Control System (ECE411)
Lectures 13 & 14

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Steady-State Error Analysis

Remark: For a *unity feedback system* ($H(s) = 1$):

$e(t) = r(t) - c(t)$

$E(s) = R(s) - C(s) = R(s) - R(s)M(s) = E(s) = [1 - M(s)]R(s)$

where $M(s)$ is the closed loop transfer function.

Thus, $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s[1 - M(s)]R(s)$

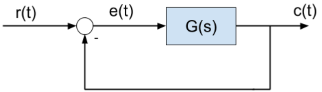
For a unit step $R(s) = \frac{1}{s}$, we get

$$e_{ss} = [1 - M(0)]$$

Note: The above results could sometimes be used for cases when $H(s) \neq 1$ (*tracking error*).

Example

Given a unity feedback system shown below with closed loop transfer function $M(s) = \frac{K}{(s^2+2s+2)(s+a)}$,



Steady-State Error Analysis-Cont.

(a) find K and a such that $e_{ss} = 1.5$ to unit ramp input,
(b) find e_{ss} for unit step input.

Part (a): First, we find the open-loop transfer function $G(s)$ from $M(s)$ using,

$$M(s) = \frac{G(s)}{1+G(s)} = \frac{K}{(s^2+2s+2)(s+a)} \implies$$

$$G(s) = \frac{K}{(s^2+2s+2)(s+a)-K} = \frac{K}{s^3+(a+2)s^2+(2a+2)s+(2a-K)}$$

Now, in order to avoid a Type 0 system which yields $e_{ss} \to \infty$ for ramp input, $2a - K = 0 \implies K = 2a$.

For a unit ramp input:

$e_{ss} = \frac{1}{K_v}$
where $K_v = \lim_{s\to\infty} sG(s)$

Thus, $e_{ss} = \frac{1}{K_v} = 1.5 \implies K_v = \frac{2}{3}$

Steady-State Error Analysis-Cont.

Using $G(s)$ in $K_v = \lim_{s\to\infty} sG(s)$ and $K_v = \frac{2}{3}$ gives,

$$K_v = \lim_{s\to\infty} \frac{K}{s^2 + (a+2)s + (2a+2)} \implies \frac{K}{2a+2} = \frac{2}{3}$$

Solving for K and a using the above equation and $K = 2a$ gives $a = 2$, and $K = 4$.

Part (b): Using the result from part (a):

$$G(s) = \frac{4}{s(s^2 + 4s + 6)}$$

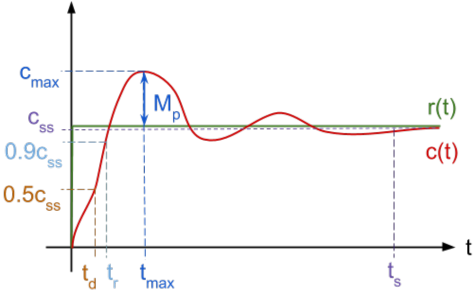
which is obviously Type 1 system $\implies e_{ss} = 0$ to unit step input.

Transient Analysis

Transient Response

Transient response allows for determining whether or not a system is stable and, if so, how stable it is (i.e. relative stability) as well as the speed of response when a step reference input is applied.

A typical time-domain response of a second order system (closed loop) to a unit step input is shown.



Transient Analysis-Cont.

Key Definitions:

1. Max Overshoot (M_p) $M_p = \frac{c_{max} - c_{ss}}{c_{ss}}$
   c_{max}: max value of $c(t)$, c_{ss}: steady-state value of $c(t)$
   $\%$max overshoot $= 100 \times M_p$
   M_p determines relative stability: Large $M_p \iff$ less stable
2. Delay time (t_d): Time for $c(t)$ to reach 50% of its final value.
3. Rise time (t_r): Time for $c(t)$ to rise from 10% to 90% of its final value.
4. Settling time (t_s): Time for $c(t)$ to decrease and stay within a specified (typically 5%) of c_{ss}.

Desirable characteristics: Small M_p, small t_d, quick t_r and fast t_s (cannot be accomplished simultaneously).

