac{(z-1)}{(z+1)}\right]=A\frac{z-1}{z+1}\qquad A=0.2$$

so that

$$H(z)=\frac{1}{\left(A^2\left[\frac{z-1}{z+1}\right]^2+0.7654A\left[\frac{z-1}{z+1}\right]+1\right)}\times\frac{1}{\left(A^2\left[\frac{z-1}{z+1}\right]^2+1.8478A\left[\frac{z-1}{z+1}\right]+1\right)}$$

Expand this expression to find

$$H(z)=\frac{0.5946(z+1)^4}{(z^2+0.9943z+0.7434)(z^2+1.3621z+0.4756)}$$ ■

■ ■ ■

†11-11 PREWARPING

Because of frequency warping, as discussed earlier, the bilinear z-transformation is most useful in obtaining filter approximations for continuous filters whose magnitude characteristics can be divided along the frequency scale into successive stop and passbands, where the loss or gain is essentially constant in the band. Compensation can be made for the effect of warping by prewarping the continuous filter design in such a way that upon applying the bilinear transformation, the critical frequencies will be shifted back to the desired values.

To examine the prewarping process in general terms (see Example 11.9) suppose that the system function $H(s)$ is expressed in partial-fraction form, with a typical term being $H_1/(s-p_1)$. By applying the bilinear z-transformation (11.59) to such a term, we obtain

$$\frac{H_1}{s-p_1}\rightarrow\frac{H_1T(z+1)}{(2-p_1T)z-2-p_1T}\qquad(11.65)$$

A partial-fraction expansion of the right-hand expression gives

$$H_b(z)=\frac{H_1T}{2-p_1T}+\frac{\left[\frac{4H_1T}{4-p_1^2T^2}\right]}{\left[z-\frac{2+p_1T}{2-p_1T}\right]}\qquad(11.66)$$

This shows that the bilinear z-transformation results in the pole in the z-plane being at $(2+p_1T)/(2-p_1T)$ rather than at e^{p_1T}, as in the impulse invariant response method. To prewarp the function, we wish to apply a transformation that moves the pole location of the digital filter to e^{p_1T}, the pole of the original filter. The transformation sequence is the following: first the transfer function is modified by moving the pole [see (11.58)]

$$\frac{H_1}{s-p_1} \rightarrow \frac{H_1}{s - \dfrac{2}{T}\tanh\dfrac{p_1 T}{2}} \tag{11.67}$$

and then applying the bilinear z-transformation to this modified transfer function. This leads, using (11.59), to

$$\frac{H_1}{s - \dfrac{2}{T}\tanh\dfrac{p_1 T}{2}} \rightarrow \frac{H_1 T(z+1)}{2\left[z - 1 - (z+1)\tanh\dfrac{p_1 T}{2}\right]} \tag{11.68}$$

By simple arithmetic manipulation it can be shown that the transfer function becomes

$$H_{pb}(z) = \frac{H_1 T(1+e^{p_1 T})}{4}\left[\frac{z+1}{z - e^{+p_1 T}}\right] \tag{11.69}$$

We will consider applying the bilinear z-transformation to the digitization of the Butterworth low-pass filter. The magnitude-squared characteristic of these filters is written [see (11.2)]

$$H(s)H(-s) = \frac{1}{1 + (-1)^n\left(\dfrac{s}{\omega_c}\right)^{2n}} \tag{11.70}$$

where ω_c is the cutoff frequency. From (11.61) replace ω_c by its prewarped value

$$\omega_c = \frac{2}{T}\tan\frac{(\omega_c T)}{2} \tag{11.71}$$

Then the magnitude squared characteristic of the filter is

$$H(s) \times H(-s) = \frac{1}{1 + (-1)^n\left[\dfrac{s}{\dfrac{2}{T}\tan\dfrac{\omega_c T}{2}}\right]^{2n}} \quad \text{(a)} \tag{11.72}$$

which becomes, by (11.59),

$$|H(z)|^2 = \frac{1}{1 + (-1)^n\left[\dfrac{\dfrac{z-1}{z+1}}{\tan\dfrac{\omega_c T}{2}}\right]^{2n}} \quad \text{(b)}$$

To find the system function $H(z)$ from this, we write the expression

$$|H(z)|^2 = \frac{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right)}{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right) + (-1)^n\left[\dfrac{z-1}{z+1}\right]^{2n}} \tag{11.73}$$

or

$$|H(z)|^2 = \frac{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right)[z+1]^{2n}}{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right)[z+1]^{2n} + (-1)^n[z-1]^{2n}}$$

The denominator polynomial in this expression can be expanded to the form

$$\left[\tan^{2n}\left(\frac{\omega_c T}{2}\right) + (-1)^n\right]\left[z^{2n} + \beta\binom{2n}{1}z^{2n-1} + \binom{2n}{2}z^{2n-2} + \beta\binom{2n}{3}z^{2n-3} + \binom{2n}{4}z^{2n-4} + \cdots + \beta\binom{2n}{2n-1}z + 1\right] \tag{11.74}$$

where

$$\beta = \frac{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right) - (-1)^n}{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right) + (-1)^n} \qquad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

A calculation shows that the $2n$ roots of (11.74) are given by

$$p_i = \frac{1 - \tan^2\left(\dfrac{\omega_c T}{2}\right) + j2\tan\left(\dfrac{\omega_c T}{2}\right)\sin\theta_i}{1 - 2\tan\left(\dfrac{\omega_c T}{2}\right)\cos\theta_i + \tan^2\left(\dfrac{\omega_c T}{2}\right)} \qquad i = 1, 2, \ldots, 2n \tag{11.75}$$

where

$$\theta_i = \begin{cases}(i-1)\pi/n & n \text{ odd}\\ (2i-1)\pi/2n & n \text{ even}\end{cases}$$

The squared gain factor is then

$$|H(z)|^2 = \frac{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right)}{\tan^{2n}\left(\dfrac{\omega_c T}{2}\right) + (-1)^n}\left[\frac{(z+1)^{2n}}{(z-p_1)(z-p_2)\cdots(z-p_{2n})}\right] \tag{11.76}$$

