Control System (ECE411)
Lectures 13 & 14

M.R. Azimi, Professor

Department of Electrical and Computer Engineering
Colorado State University

Fall 2016
Steady-State Error Analysis

Remark: For a unity feedback system \( (H(s) = 1) \):

\[
e(t) = r(t) - c(t)
\]

\[
E(s) = R(s) - C(s) = R(s) - R(s)M(s) = E(s) = [1 - M(s)]R(s)
\]

where \( M(s) \) is the closed loop transfer function.

Thus, \( e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s[1 - M(s)]R(s) \)

For a unit step \( R(s) = \frac{1}{s} \), we get

\[
e_{ss} = [1 - M(0)]
\]

Note: The above results could sometimes be used for cases when \( H(s) \neq 1 \) (tracking error).

Example

Given a unity feedback system shown below with closed loop transfer function

\[
M(s) = \frac{K}{(s^2 + 2s + 2)(s + a)},
\]

\[ r(t) \quad e(t) \quad G(s) \quad c(t) \]
Time-Domain Analysis

Steady-State Error Analysis-Cont.

(a) find $K$ and $a$ such that $e_{ss} = 1.5$ to unit ramp input,
(b) find $e_{ss}$ for unit step input.

Part (a): First, we find the open-loop transfer function $G(s)$ from $M(s)$ using,

$$
M(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{(s^2 + 2s + 2)(s + a)} \implies
$$

$$
G(s) = \frac{K}{(s^2 + 2s + 2)(s + a) - K} = \frac{K}{s^3 + (a + 2)s^2 + (2a + 2)s + (2a - K)}
$$

Now, in order to avoid a Type 0 system which yields $e_{ss} \to \infty$ for ramp input,
$2a - K = 0 \implies K = 2a$.

For a unit ramp input:

$$
e_{ss} = \frac{1}{K_v}
$$

where $K_v = \lim_{s \to \infty} sG(s)$

Thus, $e_{ss} = \frac{1}{K_v} = 1.5 \implies K_v = \frac{2}{3}$
Using $G(s)$ in $K_v = \lim_{s \to \infty} sG(s)$ and $K_v = \frac{2}{3}$ gives,

$$K_v = \lim_{s \to \infty} \frac{K}{s^2 + (a + 2)s + (2a + 2)} \implies \frac{K}{2a + 2} = \frac{2}{3}$$

Solving for $K$ and $a$ using the above equation and $K = 2a$ gives $a = 2$, and $K = 4$.

Part (b): Using the result from part (a):

$$G(s) = \frac{4}{s(s^2 + 4s + 6)}$$

which is obviously Type 1 system $\implies e_{ss} = 0$ to unit step input.
Transient Analysis

Transient Response

Transient response allows for determining whether or not a system is stable and, if so, how stable it is (i.e. relative stability) as well as the speed of response when a step reference input is applied.

A typical time-domain response of a second order system (closed loop) to a unit step input is shown.

![Time-Domain Response Diagram]

- $c_{\text{max}}$:
- $c_{ss}$:
- $0.9c_{ss}$:
- $0.5c_{ss}$:
- $r(t)$:
- $c(t)$:
- $t_d$, $t_r$, $t_{\text{max}}$, $t_s$:
- $M_p$:
Key Definitions:

1. **Max Overshoot ($M_p$)**
   
   \[ M_p = \frac{c_{max} - c_{ss}}{c_{ss}} \]
   
   - $c_{max}$: max value of $c(t)$, $c_{ss}$: steady-state value of $c(t)$
   - $\%$ max overshoot $= 100 \times M_p$
   - $M_p$ determines relative stability: Large $M_p$ $\iff$ less stable

2. **Delay time ($t_d$)**: Time for $c(t)$ to reach 50% of its final value.

3. **Rise time ($t_r$)**: Time for $c(t)$ to rise from 10% to 90% of its final value.

4. **Settling time ($t_s$)**: Time for $c(t)$ to decrease and stay within a specified (typically 5%) of $c_{ss}$.

