

Control Systems (ECE411)

Lectures 7 & 8

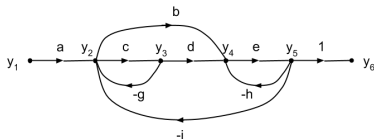
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Fall 2016

Signal Flow Graph Examples

Example 3:



Find $\frac{y_6}{y_1}$ and $\frac{y_5}{y_2}$.

Part (a): Input: y_1 Output: y_6

$$M_1 = abe \quad M_2 = acde$$

Loops:

$$P_{11} = -cg \quad P_{21} = -eh$$

$$P_{31} = -cdei \quad P_{41} = -bei$$

Loops 1 and 2 are non-touching:

$$P_{12} = P_{11}P_{21} = cgeh$$

$$\begin{aligned} \Delta &= 1 - P_{11} - P_{21} - P_{31} - P_{41} + P_{12} \\ &= 1 + cg + eh + cdei + bei + cgeh \end{aligned}$$

Signal Flow Graph Examples-Cont.

Both paths touch all loops, hence $\Delta_1 = 1$ and $\Delta_2 = 1$. Then we have,

$$\frac{y_6}{y_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{abe + acde}{\Delta}$$

Part (b): Note that y_2 is not an input node. Thus, to find $\frac{y_5}{y_1}$, we find $\frac{y_5}{y_1}$ and $\frac{y_2}{y_1}$ and then divide them. But since $y_5 = y_6$ we only need to find $\frac{y_2}{y_1}$.

For $\frac{y_2}{y_1}$: Output: y_2 Input: y_1

$$M_1 = a \quad \Delta_1 = 1 + eh$$

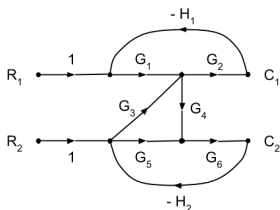
$$\frac{y_2}{y_1} = \frac{a(1+eh)}{\Delta}$$

Using the result in part (a) we get

$$\frac{y_5}{y_2} = \frac{y_5}{y_1} / \frac{y_2}{y_1} = \frac{abe+acde}{a(1+eh)}$$

Signal Flow Graph Examples-Cont.

Example 4



$$G_{11}(s) = \left. \frac{C_1}{R_1} \right|_{R_2=0}$$

Input: $R_1(s)$ Output: $C_1(s)$

$$M_{11} = G_1 G_2$$

$$P_{11} = -G_1 G_2 H_1 \quad P_{21} = -G_5 G_6 H_2$$

$$P_{31} = -G_3 G_4 G_6 H_2$$

Loops 1 and 2 are non-touching, thus $P_{12} = P_{11} P_{21} = G_1 G_2 G_5 G_6 H_1 H_2$

$$\Delta = 1 - P_{11} - P_{21} - P_{31} + P_{12}$$

Signal Flow Graph Examples-Cont.

Path 1 is non-touching with loop 2, $\Delta_1 = 1 + G_5G_6H_2$

$$G_{11}(s) = \left. \frac{C_1}{R_1} \right|_{R_2=0} = \frac{G_1G_2(1 + G_5G_6H_2)}{\Delta}$$

$$G_{12}(s) = \left. \frac{C_1}{R_2} \right|_{R_1=0}$$

$M_1 = G_2G_3$ $\Delta_1 = 1$. Thus

$$G_{12}(s) = \left. \frac{C_1}{R_2} \right|_{R_1=0} = \frac{G_2G_3}{\Delta}$$

For $G_{21}(s) = \left. \frac{C_2}{R_1} \right|_{R_2=0}$:

$M_1 = G_1G_4G_6$ $\Delta_1 = 1$. Thus,

$$G_{21}(s) = \left. \frac{C_2}{R_1} \right|_{R_2=0} = \frac{G_1G_4G_6}{\Delta}$$

Signal Flow Graph Examples-Cont.