Now, if $p = re^{j\theta}$ is a root of the denominator polynomial, then so are $re^{-j\theta}$, $(1/r)e^{j\theta}$ and $(1/r)e^{-j\theta}$. Hence $|H(z)|^2$ will have n poles inside and n poles outside the unit circle ($r = 1$). If $p_1, \ldots, p_n$ denote the poles inside the unit circle, then

$$H(z) = \frac{b(z+1)^n}{(z-p_1)(z-p_2)\cdots(z-p_n)} \quad \text{(a)} \tag{11.77}$$

where b is selected so that $H(1) = 1$; therefore

$$b = \frac{(1 - p_1)(1 - p_2)\cdots(1 - p_n)}{2^n} \qquad \text{(b)}$$

The remaining poles, $p_{n-1}, \ldots, p_{2n}$ associated with $|H(z)|^2$ are outside the unit circle. By selecting $H(z)$ according to (11.77) for bounded input signal, the output signal goes to zero for $n \to \infty$; this establishes the bounded-input, bounded-output (bibo) stability property, a matter to be considered in Section 12-17.

EXAMPLE 11.11

Determine the characteristics of a digital Butterworth filter that has 20 dB attenuation at a frequency of 2.6 times its cutoff frequency. The sampling frequency is 10 times the cutoff frequency. Assume that prewarping has been employed.

Solution: The product

$$\omega_c T = 2\pi \times 3000 \frac{1}{30{,}000} = \frac{2\pi}{10} = \frac{\pi}{5}$$

From (11.72) we write

$$|H(e^{j\omega T})|^2 = \frac{1}{1 + (-1)^n \left[\dfrac{e^{j\omega T} - 1}{e^{j\omega T} + 1} \Big/ \tan\left(\dfrac{\omega_c T}{2}\right) \right]^{2n}}$$

$$= \frac{1}{1 + (-1)^{2n} \left[\dfrac{\tan^{2n}\left(\dfrac{\omega T}{2}\right)}{\tan^{2n}\left(\dfrac{\pi}{10}\right)} \right]}$$

From the requirement that the attenuation is 20 dB down (= 0.01) when $\omega T = 2.6\pi/5$, we have

$$|H(e^{j\omega T})|^2 = \frac{1}{1 + \dfrac{\tan^{2n} 0.26\pi}{\tan^{2n} 0.1\pi}} = |0.01|^2$$

from which

$$10^4 \leq 1 + (3.28)^{2n}$$

From this we find that $n \geq 4$. Also, from (11.75) we find the four roots contained within the unit circle to be:

$$z_1, z_2 = 0.6604 \pm j0.4432$$

$$z_3, z_4 = 0.5243 \pm j0.1458$$

Thus the Butterworth function is, from (11.77),

$$H(z) = \frac{b(z + 1)^4}{(z^2 - 1.3208z + 0.6325)(z^2 - 1.0486z + 0.2972)}$$

where b is the value specified by (11.77b). ■

■ ■ ■

11-12 FINITE IMPULSE RESPONSE (FIR) FILTERS

The design considerations in the previous two sections were based largely on the impulse response of analog filters, and these led to transfer functions of the IIR (recursive filter) type. We now wish to present two methods for designing FIR filters (nonrecursive), whose present output is computed by using only the present and past inputs, but none of its previous outputs. Because no feedback is present, this type of filter is stable. Furthermore, such filters are associated with linear phase characteristics, and so phase distortion in the output may be eliminated.

Two related methods will be discussed: (a) Fourier series method, and (b) the DFT method. Our discussion will be confined to a low-pass filter. Transformations for other types are discussed below. Refer to Fig. 11.20.

In the Fourier series method, the specified $H(\omega)$ is expanded into a Fourier expansion, assuming zero axis symmetry, thereby including only cosine terms. The procedure now continues as follows:

1. Truncate the series expansion to N terms and evaluate the coefficients.
2. Write the cosine terms in the expansion in exponential form and then write the exponential terms as functions of z through the transformation $z = \exp(jn\theta)$. This yields an expression for $H'(z)$.
3. Multiply the expression for $H'(z)$ by z^{-N} to yield $H(z)$, the system function for the FIR filter. Recall that z^{-N} is just a phase factor which will not alter the amplitude expression for $H(z)$.

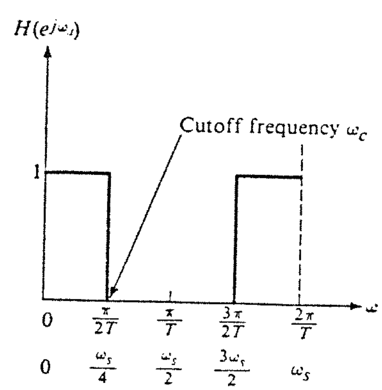


Figure 11.20
Ideal low-pass filter characteristics.

In the DFT method, we sample the frequency response function at distances $1/NT$. This provides a sequence $\{H(n\Omega)\}$ which will be the DFT of an impulse sequence $\{h(kT)\}$. From the results obtained in Chapter 10, we write (10.4)

$$H(n\Omega) = \sum_{k=0}^{N-1} h(kT)e^{-j(2\pi kn/N)} = \sum_{k=0}^{N-1} h(kT)e^{-jn\Omega kT}$$

$$k = 0, 1, \ldots, N-1 \qquad \Omega = \frac{\omega_s}{N} = \frac{2\pi}{NT} \qquad \omega_s = \frac{2\pi}{T} \tag{11.78}$$

$$h(kT) = \frac{1}{N}\sum_{n=0}^{N-1} H(n\Omega)e^{jn\Omega kT} \qquad n = 0, 1, \ldots, N-1 \tag{11.79}$$

The steps required to design the appropriate discrete filter are:

1. From the given amplitude frequency characteristics, obtain the desired sampled values $H(n\Omega)$ for a specified N.
2. Incorporate into (11.79) the values found in Step 1, and find the sequence $\{h(kT)\}$.
3. Use the values $\{h(kT)\}$ in the Z-transform relationship given in (9.3) to find $H(z)$.
4. Plot $|H(e^{j\omega T})|$ versus ω to obtain the amplitude characteristics.