**Desirable characteristics:** Small $M_p$, small $t_d$, quick $t_r$ and fast $t_s$ (cannot be accomplished simultaneously).
Consider a control system with closed-loop transfer function,

\[ M(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \quad M(0) = 1 \]

Characteristic Equation: \( \rho(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \)
has the following roots,

\[ s_{1,2} = -\xi \omega_n \pm j\omega_n \sqrt{1 - \xi^2} = -\alpha \pm j\omega \]

These are depicted in the following figure.

\[ \cos \theta = -\xi, \quad \tan \theta = \frac{\sqrt{1-\xi^2}}{-\xi} \]
Transient Response of 2\textsuperscript{nd}-Order Control System-Cont.

Response to unit step input ($R(s) = \frac{1}{s}$) is

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)(s+\alpha)^2 + \omega^2}$$

Use PFE, time-domain response is found to be

$$c(t) = c_{ss} + \frac{1}{\sqrt{1 - \xi^2}} e^{-\alpha t} \sin[\omega t - \theta], \quad \forall t \geq 0$$

$$\alpha = \xi \omega_n: \text{ Damping Factor- Controls the rate of rise time and decay time i.e. } \alpha \text{ controls damping and speed of response.}$$

Can control oscillations by changing $\omega$.
Can control damping by changing $\xi$. 
\[ \tau = \frac{1}{\alpha} \]: Time Constant

Large \( \alpha \) \( \Rightarrow \) small \( \tau \) \( \Rightarrow \) signal decays quickly.

\( \xi \): Damping Ratio (ratio between actual damping factor and the damping factor for critically damped (\( \xi = 1 \) \( \Rightarrow \) \( s_{1,2} = -\omega_n \)).

\( \omega_n \): Natural Undamped Frequency (\( \xi = 0 \) \( \Rightarrow \) \( s_{1,2} = \pm j\omega_n \) i.e. purely oscillatory with frequency \( \omega_n \))

\[ \omega = \omega_n \sqrt{1 - \xi^2} \]: Conditional Frequency
Transient Response - Different Damping Cases

(a) **Underdamped:**

\[ 0 < \xi < 1, \quad s_{1,2} = -\xi \omega_n \pm j\omega_n \sqrt{1 - \xi^2} \]

*Characteristics:* Small rise time \((t_r)\), large overshoot \((M_p)\).

(b) **Critically Damped:**

\[ \xi = 1, \quad s_{1,2} = -\omega_n \text{ (repeated real roots)} \]

*Characteristics:* No overshoot, slow/large rise time.
(c) **Overdamped:**

\[ \xi > 1 \quad s_{1,2} = -\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1} \text{ (two real distinct roots)} \]

*Characteristics:* No overshoot, very large rise time.

(d) **Undamped (Oscillatory):**

\[ \xi = 0, \quad s_{1,2} = \pm j\omega_n \]
(e) Negatively damped (unstable):

\[ \xi < 0, \quad s_{1,2} = -\xi \omega_n \pm j\omega_n \sqrt{1 - \xi^2} \]
1. Peak Time ($t_{max}$)

To find the peak time (time at which the step response reaches its maximum), we take the derivative of the step response and set it to zero.

$$\frac{dc(t)}{dt} = 0$$

$$\frac{dc(t)}{dt} = -\xi e^{-\xi \omega_n t} \frac{\sin(\omega t - \theta)}{\sqrt{1-\xi^2}} + \frac{e^{-\xi \omega_n t} \omega \cos(\omega t - \theta)}{\sqrt{1-\xi^2}}$$

Using $\omega = \omega_n \sqrt{1 - \xi^2}$ and trig identities, we can simplify the above equation as:

$$\frac{dc(t)}{dt} = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t} \sin(\omega t), \quad \forall t \geq 0,$$

Now, $\frac{dc(t)}{dt} = 0 \implies \sin(\omega t) = 0$ or when $t \to \infty$ (i.e. final value)

The first condition gives the extrema points (maxima and minima) of $c(t)$, i.e. $\omega t = n\pi \implies t = \frac{n\pi}{\omega_n \sqrt{1-\xi^2}}$

The first maximum (Max overshoot) of $c(t)$ happens for $n = 1$. Thus,

$$t_{max} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$
Note: Although Max and Min of $c(t)$ occur at periodic interval, the response is NOT periodic due to damping (unless $\xi = 0$).