Transient Response of 2^{nd}-Order Control System

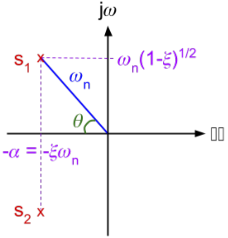
Consider a control system with closed-loop transfer function,

$$M(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \qquad M(0) = 1$$

Characteristic Equation : $\rho(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = 0$
has the following roots,

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -\alpha \pm j\omega$$

These are depicted in the following figure.



$$\cos\theta = -\xi, \qquad \tan\theta = \frac{\sqrt{1-\xi^2}}{-\xi}$$

Transient Response of 2^{nd}-Order Control System-Cont.

Response to unit step input ($R(s) = \frac{1}{s}$) is

$$C(s) = \frac{\omega_n^2}{s(\underbrace{s^2 + 2\xi\omega_n s + \omega_n^2}_{(s+\alpha)^2+\omega^2})}$$

Use PFE, time-domain response is found to be

$$c(t) = \underbrace{1}_{c_{ss}} + \underbrace{\overbrace{\frac{e^{-\alpha t}}{\sqrt{1-\xi^2}}}^{\text{damping}} \sin[\omega t - \theta]}_{c_{tr}(t)}, \quad \forall t \geq 0$$

$\alpha = \xi\omega_n$: Damping Factor- Controls the rate of rise time and decay time i.e. α controls damping and speed of response.

Can control oscillations by changing ω.
Can control damping by changing ξ.

Transient Response of 2^{nd}-Order Control System-Cont.

$\tau = 1/\alpha$: Time Constant
Large $\alpha \implies$ small $\tau \implies$ signal decays quickly.

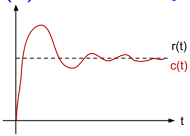
ξ: Damping Ratio (ratio between actual damping factor and the damping factor for critically damped ($\xi = 1 \Longrightarrow s_{1,2} = -\omega_n$).

ω_n: Natural Undamped Frequency ($\xi = 0 \Longrightarrow s_{1,2} = \pm j\omega_n$ i.e. purely oscillatory with frequency ω_n)

$\omega = \omega_n\sqrt{1-\xi^2}$: Conditional Frequency

Transient Response-Different Damping Cases

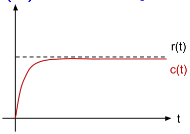
(a) **Underdamped:**



$0 < \xi < 1, \qquad s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

Characteristics: Small rise time (t_r), large overshoot (M_p).

(b) **Critically Damped:**

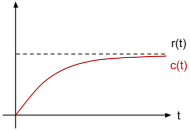


$\xi = 1, \qquad s_{1,2} = -\omega_n$ (repeated real roots)

Characteristics: No overshoot, slow/large rise time.

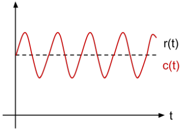
Transient Response-Different Damping Cases

(c) **Overdamped:**



$\xi > 1 \qquad s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$ (two real distinct roots)
Characteristics: No overshoot, very large rise time.

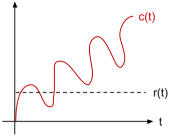
(d) **Undamped (Oscillatory):**



$\xi = 0, \qquad s_{1,2} = \pm j\omega_n$

Transient Response-Different Damping Cases

(e) **Negatively damped (unstable):**



$$\xi < 0, \qquad s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

Transient Response: Performance Measures

1. Peak Time (t_{max})

To find the peak time (time at which the step response reaches its maximum), we take the derivative of the step response and set it to zero.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = \frac{-\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\omega t - \theta\right] + \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega\cos[\omega t - \theta]$$

Using $\omega = \omega_n\sqrt{1-\xi^2}$ and trig identities, we can simplify the above equation as:

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin(\omega t), \qquad \forall t \geq 0,$$

Now, $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0 \implies \sin(\omega t) = 0$ or when $t \to \infty$ (i.e. final value)

The first condition gives the extrema points (maxima and minima) of $c(t)$, i.e.
$\omega t = n\pi \implies t = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}$

The first maximum (Max overshoot) of $c(t)$ happens for $n = 1$. Thus,

$$\boxed{t_{max} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Transient Response-Performance Measures-Cont.