EXAMPLE 11.12
Find the nonrecursive filter corresponding to the ideal filter shown in Figure 11.21a for $N = 16$.

Solution: From Figure 11.21a the sampled values of $H(e^{j\omega T})$ are:

$\{1, 1, 0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.5, 1\}$

Observe that we have used the average value of the function at the point of discontinuity. Next apply (11.79) to obtain

$$h(0T) = \frac{1}{16}(1 + 1 \times e^{j1(\pi/8)0} + 0.5 \times e^{j2(\pi/8)0} + 0 \times e^{j3(\pi/8)0} + \cdots$$
$$+ 0 \times e^{j13(\pi/8)0} + 0.5 \times e^{j14(\pi/8)0} + 1 \times e^{j15(\pi/8)0}$$
$$= \frac{4}{16} = 0.25$$

$$h(1T) = \frac{1}{16}(1 + 1e^{j1(\pi/8)1} + 0.5e^{j2(\pi/8)1} + 0 + \cdots + 0.5e^{j14(\pi/8)1}$$
$$+ 1e^{j15(\pi/8)1})$$
$$= \frac{1}{16}(1 + 0.924 + j0.383 + 0.354 + j0.354 + \cdots + 0.354 - j0.354$$
$$+ 0.924 - j0.383) = 0.222$$

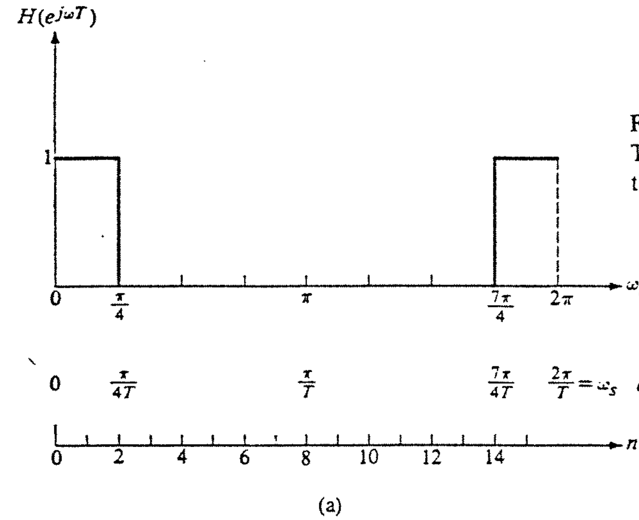
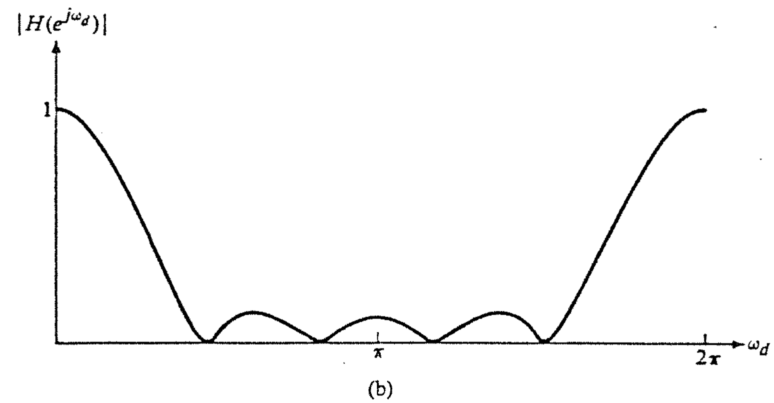


Figure 11.21
The frequency response of the FIR filter under review.



$$h(2T) = \frac{1}{16}(1 + 1e^{j1(\pi/8)2} + 0.5e^{j2(\pi/8)2} + 0 + \cdots + 0.5e^{j14(\pi/8)2}$$
$$+ 1e^{j15(\pi/8)2})$$
$$= 0.151$$

From Step 3, we obtain

$$H'(z) = 0.151z^2 + 0.222z^1 + 0.25z^0 + 0.222z^{-1} + 0.151z^{-2}$$

The positive power on z indicates a time advance, which in turn requires that the filter must have input data for $t < 0$. To obtain a causal filter, we multiply $H'(z)$ by z^{-2}, which yields, for the desired filter,

$$H(z) = 0.151 + 0.222z^{-1} + 0.25z^{-2} + 0.222z^{-3} + 0.151z^{-4}$$

The amplitude characteristics $|H(e^{j\omega T})|$ are plotted in Figure 11.21b.

The time shift does not alter the amplitude characteristics of the filter but only its phase, which here is linear; this feature is often desired. ■

■ ■ ■

11-13 USE OF WINDOW FUNCTIONS FOR FIR FILTERS

In this section we will develop the Fourier series method of filter design. Also, we will discuss the use of window functions to produce a smoother response function than would be possible without the use of an appropriate window.

We know that any periodic function can be expanded into a Fourier series. Thus $H(e^{j\omega T})$, which is periodic, can be written in the form

$$H(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h(kT)e^{jk\omega T} \tag{11.80}$$

where

$$h(kT) = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} H(e^{j\omega T})e^{jk\omega T}\,d\omega \tag{11.81}$$

If we set $z = e^{j\omega T}$ [see (11.47)], then (11.80) becomes

$$H(z) = \sum_{k=-\infty}^{\infty} h(kT)z^{-k} \tag{11.82}$$

In practice, however, we use a finite number of samples, and we therefore set

$$h(kT) = 0 \qquad \text{for } k > \frac{N-1}{2} \quad \text{and} \quad k < -\frac{N-1}{2} \tag{11.83}$$