2. Max Overshoot ($M_p$)

To find $M_p$, we substitute $t_{max}$ in expression for $c(t)$. This yields,

$$c_{max} = c(t) \bigg|_{t=t_{max}} = 1 + e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}}$$

Thus, using the fact that $c_{ss} = 1$, we get

$$M_p = c_{max} - 1 = e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}}$$

Or in percentage,

$$\% \text{Max Overshoot} = 100e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}}$$

As can be seen, Max Overshoot is solely a function of $\xi$. Hence, Larger $\xi \implies$ smaller $M_p$ ($\xi < 1$). But this would increase the delay time and rise time as seen next.

For $t_d$, $t_r$, and $t_s$ only approximate equations can be obtained. These are given next.
3. Delay Time ($t_d$)

For $t_d$, we set $c(t) = 0.5$ and solve for $t_d$,

$$t_d \approx \frac{1 + 0.7\xi}{\omega_n}, \quad 0 < \xi < 1$$

$$t_d \approx \frac{1+0.6\xi+0.15\xi^2}{\omega_n}$$: wider range of $\xi$ and more accurate.

4. Rise Time ($t_r$)

$$t_r \approx \frac{0.8 + 2.5\xi}{\omega_n}, \quad 0 < \xi < 1$$

$$t_r = \frac{1+1.1\xi+1.4\xi^2}{\omega_n}$$, wider range of $\xi$ and more accurate.

5. Settling Time ($t_s$)

$$t_s \approx \frac{4}{\xi\omega_n}$$

As can be seen, Small $\xi \implies$ smaller $t_d$ and $t_r$ but larger $t_s$.

**Optimum Range for $\xi$: $0.5 \leq \xi \leq 0.8$**
1. Gain Controller

Consider servo control system below:

\[ G(s) = \frac{K}{s(Js+B)} \Rightarrow \text{Type 1} \Rightarrow e_{ss} = 0 \text{ for } r(t) = u_s(t) \]

For a unit ramp input \( e_{ss} = \frac{1}{K_v} \) and

\[ K_v = \lim_{s \to 0} sG(s)H(s) = \frac{K}{B} \]

Thus,

\[ e_{ss} \text{ to unit ramp} \Rightarrow e_{ss} = \frac{B}{K} \]

which implies Small \( e_{ss} \) Requires Large Gain \( K \).
**Transient Analysis:**
The closed-loop transfer function

\[
M(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(sJ+B)}}{1+\frac{K}{s(sJ+B)}} = \frac{K/J}{s^2 + Bs/J + K/J}
\]

Comparing with standard case,

\[
M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2} \implies \omega_n = \sqrt{K/J} \text{ and } 2\omega_n\xi = B/J \implies \xi = \frac{B}{2\sqrt{KJ}}
\]

Since \( B \) and \( J \) cannot be tweaked (motor parameters), Large \( K \) \( \implies \) Reduced \( \xi \) \( \implies \) large \( M_p \) \( \implies \) i.e. less stable.

Thus, a simple gain controller \( K \) won’t produce desirable steady-state and transient behavior as a compromise between small steady state error and good relative stability and fast response cannot be achieved.
2. Proportional-Derivative Controller (PD)

Controller transfer function $G_c(s) = K_P + K_D s$, where $K_P$ is the proportional constant, and $K_D$ is the derivative constant.