Finally, for $G_{22}(s) = \left. \frac{C_2}{R_2} \right|_{R_1=0}$

$$M_1 = G_5 G_6 \quad \Delta_1 = 1 + G_1 G_2 H_1, \text{ and}$$

$$M_2 = G_3 G_4 G_6 \quad \Delta_2 = 1.$$

Thus

$$G_{22}(s) = \left. \frac{C_2}{R_2} \right|_{R_2=0} = \frac{G_5 G_6 (1 + G_1 G_2 H_1) + G_3 G_4 G_6}{\Delta}$$

State-Space Representation

Idea: Convert an n^{th} -order differential equation of an LTI system to n 1^{st} -order simultaneous differential equations where variables are called *state-variables* and specification of their values at same time instant is the *state* of the system at that time. Given the state of the system at time t_0 and all the inputs from time t_0 to t , the behaviour of the system can be completely determined for all $t > t_0$.

State Variables: Smallest set of variables that determine the behaviour of an LTI system. We use the notation x_1, \dots, x_n .

Note: They might not be physically measurable or observable quantities.

State Vector: State variables arranged in a $n - D$ vector:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathcal{R}^n$$

State-Space: n -dimensional space whose coordinates axis are x_1, \dots, x_n .

State-Space Representation-Cont.

State-Space Equations (SISO):

(a) LTI System

Let $u(t)$: input, $y(t)$: output, and $\underline{x}(t)$: state vector. Then, state equations for LTI systems are

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) \quad (\text{state equation})$$

$$y(t) = C\underline{x}(t) + du(t) \quad (\text{output equation})$$

where A is a $n \times n$ matrix, B is a $n \times 1$ vector, C is $1 \times n$, and d is a scalar. Also,

$$\dot{\underline{x}}(t) = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_N(t)}{dt} \end{bmatrix}$$

(b) Linear Time-Varying (LTV) System

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)u(t)$$

$$y(t) = C(t)\underline{x}(t) + d(t)u(t)$$

where A , B , C , d are now time-dependent.

State-Space Representation-Cont.

(c) Nonlinear System

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), u(t); t)$$

$$y(t) = g(\underline{x}(t), u(t); t)$$

where $\underline{f}(\cdot) = [f_1(\cdot) \ f_2(\cdot), \dots \ f_n(\cdot)]^T$ i.e. a vectorial function of state vector $\underline{x}(t)$ and $g(\cdot)$ is a scalar (for SISO) function of $\underline{x}(t)$.

Advantages of State-Space Representation:

- 1 Unified way of representing all types systems e.g., nonlinear, time-varying, MIMO and large scale systems.
- 2 Gives more insight into structure of the system (via state variables).
- 3 Controller design using *state feedback* is a more versatile and effective strategy when compared to the traditional methods e.g., PID control.

State-Space Representation-Cont.

Then

$$\frac{dv_c(t)}{dt} = \frac{dx_1(t)}{dt} = \frac{1}{C}x_2(t) - \frac{1}{C}x_3(t)$$

$$\frac{di_1(t)}{dt} = \frac{dx_2(t)}{dt} = -\frac{1}{L_1}x_1(t) - \frac{R_1}{L_1}x_2(t) + \frac{1}{L_1}v_s(t)$$

$$\frac{di_2(t)}{dt} = \frac{dx_3(t)}{dt} = \frac{1}{L_2}x_1(t) - \frac{R_2}{L_2}x_3(t)$$

Thus, we get the following state equation

$$\dot{\underline{x}}(t) = \begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} & -\frac{1}{C} \\ -\frac{1}{L_1} & -\frac{R_1}{L_1} & 0 \\ \frac{1}{L_2} & 0 & -\frac{R_2}{L_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1} \\ 0 \end{bmatrix} v_s(t)$$

If we want to get the voltage across R_2 as output, then:

$$y(t) = v_{R_2} = R_2 i_2(t) = \begin{bmatrix} 0 & 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

State-Space Representation-Cont.

Now, suppose we want to get both v_{R_2} and v_c as outputs, then :

$$\underline{y}(t) = \begin{bmatrix} v_{R_2} \\ v_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Note: The *states*, voltages across capacitors and currents through inductors, capture everything to know about this RLC circuit.