For a finite number of samples, (11.82) becomes

$$H'(z) = h(0) + \sum_{k=1}^{(N-1)/2} [h(-kT)z^k + h(kT)z^{-k}] \tag{11.84}$$

To create a causal filter, we multiply $H'(z)$ by $z^{-(N-1)/2}$, which then gives as a final configuration the expression

$$H(z) = z^{-(N-1)/2}H'(z) \tag{11.85}$$

The truncation of the infinite series specified in (11.80) which implies the use of a rectangular window results in a modified FIR filter that closely resembles the exact one. The truncation operation creates Gibbs phenomenon overshoots at the points of discontinuities of $H(e^{j\omega T})$ with the ripples existing in the neighborhood of the discontinuities. Here, as in several other points of our study, **weighting sequences** (windows) are used to modify the truncated function in order to decrease the overshoot. Thus the modified truncated sequence of the Fourier coefficients will be specified by

$$h_w(kT) = \begin{cases} h(kT)\,w(kT) & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{11.86}$$

where

$$w(kT) = \begin{cases} w(kT) & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases} \tag{11.87}$$

To design an FIR filter using the DFT approach, the following steps are required:

1. From the given amplitude characteristics, obtain the desired sampled values $H(n\Omega)$ for a specified N.
2. Incorporate in (11.81) the values found in Step 1 and find the sequence $\{h(kT)\}$.
3. Apply an appropriate window function to the finite sequence found in Step 2.
4. Use (11.84) to obtain $H'(z)$.
5. Multiply $H'(z)$ by $z^{-(N-1)/2}$.
6. Plot $|H(e^{j\omega T})|$ versus ω to obtain the amplitude characteristic of the resulting filter.

EXAMPLE 11.12

Apply the Hamming window [see (10.72)] to the impulse response sequence $h(kT)$ that corresponds to the desired frequency characteristics of the filter shown in Figure 11.20. Choose the values: $0 \le k \le 4 = N-1$ and $T = 1$.

Solution: From (11.81) we obtain

$$h(kT) = \frac{T}{2\pi}\int_{-\pi/2T}^{\pi/2T} e^{jk\omega T}\,d\omega = \frac{1}{\pi k}\sin\frac{\pi k}{2}$$

$$= \frac{1}{\pi k}\sin\left(T\frac{1}{4}\times\frac{2\pi}{T}k\right) = \frac{1}{\pi k}\sin(\omega_c kT)$$

where ω_c is the cutoff frequency of the filter and $\omega_s = 2\pi/T$ is the sampling frequency. If we set $T = 1$, a given condition, the impulse sequence is

$$\{h(k)\} = \frac{1}{2}, \frac{1}{\pi}, 0, -\frac{1}{3\pi}, 0$$

For the Hamming window in a form that is symmetric around the origin, we must write (10.72) in the form

$$w_{hm}(k) = 0.54 + 0.46\cos\frac{k\pi}{K}$$

where the constant K is equal to the number of terms to be included in the range on each side of $h(0T)$. Thus we find, (where $k = 4$)

$$w_{hm}(0) = 1 \qquad w_{hm}(1) = 0.865 \qquad w_{hm}(2) = 0.541$$
$$w_{hm}(3) = 0.215 \qquad w_{hm}(4) = 0.081$$

The resulting windowed sequence is

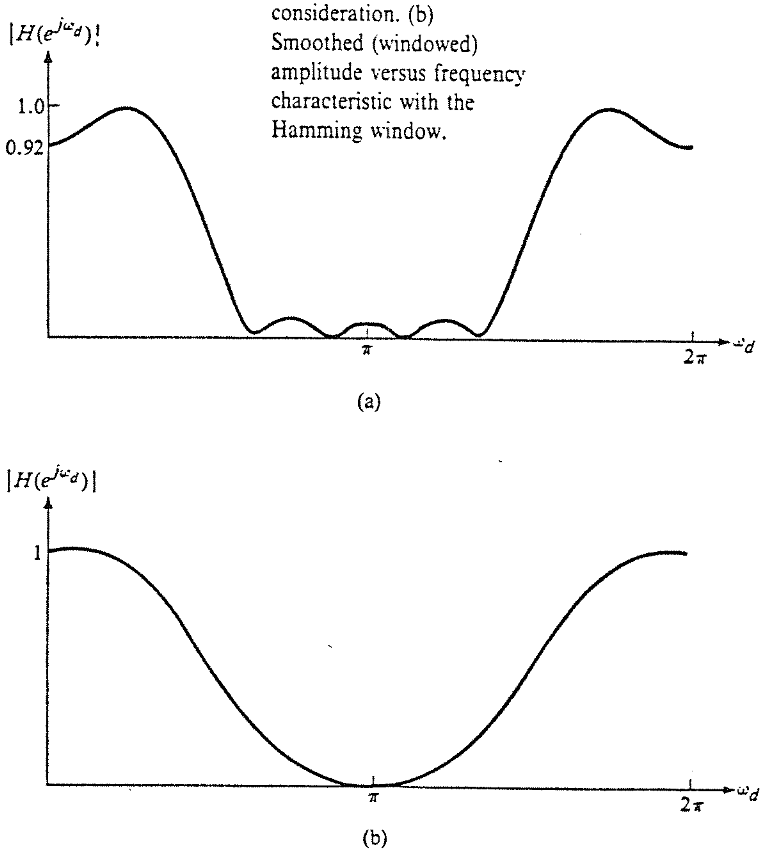


Figure 11.22
(a) Unsmoothed amplitude versus frequency characteristic of the filter under consideration. (b) Smoothed (windowed) amplitude versus frequency characteristic with the Hamming window.

$$\{h_w(k)\} = \{h(k)w_{hm}(k)\} = \frac{1}{2}, \frac{0.865}{\pi}, 0, -\frac{0.215}{3\pi}, 0$$

This sequence is used in (11.84) to obtain

$$H'(z) = \frac{1}{2}z^0 + \frac{0.865}{\pi}z^{-1} + 0 \times z^{-2} - \frac{0.215}{3\pi}z^{-3} + 0 \times z^{-4}$$
$$+ \frac{0.865}{\pi}z^1 + 0 \times z^2 - \frac{0.215}{3\pi}z^3 + 0 \times z^4$$

Since our largest positive exponent is 3, we multiply $H'(z)$ by z^{-3} to find

$$H(z) = z^{-3}H'(z)$$
$$= -\frac{0.215}{3\pi} + \frac{0.865}{\pi}z^{-2} + \frac{1}{2}z^{-3} + \frac{0.865}{\pi}z^{-4} - \frac{0.215}{3\pi}z^{-6}$$

A plot of $|H(e^{j\omega T})|$ with $T = 1$ is given in Figure 11.22 for both the unmodified and the windowed cases. It is apparent from the plots that the effect of the window tends to smooth the ripples.