Note: Although Max and Min of $c(t)$ occur at periodic interval, the response is NOT periodic due to damping (unless $\xi = 0$).

2. Max Overshoot (M_p)

To find M_p, we substitute t_{max} in expression for $c(t)$. This yields,

$$c_{max} = c(t)\,|_{t=t_{max}} = 1 + e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}$$

Thus, using the fact that $c_{ss} = 1$, we get

$$\boxed{M_p = c_{max} - 1 = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}}$$

Or in percentage,

$$\boxed{\%\text{Max Overshoot} = 100e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}}}$$

As can be seen, Max Overshoot is solely a function of ξ. Hence, Larger $\xi \implies$ smaller M_p ($\xi < 1$). But this would increase the delay time and rise time as seen next.

For t_d, t_r, and t_s only approximate equations can be obtained. These are given next.

Transient Response-Performance Measures-Cont.

3. Delay Time (t_d)

For t_d, we set $c(t) = 0.5$ and solve for t_d,

$$\boxed{t_d \approx \frac{1 + 0.7\xi}{\omega_n}}, \qquad 0 < \xi < 1$$

$t_d \approx \frac{1+0.6\xi+0.15\xi^2}{\omega_n}$: wider range of ξ and more accurate.

4. Rise Time (t_r)

$$\boxed{t_r \approx \frac{0.8 + 2.5\xi}{\omega_n}}, \qquad 0 < \xi < 1$$

$t_r = \frac{1+1.1\xi+1.4\xi^2}{\omega_n}$, wider range of ξ and more accurate.

5. Settling Time(t_s)

$$\boxed{t_s \approx 4/\xi\omega_n}$$

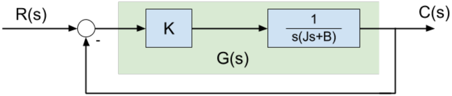
As can be seen, Small $\xi \implies$ smaller t_d and t_r but larger t_s.

Optimum Range for ξ: $0.5 \le \xi \le 0.8$

Analyzing Simple Controllers for 2^{nd} Order Systems

1. Gain Controller

Consider servo control system below:



Steady State Error Analysis:

Loop transfer function:
$G(s) = \frac{K}{s(Js+B)} \implies$ Type 1 $\implies e_{ss} = 0$ for $r(t) = u_s(t)$

For a unit ramp input $e_{ss} = \frac{1}{K_v}$ and

$K_v = \lim_{s\to 0} sG(s)H(s) = \frac{K}{B}$

Thus,
e_{ss} to unit ramp $\implies e_{ss} = \frac{B}{K}$

which implies Small e_{ss} Requires Large Gain K.

Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

Transient Analysis:

The closed-loop transfer function

$$M(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(sJ+B)}}{1+\frac{K}{s(Js+B)}} = \frac{K/J}{s^2+Bs/J+K/J}$$

Comparing with standard case,

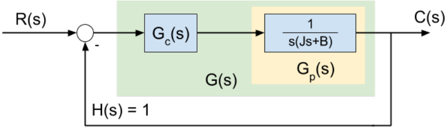
$$M(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2} \implies \omega_n = \sqrt{K/J} \text{ and } \quad 2\omega_n\xi = B/J \implies \xi = \frac{B}{2\sqrt{KJ}}$$

Since B and J cannot be tweaked (motor parameters), Large $K \implies$ Reduced $\xi \implies$ large $M_p \implies$ i.e. less stable.

Thus, a simple gain controller (K) won't produce desirable steady-state and transient behavior as a compromise between small steady state error and good relative stability and fast response cannot be achieved.

Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

2. Proportional-Derivative Controller (PD)



Controller transfer function $G_c(s) = K_P + K_D s$, where K_P is the proportional constant, and K_D is the derivative constant.

Steady State Error Analysis:

Loop transfer function:
$G(s) = G_c(s)G_p(s) = \frac{K_P+K_D s}{s(Js+B)} \implies$ Still Type 1 $\implies e_{ss} = 0$ for $r(t) = u_s(t)$

For a unit ramp input $e_{ss} = \frac{1}{K_v}$ and

$K_v = \lim_{s\to 0} sG(s)H(s) = \frac{K_P}{B} \implies e_{ss} = \frac{B}{K_P}$

i.e. it is possible to make e_{ss} to a unit ramp as small as possible by increasing proportional Gain K_P.

Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

Transient Analysis:

The closed-loop transfer function

$$M(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} = \frac{(K_P+K_D s)/J}{s^2+\underbrace{\frac{(B+K_D)s}{J}}_{2\xi\omega_n}+\underbrace{\frac{K_P}{J}}_{\omega_n^2}}$$

Again, comparing with standard case,

$M(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2} \implies \omega_n = \sqrt{\frac{K_P}{J}}$ and

$2\omega_n\xi = \frac{(B+K_D)}{J} \implies \xi = \frac{B+K_D}{2\sqrt{K_P J}}$

Thus, we can choose:

(a) Large K_P for small e_{ss} to unit ramp, and

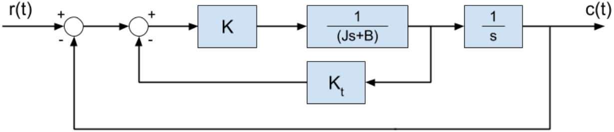
(b) Appropriate K_D to have $0.5 < \xi < 0.8$.

However, PD controller adds a zero at $s = -K_P/K_D$ which could have an impact in changing the shape of the response to unit step. Additionally, PD controller is susceptible to noise and difficult to realize.

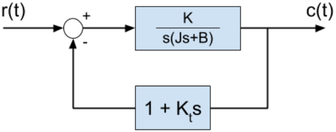
Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

3. Tachometer Control

Using $\frac{1}{s(Js+B)} \Longrightarrow \left(\frac{1}{Js+B}\right)\left(\frac{1}{s}\right)$, we design a rate feedback (tachometer) control as shown.



Alternatively, the above block diagram can be reduced to the typically used tachometer control system.



Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

Steady State Error Analysis:

Loop transfer function:
$G(s)H(s) = \frac{K(1+K_t s)}{s(Js+B)} \implies$ Still Type 1 $\implies e_{ss} = 0$ for $r(t) = u_s(t)$

For a unit ramp input $e_{ss} = \frac{1}{K_v}$ and

$K_v = \lim_{s \to 0} sG(s)H(s) = \frac{K}{B} \implies e_{ss} = \frac{B}{K}$

i.e. it is possible to make e_{ss} to a unit ramp as small as possible by increasing Gain K.

Transient Analysis:

The closed-loop transfer function
$M(s) = \frac{\frac{K}{J}}{s^2 + \frac{(B+KK_t)}{J}s + \frac{K}{J}}$
Comparing with standard case,

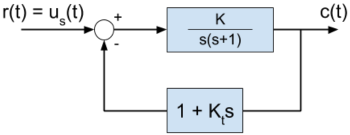
$M(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2} \implies \omega_n = \sqrt{\frac{K}{J}}$ and
$2\omega_n \xi = \frac{(B+KK_t)}{J} \implies \xi = \frac{B+KK_t}{2\sqrt{KJ}}$

Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

Thus, we can choose:
(a) Large Gain K for small e_{ss} to unit ramp, and
(b) Appropriate K_t to have $0.5 < \xi < 0.8$.

Note: Tachometer control doesn't have the same issues of the PD controller. Hence, used widely for servo control.

Example: For the tach control system below, find K and K_t such that max overshoot, M_p, to unit step is 0.2 and peak time is 1 second. Then, using these values of K and K_t, find t_r and t_s.



$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.2 \implies \xi = 0.456$$

$$t_{max} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1 \implies \omega_n = 3.53$$

Analyzing Simple Controllers for 2^{nd} Order Systems-Cont.

But,

$K = \omega_n^2 \implies K = 12.5$

Also,

$\xi = \frac{1+KK_t}{2\sqrt{K}} = 0.456 \implies K_t = 0.178$

$t_r = \frac{0.8+2.5\xi}{\omega_n} = 0.55\text{sec}$

$t_s = \frac{4}{\xi\omega_n} = 2.48\text{sec}$