Transformation from Differential Equation (or Transfer Function) to State-Space Formulation:

Given an LTI system described by,

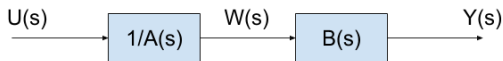
$$\sum_{i=0}^n a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^m b_j \frac{d^j u(t)}{dt^j}$$

$y(t)$: output, $u(t)$: input, n : order. Let $m = n$ and $a_n = 1$ (normalize if not). Alternatively, the transfer function (all ICs=0) is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{j=0}^n b_j s^j}{\sum_{i=0}^n a_i s^i} = \frac{B(s)}{A(s)}$$

State-Space Representation-Cont.

Decompose the transfer function into two cascaded parts with an All-Pole, $\frac{1}{A(s)}$, and All-Zero, $B(s)$, parts as shown.



All-pole part:

$$\frac{W(s)}{U(s)} = \frac{1}{A(s)} = \frac{1}{\sum_{i=0}^n a_i s^i}$$

$$\left(\sum_{i=0}^n a_i s^i\right) W(s) = U(s) \implies (s^n + a_{n-1}s^{n-1} + \dots + a_0) W(s) = U(s)$$

Convert to time-domain by taking inverse LT:

$$\frac{d^n w(t)}{dt^n} + a_{n-1} \frac{d^{n-1} w(t)}{dt^{n-1}} + \dots + a_0 w(t) = u(t)$$

State-Space Representation-Cont.

Assign state variables as follows:

$$x_1(t) = w(t)$$

$$x_2(t) = \frac{dw(t)}{dt} \implies \frac{dx_1(t)}{dt} = x_2(t)$$

$$\vdots$$

$$x_n(t) = \frac{d^{n-1}w(t)}{dt^{n-1}} \implies \frac{dx_n(t)}{dt} = \frac{d^n w(t)}{dt^n} = -a_{n-1}x_n(t) - \dots - a_0x_1(t) + u(t)$$

in matrix form,

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{d^N x(t)}{dt^N} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

or

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$

This structure is known as *Phase Variable Canonical Form (PVCF)*.

State-Space Representation-Cont.

All-zero part:

$$Y(s) = B(s)W(s) = \left(\sum_{j=0}^n b_j s^j \right) W(s)$$

Convert to time-domain by taking inverse LT:

$$y(t) = b_n \frac{d^n w(t)}{dt^n} + b_{n-1} \frac{d^{n-1} w(t)}{dt^{n-1}} + \cdots + b_0 w(t)$$

Substituting terms from previous state equations:

$$y(t) = b_n (-a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t)) + b_{n-1} x_n(t) + \cdots + b_0$$

$$y(t) = (b_0 - b_n a_0) x_1(t) + (b_1 - b_n a_1) x_2(t) + \cdots + (b_{n-1} - b_n a_{n-1}) x_{n-1}(t) + b_n u(t)$$

$$y(t) = c_0 x_1(t) + c_1 x_2(t) + \cdots + c_{n-1} x_{n-1}(t) + du(t) = C \underline{x}(t) + du(t)$$

where $c_i = b_i - b_n a_i$ for all $i \in [0, n-1]$ and $d = b_n$.

Remark: If $m < n$ (strictly proper), then $b_n = 0$, $c_i = b_i$.

Note: Given a transfer function you can always using these general equations (without derivation) to arrive at PVCF form.

State-Space Representation-Cont.

Example: An LTI system is given by:

$$\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 2y(t) + \int_0^t y(\tau)d\tau = u(t)$$

Convert to PVCF.

We first take LT and apply the properties,

$$(s^3 + 3s^2 + 2 + \frac{1}{s})Y(s) = U(s)$$

Thus, the transfer function: $H(s) = \frac{Y(s)}{U(s)} = \frac{s}{s^4 + 3s^3 + 2s + 1}$

From this transfer function, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$, $a_3 = 3$ while $b_1 = 1$ and $b_0 = b_2 = b_3 = 0$. Thus, we get

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & 0 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

and

$$y(t) = [0 \ 1 \ 0 \ 0]\underline{x}(t).$$