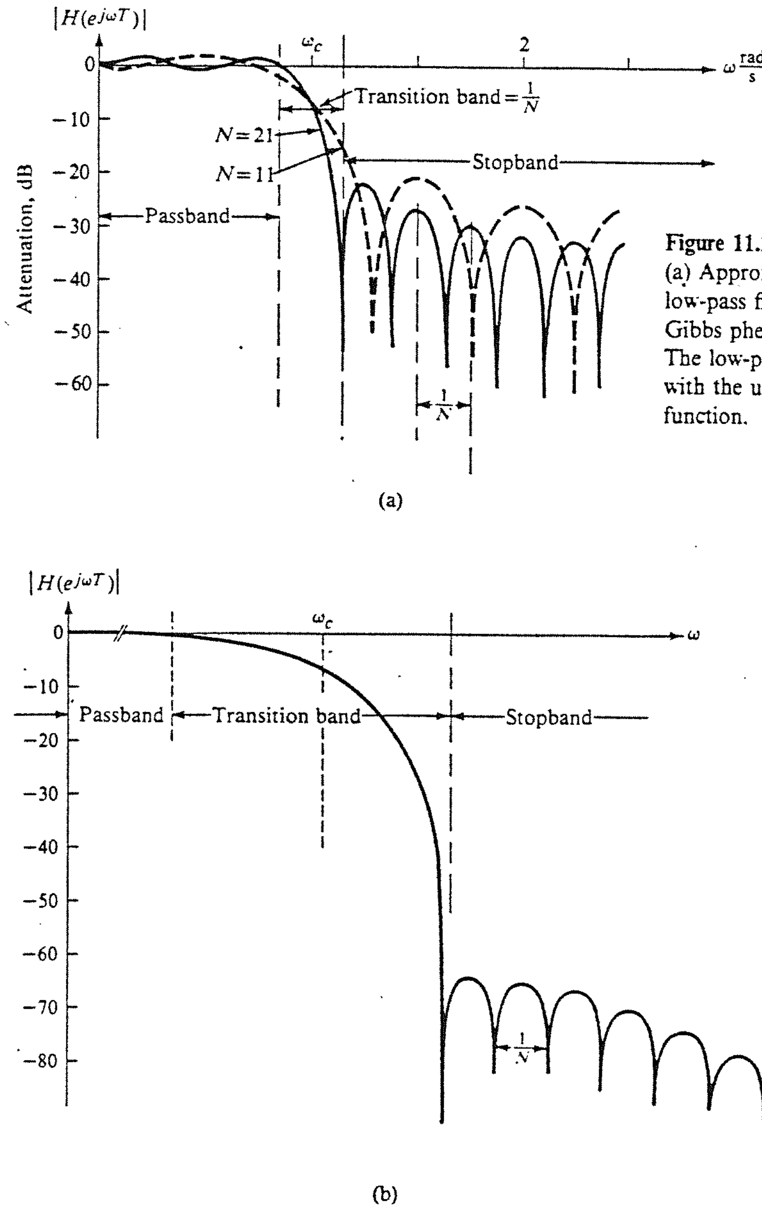


Figure 11.23
(a) Approximation to ideal low-pass filter, exhibiting Gibbs phenomenon. (b) The low-pass filter designed with the use of a window function.

The data contained in Figure 11.22 are often presented by plotting the magnitude function $|H(e^{j\omega T})|$ on a dB scale. Figure 11.23a shows such a plot for a low-pass filter with $N = 11$ and $N = 21$. We see that the points for $|H(e^{j\omega T})| = 0$ are placed in sharp perspective since the magnitude function becomes $-\infty$ at these points. Figure 11.23b shows by comparison the effect obtainable using an appropriate window for the filter (with $N = 11$) shown in Figure 11.23a. The clearly evident. As shown here, the transition bandwidth is approximately $5/N$.

The three most commonly used window functions for modifying nonrecursive digital filters are:

1. Hamming window

$$w_{hm} = 0.54 + 0.46\cos(k\pi/K) \tag{11.88}$$

2. Blackman window

$$w_b = 0.42 + 0.5\cos(k\pi/K) + 0.08\cos(2k\pi/K) \tag{11.89}$$

3. Hann window

$$w_h = 0.5 + 0.5\cos(k\pi/K) \tag{11.90}$$

The constant K is equal to the number of terms to be included in the expansion for $h(0T)$.

■ ■ ■

11-14 THE DFT AS A FILTER

The discrete Fourier transform (DFT) can be used in digital filter design. This requires that the discrete signal to be filtered is transformed to the frequency domain via an FFT algorithm, and the frequency samples of the signal are then multiplied by the desired frequency response characteristic of the filter. The filtered frequency samples are then transformed to the time domain through an inverse FFT procedure. Since the filtering operation is accomplished in the frequency domain, there is no need for determining the coefficients of the DFT. We first assume that the desired frequency response is specified at discrete frequency points, and the filter coefficients are found from the discrete frequency samples through the inverse DFT. Suppose therefore that

$$H(n) \triangleq H\left(\frac{2\pi n}{NT}\right) = H(n\Omega) \qquad n = 0, 1, 2, \ldots, N-1$$

denotes the samples of the frequency response at equally spaced frequency points. The unit sample response sequence is

$$h(k) = \frac{1}{N}\sum_{n=0}^{N-1} H(n)e^{j2\pi kn/N}$$

$$\triangleq \frac{1}{N}\sum_{n=0}^{N-1} H(n\Omega)e^{jn\Omega kT} \qquad k = 0, 1, \ldots, N-1 \tag{11.91}$$

The transfer function $H(z)$ is then

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k} = \sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} H(n)e^{j2\pi kn/N}\right)z^{-k}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} H(n)\sum_{k=0}^{N-1} (e^{j2\pi kn/N})z^{-k} \tag{11.92}$$