**Steady State Error Analysis:**

Loop transfer function:

$$G(s) = G_c(s)G_p(s) = \frac{K_P+K_D s}{s(Js+B)} \quad \Rightarrow \quad \text{Still Type 1} \quad \Rightarrow \quad e_{ss} = 0 \text{ for } r(t) = u_s(t)$$

For a unit ramp input $e_{ss} = \frac{1}{K_v}$ and

$$K_v = \lim_{s \to 0} sG(s)H(s) = \frac{K_P}{B} \quad \Rightarrow \quad e_{ss} = \frac{B}{K_P}$$

i.e. it is possible to make $e_{ss}$ to a unit ramp as small as possible by increasing proportional Gain $K_P$. 
Transient Analysis:

The closed-loop transfer function

\[ M(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} = \frac{(K_P+K_D s)/J}{s^2 + \frac{(B + K_D)s}{J} + \frac{K_P}{J}} + \frac{2\xi\omega_n}{J} + \frac{\omega_n^2}{J} \]

Again, comparing with standard case,

\[ M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2} \implies \omega_n = \sqrt{\frac{K_P}{J}} \quad \text{and} \]

\[ 2\omega_n\xi = \frac{(B+K_D)}{J} \implies \xi = \frac{B+K_D}{2\sqrt{K_P J}} \]

Thus, we can choose:

(a) Large \( K_P \) for small \( e_{ss} \) to unit ramp, and
(b) Appropriate \( K_D \) to have \( 0.5 < \xi < 0.8 \).

However, PD controller adds a zero at \( s = -K_P/K_D \) which could have an impact in changing the shape of the response to unit step. Additionally, PD controller is susceptible to noise and difficult to realize.
3. Tachometer Control

Using \( \frac{1}{s(Js+B)} \Rightarrow \left( \frac{1}{Js+B} \right) \left( \frac{1}{s} \right) \), we design a rate feedback (tachometer) control as shown.

Alternatively, the above block diagram can be reduced to the typically used tachometer control system.
Analyzing Simple Controllers for 2\textsuperscript{nd} Order Systems-Cont.

**Steady State Error Analysis:**

Loop transfer function:
\[ G(s)H(s) = \frac{K(1+Kts)}{s(Js+B)} \implies \text{ Still Type 1 } \implies e_{ss} = 0 \text{ for } r(t) = u_s(t) \]

For a unit ramp input \( e_{ss} = \frac{1}{K_v} \) and
\[ K_v = \lim_{s \to 0} sG(s)H(s) = \frac{K}{B} \implies e_{ss} = \frac{B}{K} \]

i.e. it is possible to make \( e_{ss} \) to a unit ramp as small as possible by increasing Gain \( K \).

**Transient Analysis:**

The closed-loop transfer function
\[ M(s) = \frac{KJ}{s^2 + (B+KK_t)s + JK} \]

Comparing with standard case,
\[ M(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \implies \omega_n = \sqrt{\frac{K}{J}} \text{ and } \]
\[ 2\omega_n \xi = \frac{(B+KK_t)}{J} \implies \xi = \frac{B+KK_t}{2\sqrt{KJ}} \]
Analyzing Simple Controllers for $2^{nd}$ Order Systems-Cont.

Thus, we can choose:
(a) Large Gain $K$ for small $e_{ss}$ to unit ramp, and
(b) Appropriate $K_t$ to have $0.5 < \xi < 0.8$.

**Note:** Tachometer control doesn't have the same issues of the PD controller. Hence, used widely for servo control.

**Example:** For the tach control system below, find $K$ and $K_t$ such that max overshoot, $M_p$, to unit step is 0.2 and peak time is 1 second. Then, using these values of $K$ and $K_t$, find $t_r$ and $t_s$.

\[
M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.2 \implies \xi = 0.456
\]
\[
t_{max} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 1 \implies \omega_n = 3.53
\]
But,

\[ K = \omega_n^2 \implies K = 12.5 \]

Also,

\[ \xi = \frac{1+KK_t}{2\sqrt{K}} = 0.456 \implies K_t = 0.178 \]

\[ t_r = \frac{0.8+2.5\xi}{\omega_n} = 0.55\text{sec} \]

\[ t_s = \frac{4}{\xi\omega_n} = 2.48\text{sec} \]