

Root Locus Procedure

ECE 411

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Overview of Root Loci: In the design and analysis of control systems, it is often required to investigate the sensitivity due to the variations of the parameters in the system/controller. The root locus method plots the trajectories of the roots of the characteristic equation when certain parameters in the system vary. Given the pole-zero configuration, the construction of complete root loci diagram requires:

- Search for all the s_i points in the s-plane that satisfy the equations
- The determination of the values of K, i.e. either on root loci or on complementary root loci

Expanding on bullets, can develop rules to aid in plotting root loci.

Construction of the Complete Root Loci:

1. **K = 0 Points:** on complete root loci, corresponds to the poles of $G(s)H(s)$
2. **K = $\pm\infty$ Points:** on complete root loci, corresponds to the zeros of $G(s)H(s)$
3. **Number of Branches on the Complete Root Loci:**

- (a) *Branch:* locus of one root when K takes values between $-\infty$ and ∞
- (b) Number of branches = $\max(m,n)$
- (c) *Example:* Consider $s(s+2)(s+3) + K(s+1) = 0$

$$\frac{s+1}{s(s+2)(s+3)} = -\frac{1}{K} \text{ or } K = -\frac{s(s+2)(s+3)}{s+1}$$

$$K = 0 \implies s = 0, s = -2, s = -3$$

$$K = I\infty \implies s = -1, s = \infty, s = \infty$$

$$n = 3, m = 1 \implies \text{Number of branches} = \max(m,n) = 3$$

4. **Symmetry of Complete Root Loci:** symmetrical about real axis (real coefficients). Also symmetrical with respect to the axis of symmetry of the poles and zeros of $G(s)H(s)$
5. **Asymptotes of the Complete Root Loci (behavior at $s = \infty$):** As $s \rightarrow \infty$,
 - (a) The root loci for $K \geq 0$ are asymptotic to straight lines with angles of:

$$\Theta_k = \frac{(2k+1)\pi}{n-m}, \quad k = 0, 1, 2, \dots, |n-m|-1$$

- (b) The complementary root loci, i.e. $K \leq 0$, the angles of the asymptotes are:

$$\Theta_k = \frac{2k\pi}{n-m}, \quad k = 0, 1, 2, \dots, |n-m|-1$$

Thus, there are $|n-m|$ asymptotes for each type of loci.

6. **Intersection of Asymptotes (centroid):** the intersection of $2|n - m|$ asymptotes of the complete root loci lies on the real axis of the s-plane and is given by:

$$\sigma = \frac{b_1 - a_1}{n - m}$$

where b_1 and a_1 are coefficients of $\rho(s)$. Note that since poles and zeros are either real or complex conjugate pairs, the imaginary parts will be cancelled!

7. **Root Loci on the Real Axis:**

- (a) For $K \geq 0$, on a given section of real axis, root locus is found if the total number of real poles and zeros of $G(s)H(s)$ to the right of the section is odd.
- (b) For $K \leq 0$, on a given section of real axis, complex root locus is found if the total number of real poles and zeros of $G(s)H(s)$ to the right of the section is even.

Note that the complex poles and zeros of $G(s)H(s)$ do not affect the existence properties of root locus and complex root locus on the real axis.

8. **Angles of Departure (from poles) and Angles of Arrival (at zeros):** these angles describe the behavior of the root loci near the pole (zero). To find these angles, choose s , close to poles or zeros, and use:

$$\begin{aligned} \angle G_1(s_1)H_1(s_1) &= \sum_{i=1}^m \angle s_1 + z_i - \sum_{j=1}^n \angle s_i + p_j \\ &= (2k + 1)\pi, \quad K \geq 0 \text{ root locus} \\ &= 2k\pi, \quad K \leq 0 \text{ complex root locus} \end{aligned}$$

9. **Intersection of the Root Loci with the Imaginary Axis:** the points where the complete root loci intersect with the imaginary axis and the corresponding values of K may be determined by means of Routh-Hurwitz criterion and solving the auxillary equation.

10. **Break-in Points and Breakaway Points (Saddle Points):** corresponds to multiple-order roots of the equation. These points on the root loci correspond to two or more roots *meeting* on locus or *departing* from locus.

$$\# \text{ branches enter} = \# \text{ branches leave}$$

Breakaway points may not necessarily be on real axis. To find the breakaway points, two methods can be used:

- (a) Method 1: Find the roots of:

$$\begin{aligned} \frac{dG_1(s)H_1(s)}{ds} &= 0, \text{ which also satisfy:} \\ 1 + KG_1(s)H_1(s) &= 0, \text{ or use:} \\ \frac{dK}{ds} &= 0 \end{aligned}$$

Remarks: All real solutions of $\frac{dG_1(s)H_1(s)}{ds} = 0$ are saddle points on the root loci ($-\infty < K < \infty$). Complex conjugate solutions are breakaway points only if they satisfy both equations. The value of K corresponding to saddle points is obtained using:

$$K = -\frac{1}{G_1(s)H_1(s)}$$

- (b) Method 2: Let $\rho(s) = A_0s^n + A_1s^{n-1} + \dots + A_n = 0$ and $\rho'(s) = \frac{d\rho(s)}{ds} = B_0s^{n-1} + B_1s^{n-2} + \dots + B_{n-1} = 0$. Form the table:

s^n	A_0	A_1	\dots	A_{n-1}	A_n
s^{n-1}	B_0	B_1	\dots	B_{n-1}	0
s^{n-1}	$\frac{B_0 A_1 - A_0 B_1}{B_0} = C_0$	\dots	\dots	\dots	\dots
s^{n-1}	B_0	B_1	\dots	B_{n-1}	0
s^{n-2}	$\frac{B_0 C_1 - B_1 C_0}{B_0} = D_0$	\dots	\dots	\dots	\dots
s^{n-2}	$\frac{D_0 B_1 - B_0 D_1}{D_0} = E_0$	\dots	\dots	\dots	\dots
s^{n-2}	D_0	D_1	\dots	D_{n-1}	0