But the summation

$$\sum_{k=0}^{N-1} e^{j2\pi nk/N}z^{-k} = 1 + e^{j(2\pi n)/N}z^{-1} + \cdots + e^{j(2\pi n)(N-1)/N}z^{-(N-1)}$$

$$= \frac{1-(e^{j2\pi n/N}z^{-1})^N}{1-e^{j2\pi n/N}z^{-1}} = \frac{1-e^{j2\pi n}z^{-N}}{1-e^{j2\pi n/N}z^{-1}}$$

$$= \frac{1-z^{-N}}{1-e^{j2\pi n/N}z^{-1}} = \frac{z^N-1}{z^{N-1}(z-e^{j2\pi n/N})}$$

and (11.92) becomes

$$H(z) = \frac{1}{N}\sum_{n=0}^{N-1} H(n)\frac{1-z^{-N}}{1-e^{j2\pi n/N}z^{-1}} \tag{11.93}$$

This shows that the transfer function can be found, given the samples of the frequency response of the FIR filter. This equation has N poles and N zeros. The N zeros are located at the principal roots of 1. There are $N-1$ poles located at $z = 0$, and there is one pole located at $z = \exp[j2\pi n/N]$. This pole will cancel the zero for the value $n = N$. Thus the resulting sequence has $N-1$ poles, all located at $z = 0$ and $N-1$ zeros located on the unit circle. This means, of course, that $H(z)$ is given as the ratio of two polynomials. Therefore the FIR filter that is realized from this transfer function is a recursive one.

EXAMPLE 11.13

Show that each term in the summation of (11.93) specifies a filter. Choose $N = 3$, and consider the function for $n = 8$.

Solution: We examine the specific function

$$H_n(z) = \frac{1-z^{-N}}{1-e^{j2\pi n/N}z^{-1}} = \frac{z^N-1}{z^{N-1}(z-e^{j2\pi n/N})}$$

For $T = 1/f_s$,

$$|H_n(e^{j\omega T})| = \left|\frac{e^{j\omega NT}-1}{e^{j\omega T}-e^{j2\pi n/N}}\right| = \left|\frac{e^{j2\pi Nf/f_s}-1}{e^{j2\pi f/f_s}-e^{j2\pi n/N}}\right|$$

$$= \frac{|e^{j\pi Nf/f_s}-e^{-j\pi Nf/f_s}|}{|e^{j\pi(f/f_s+n/N)}-e^{-j\pi(f/f_s+n/N)}|} = \frac{\sin \pi Nf/f_s}{\sin \pi\left(\dfrac{f}{f_s}+\dfrac{n}{N}\right)}$$

We observe that $H_n(z)$ includes $N-1$ poles at $z = 0$ and N zeros on the unit circle. However, one of the poles and one of the zeros cancel each other. The situations for the zero-poles constellation and the response function are illustrated in Figure 11.24. ■

The frequency response of this filter is obtained from (11.93) by setting $z = \exp[j\omega T]$. We then have

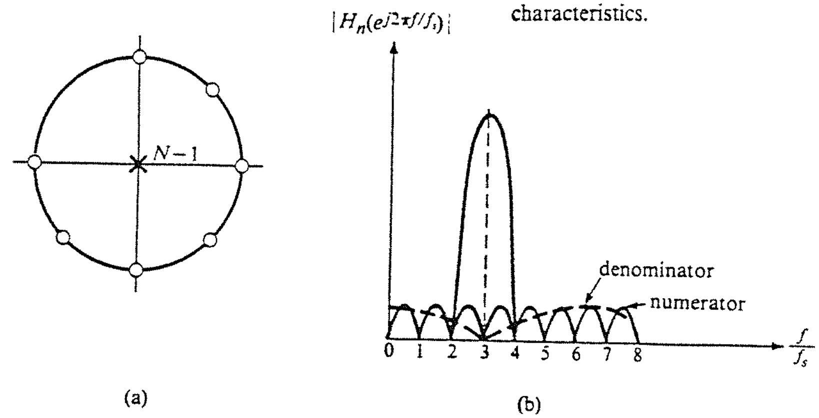


Figure 11.24
The DFT filter function.
(a) Pole-zero configuration.
(b) Frequency response characteristics.

$$H(e^{j\omega T}) = \frac{1}{N}\sum_{n=0}^{N-1} H(n)\frac{1-e^{-jN\omega T}}{1-e^{j2\pi n/N}e^{-j\omega T}}$$
$$= \frac{1}{N}\sum_{n=0}^{N-1} H(n)\frac{1-e^{-jN(\omega T-2\pi n/N)}}{1-e^{-j(\omega T-2\pi n/N)}}$$

This can be expanded to

$$H(e^{j\omega T}) = \frac{1}{N}\sum_{n=0}^{N-1} H(n)\frac{e^{-jN(\omega T-2\pi n/N)/2}}{e^{-j(\omega T-2\pi n/N)/2}}\frac{\sin\left[N\dfrac{\omega T-\dfrac{2\pi n}{N}}{2}\right]}{\sin\left[\dfrac{\omega T-\dfrac{2\pi n}{N}}{2}\right]}$$

$$= e^{-j(N-1)\omega T/2}\sum_{n=0}^{N-1} H(n)e^{-j\pi n/N}\frac{\sin\left(\dfrac{N\omega T}{2}\right)}{N\sin\left[\dfrac{\left(\omega T-\dfrac{2\pi n}{N}\right)}{2}\right]} \tag{11.94}$$

As we know from DFT properties, for the unit sample response sequence $h(k)$ of the FIR filter to be a real valued quantity, the frequency samples $H(n)$ must satisfy the symmetry property; that is, the real part must be an even function and the imaginary part must be an odd function. Therefore the frequency samples must be so selected that $H(N-n) = H(n)$. This requires that the amplitudes and phase be selected so that

$$A(N-n) = A(n) \qquad \theta(N-n) = 2m\pi - \theta(n)$$
$$\text{for} \quad n = 1, 2, \ldots, N-1 \tag{11.95}$$

where m is an integer. Therefore we must have

$$H(0) = \text{real} \quad \text{or} \quad \theta(0) = m\pi \qquad \text{(a)}$$
$$H\left(\frac{N}{2}\right) = A\left(\frac{N}{2}\right) \geq 0 \qquad \theta\left(\frac{N}{2}\right) = 2m\pi \qquad \text{(b)} \tag{11.96}$$

■ ■ ■

11-15 FREQUENCY TRANSFORMATION OF IIR FILTERS

In parallel with the availability of special transformations for analog filters for converting low-pass to high-pass, we can find special transformations for digital filters. These will permit transformation of a low-pass filter to another low-pass filter, a high-pass filter, a bandpass filter, or a bandstop digital filter. We consider these transformations:

1. Low-Pass to Low-Pass Transformation. This transformation is

$$z^{-1} \rightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \tag{11.97}$$

where

$$\alpha = \frac{\sin\left[\left(\dfrac{\omega_{dc}-\omega'_{dc}}{2}\right)\right]}{\sin\left[\left(\dfrac{\omega_{dc}+\omega'_{dc}}{2}\right)\right]}$$

ω_{dc} = cutoff frequency of the given filter

ω'_{dc} = cutoff frequency of the desired filter

EXAMPLE 11.14

Design a low-pass filter with $\omega'_{dc} = 0.5$ if the given filter is that specified in Example 11.9.

Solution: We first determine the value of

$$\alpha = \frac{\sin\left[\left(\dfrac{0.4\pi-0.5\pi}{2}\right)\right]}{\sin\left[\left(\dfrac{0.4\pi+0.5\pi}{2}\right)\right]} = -0.158$$

Next we combine (11.97) with the results of Example 11.9. We find

$$H(z) = \frac{2.117 + 4.234\left[\dfrac{z^{-1}+0.158}{1+0.158z^{-1}}\right] + 2.117\left[\dfrac{z^{-1}+0.158}{1+0.158z^{-1}}\right]^2}{10.232 - 3.766\left[\dfrac{z^{-1}+0.158}{1+0.158z^{-1}}\right] + 2.002\left[\dfrac{z^{-1}+0.158}{1+0.158z^{-1}}\right]^2}$$

$$= \frac{2.839 + 5.683z^{-1} + 2.839z^{-2}}{3.963 + 9.682z^{-2}}$$

2. Low-Pass to High-Pass Transformation.

$$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}} \tag{11.98}$$

where

$$\alpha = -\frac{\cos\left[\left(\frac{\omega_{dc} + \omega'_{dc}}{2}\right)\right]}{\cos\left[\left(\frac{\omega_{dc} - \omega'_{dc}}{2}\right)\right]}$$

3. Low-Pass to Bandpass Transformation.

$$z^{-1} \rightarrow -\frac{z^{-2} - \frac{2\alpha\beta}{\beta + 1} z^{-1} + \frac{\beta - 1}{\beta + 1}}{\frac{\beta - 1}{\beta + 1} z^{-2} - \frac{2\alpha\beta}{\beta + 1} z^{-1} + 1} \tag{11.99}$$

where

$$\alpha = \frac{\cos\left[\left(\frac{\omega'_{du} + \omega'_{dl}}{2}\right)\right]}{\cos\left[\left(\frac{\omega'_{du} - \omega'_{dl}}{2}\right)\right]}$$

$$\beta = \cot\left(\frac{\omega'_{du} - \omega'_{dl}}{2}\right) \tan \frac{\omega_{dc}}{2}$$

ω'_{du} = desired upper cutoff frequency

ω'_{dl} = desired lower cutoff frequency

4. Low-Pass to Bandstop Transformation.

$$z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha}{1 + \beta} z^{-1} + \frac{1 - \beta}{1 + \beta}}{\frac{1 - \beta}{1 + \beta} z^{-2} - \frac{2\alpha}{1 + \beta} z^{-1} + 1} \tag{11.100}$$

where

$$\alpha = \frac{\cos\left[\left(\frac{\omega'_{du} + \omega'_{dl}}{2}\right)\right]}{\cos\left[\left(\frac{\omega'_{du} - \omega'_{dl}}{2}\right)\right]}$$

$$\beta = \tan\left(\frac{\omega'_{du} - \omega'_{dl}}{2}\right) \tan \frac{\omega_{dc}}{2}$$

ω'_{du} = desired upper cutoff frequency

ω'_{dl} = desired lower cutoff frequency

■ ■ ■

11-16 RECURSIVE VERSUS NONRECURSIVE DESIGNS: GENERAL REMARKS

A comparison of the important features of different filter designs is helpful. In recursive filters the poles of the transfer function can be placed anywhere within the unit circle. As a result, high selectivity can be achieved using low-order transfer functions. With nonrecursive filters, on the other hand, the poles are fixed at the origin and high selectivity can be achieved only by using a relatively high order for the transfer function. For the same filter specifications, the order of the nonrecursive filter might be as much as 5 to 10 times that of the recursive structure, with the consequent need for more electronic parts. Often, however, a recursive filter might not meet the specifications, and in such cases the nonrecursive filter can be used by the designer.

An important advantage of the nonrecursive filter is that it can be implemented using the FFT method, in the manner discussed in Section 11-14.

Hardware filter implementation requires that storage of input and output data and also arithmetic operations are implemented by using finite word-length registers (for example, 8, 12, or 16 bits). As a result, certain errors will occur; these are categorized as follows:

1. Quantization errors due to arithmetic operations such as rounding off and truncation.
2. Quantization errors due to representing the input signal by a set of discrete values.
3. Quantization errors when the filter coefficients are represented by a finite number of bits.

It is left to the filter designer to decide on the various trade-offs between cost and precision in trying to reach a specified goal.

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PROBLEMS

11-2.1 A low-pass filter is to be designed to have a nominal cutoff of 5 kHz. It is to be maximally flat (Butterworth) and is to be down 1 dB at the edge of the passband. The response function is to have 3 poles.
- **a.** Locate the poles in the s-plane.
- **b.** What is the rate of attenuation remote from cutoff?

11-2.2 Derive the transfer function for a third-order Butterworth low-pass filter and locate its poles.

11-2.3 The squared normalized amplitude of a Butterworth filter is $|H_n(j\omega)|^2 = 1/(1+\omega^6)$. Find the normalized transfer function $H_n(j\omega)$.

11-2.4 Show that the high frequency roll-off of an nth-order Butterworth filter is $20n$ dB/decade. Also, show that the first $(2n-1)$ derivatives of an nth-order Butterworth filter are zero at $\omega = 0$.

11-3.1 Repeat Problem 11-2.1 for a Chebyshev filter.

11-3.2 Find the value of n for a Butterworth and a Chebyshev filter that will satisfy the conditions specified in Figure P11-3.2.

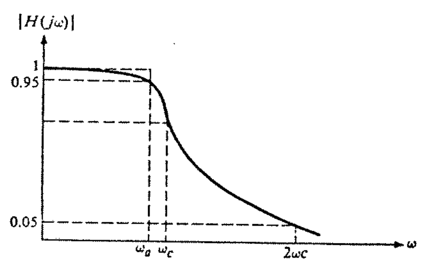

Figure P11-3.2

11-3.3 A Chebyshev low-pass filter is designed to have a passband ripple ≤ 2 dB with a cutoff frequency of 1000 rad/s. The attenuation is to be at least 50 dB at 5000 rad/s. Specify ϵ, n, and $H(j\omega)$.

11-3.4 Compare the attenuation at high frequencies of Butterworth and Chebyshev low-pass filters of the same order when the 3 dB cutoff frequencies are the same for both filters. Sketch the results on a graph of dB versus ω.

11-3.5 The characteristics to be met by a low-pass filter are:

passband: 0 to 4 kHz

attenuation: amplitude to be down 60 dB at 6 kHz

Determine the value of n for:
- **a.** Butterworth.
- **b.** Chebyshev with 1/2 dB ripple in the passband.
- **c.** Chebyshev with 1 dB ripple in the passband.

11-7.1 A bandpass filter is to have nominal cutoff points at frequencies 300 and 600 kHz. It is to have a maximally flat shape (Butterworth) and is to be down 3 dB at the edge of the band. The response function is to have 3 poles.
- **a.** Locate the poles of the response function.
- **b.** What is the ultimate rate of attenuation on each side of the band?

11-9.1 Suppose that two RC sections, as discussed in Example 11.6, are cascaded, with a buffer amplifier between the sections to avoid loading. Find the digital filter equivalent to this combination. How does it relate to the results of Example 11.6?

11-9.2 Design a low-pass digital filter consisting of a cascade of identical sections, each of which has $H(z) = \dfrac{(1-\alpha)z}{z-\alpha}$.

The nominal cutoff is 1 kHz with a gain at $\omega = 2\omega_c$ that is less than 0.35. The sampling frequency is $10f_c$. Specify the $H(e^{j\omega T})$ of this filter.

11-9.3 Repeat Example 11.4 with $T = 0.1$ s and $T = .0025$ s.

11-9.4 Find the second-order Butterworth digital filter with cutoff frequency $f_c = 100$ Hz and sampling rate of 1250 Hz.

11-10.1 Design a Butterworth low-pass filter, given the following data: half-power point at $\omega_1 = 250\pi$, sampling period $T = .0005$ s, $\omega_2 = 500\pi$, gain ≤ 0.2. Refer to Figure 11.2.

11-10.2 Locate the points $s = 1$, $s = 1 + j\omega_s/2$, $s = 1 - j\omega_s/2$, $s = 10$, and $s = \infty$ on the s-plane when mapped by a bilinear transformation onto the z-plane. Assume that $T = 0.1$.

11-10.3 Show that warping also affects the phase curves.

11-11.1 Determine analytically the frequency response of the nonrecursive filter specified by $H(z) = \sum_{n=0}^{20} z^{-n}$.

11-11.2 Show that if $p = re^{j\theta}$ is a root of the polynomial in z given in (11.67), then $re^{-j\theta}$, $(1/r)e^{j\theta}$, and $(1/r)e^{-j\theta}$ also are roots.

11-11.3 Consider the Butterworth low-pass filter given by (11.73). Show that by rotating the poles and zeros, the resulting function has the squared gain

$$|H(e^{j\omega T})|^2 = \frac{1}{1 + \dfrac{\cot^{2n}\left(\dfrac{\omega T}{2}\right)}{\cot^{2n}\left(\dfrac{\omega_2 T}{2}\right)}}$$

where $\omega_2 = (\pi/T) - \omega_1$, and where $H(-1) = 1$. This is a high-pass filter.

11-12.1 The low-pass filter of Example 11.11 is modified by the use of a Hann window. Find the resulting frequency response.

11-12.2 Refer to Figure P11-12.2.
- **a.** Determine a nonrecursive filter that will approximate the frequency response shown using 10 terms from the Fourier series design method.
- **b.** Using the Hamming window function, determine a nonrecursive filter that will approximate the given frequency response using 10 terms from the Fourier series.
- **c.** Plot the frequency response of the filters developed in **a.** and **b.** Comment on the results.

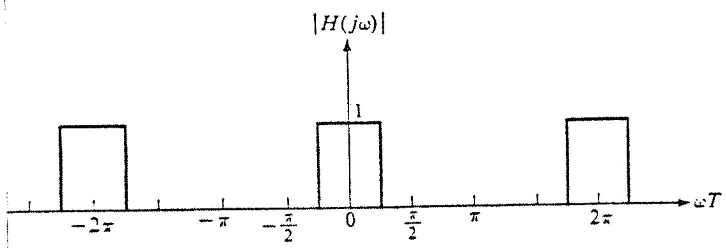

Figure P11-12.2

1-14.1 Determine the frequency response of a DFT filter that has a linear phase characteristic, with the constant delay equal to $(N-1)/2$ units of the sampling interval.

1-14.2 Show that a nonrecursive transfer function obtained by the DFT method can be realized by means of a set of parallel second-order recursive sections in cascade with an elementary Nth-order nonrecursive section.